Mixed Finite Element Multilevel Approach Enables Fast and Accurate Reservoir Simulations

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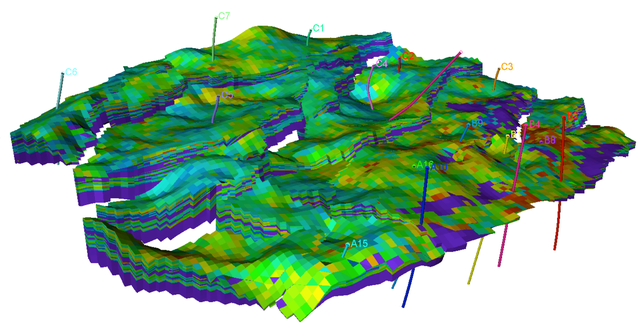

Abstract

The objective of this project is to conduct research into efficient, robust and scalable algorithmic strategies to enable high-performance massively parallel computations of reservoir simulation. The goal of this research is to open new opportunities for, e.g.

- meeting practical time to solution demands for improved engineering analysis,
- improved accuracy simulations via large-scale distributed computations,
- enabling new application areas in risk assessment and optimization.

The aim is to design an algorithmic strategy which maintains or possibly improves current state-of-the-art convergence rates while achieving scalability to (in principle) an arbitrary large number of cores.

Motivation

- **Scalable** solver using **multilevel** methods based on Finite Element discretization for **unstructured** meshes.
- Flexible framework for higher-order discretizations.
- More accurate approximations to velocities.

Governing equations

Using Darcy's law for flow of fluids in porous medium, a mass conservation law and a volume balance constraint, the following system of equations can be derived:

$$\mathbf{K}^{-1}\boldsymbol{\lambda}^{-1}\mathbf{u} + \boldsymbol{\nabla} p = \left(\sum_{\alpha} \boldsymbol{\rho}_{\alpha} \boldsymbol{f}_{\alpha}\right) \boldsymbol{g} \boldsymbol{\nabla} \boldsymbol{z} \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \sum_{\alpha} \frac{\boldsymbol{q}_{\alpha}}{\boldsymbol{\rho}_{\alpha}} \tag{2}$$

$$\boldsymbol{\phi} \frac{\boldsymbol{\partial} \boldsymbol{S}_{\alpha}}{\boldsymbol{\partial t}} + \boldsymbol{\nabla} \cdot \mathbf{u}_{\alpha}(\boldsymbol{S}_{\alpha}) = \frac{\boldsymbol{q}_{\alpha}}{\boldsymbol{\rho}_{\alpha}}, \tag{3}$$

where

- $\boldsymbol{\alpha} = \{\text{oil, water, gas}\}$ denotes the phase.
- $\boldsymbol{p}$ is the pressure.
- $\mathbf{u}$ is the total velocity equal to the sum of the phase velocities $\mathbf{u}_{\alpha}$.
- $\boldsymbol{S}_{\alpha}$ is the phase saturation.
- $\mathbf{K}$ is a tensor describing the permeability of the porous medium.
- $\boldsymbol{\lambda}$ is the total mobility:

$$\boldsymbol{\lambda} = \sum_{\alpha} \boldsymbol{\lambda}_{\alpha} = \sum_{\alpha} \frac{\boldsymbol{k}_{r,\alpha}(\boldsymbol{S}_{\alpha})}{\boldsymbol{\mu}_{\alpha}} \tag{4}$$

- $\boldsymbol{k}_{r,\alpha}$ is the phase relative permeability and $\boldsymbol{\mu}_{\alpha}$ is the phase viscosity.
- $\boldsymbol{f}_{\alpha} = \dfrac{\boldsymbol{\lambda}_{\alpha}}{\boldsymbol{\lambda}}$ is the fractional flow function and $\boldsymbol{\rho}_{\alpha}$ is the phase density.
- $\boldsymbol{\phi}$ is the porosity.
- $\boldsymbol{q}_{\alpha}$ is the source term for wells and and $\boldsymbol{g}$ is the standard gravity.

The phase velocity is related to the total velocity by

$$\mathbf{u}_{\alpha} = \boldsymbol{f}_{\alpha} \left(\mathbf{u} + \mathbf{G}_{\alpha}\right), \tag{5}$$

where

$$\mathbf{G}_{\alpha} = \sum_{\substack{\beta=o,w,g \\ \beta \neq \alpha}} \boldsymbol{\lambda}_{\beta}(\boldsymbol{\rho}_{\alpha} - \boldsymbol{\rho}_{\beta}) \mathbf{K} \boldsymbol{g} \boldsymbol{\nabla} \boldsymbol{z} \tag{6}$$

allows for counter-current flow due to gravity. Primary variables are the pressure $\boldsymbol{p}$, the total velocity $\mathbf{u}$ and the phase saturations $\boldsymbol{S}_{\alpha}$.

Discretization

Most commercial and academic reservoir simulators employ low-order Finite Volume discretization for an equivalent system of equations than stated above. Note that in that formulation, the total velocity is not solved for.

For equations (1) and (2), the **Mixed Finite Element** method is applied: Find $\mathbf{u}_{\tau} \in \mathbf{V}_{\tau} \in \boldsymbol{H}(\text{div}, \Omega)$, $\boldsymbol{p}_{\tau} \in \boldsymbol{W}_{\tau} \in \boldsymbol{L}^2(\Omega)$ such that

$$\left(\mathbf{K}^{-1}\boldsymbol{\lambda}^{-1}\mathbf{u}_{\tau}\right), \mathbf{v}) - (\boldsymbol{p}_{\tau}, \boldsymbol{\nabla} \cdot \mathbf{v}) = \left(\boldsymbol{g} \boldsymbol{\nabla} \boldsymbol{z} \sum_{\alpha} \boldsymbol{f}_{\alpha} \boldsymbol{\rho}_{\alpha}, \mathbf{v}\right), \qquad \mathbf{v} \in \mathbf{V}_{\tau} \tag{7}$$

$$(\boldsymbol{\nabla} \cdot \mathbf{u}_{\tau}, w) = \left(\sum_{\alpha} \frac{\boldsymbol{q}_{\alpha}}{\boldsymbol{\rho}_{\alpha}}, w\right), \qquad w \in \boldsymbol{W}_{\tau} \tag{8}$$

As a first step, a low-order Finite Volume method is used for discretizing equation (3), however, a future goal is to switch to the **Discontinuous Galerkin Finite Element** method. Backward Euler is used for temporal integration.

Advantages to this choice of discretization

Compared to low-order Finite Volume or Finite Difference on the whole system, this approach can

- reduce grid orientation effects,
- reduce smeared fronts due to numerical dispersion,
- enable flexible higher-order schemes,
- make more **robust** and **scalable multilevel** algorithms possible.

Typical solution approach

The typical approach and state-of-the-art in commercial and academic simulators is to:

- Linearize nonlinear system **globally** with some variant of **Newton's** method
- Solve linear system in each newton iteration with e.g. **Flexible GMRES** preconditioned with a two-stage preconditioner called "Constrained Pressure Residual":
  - First stage: Solve scalar pressure equation with **classical AMG** and correct full system
  - Second stage: Apply **Incomplete LU** to corrected full system

Challenge: For compositional simulations, the system can consist of more than 10 conservation laws and only one pressure equation, which means multilevel methods are only applied to the pressure equation and the remaining 10 conservation laws are not solved optimally.

Proposed solution approach

- Full Approximation Scheme.
- Element-based Algebraic Multigrid for mixed system for pressure and total velocity.
- Picard iterations on coarsest grid (saturations can be updated explicitly from the total velocity found for the mixed system). If Picard iterations are not strong enough, Newton iterations can be considered.
- Later: Multilevel solver for conservation laws.
- Possibly: Employ Hybridization to mixed system to be able to use smoothers suitable for Symmetric Positive Definite systems.

Element-based Algebraic Multigrid (AMGe)

- In reservoir simulation, only the finest mesh is given and we subsequently need to construct the hierarchy of coarse spaces only on the basis of that information. This fact advocates the use of Algebraic Multigrid methods over Geometric Multigrid for **unstructured** meshes.
- For mixed systems, which results in saddle point problems, classical AMG is not suitable.
- Element-based algebraic multigrid (AMGe) utilizes the information provided by the Finite Element discretizations of the equations. The individual element information, namely the element matrices and element topology, is the main input to construct the AMG hierarchy.

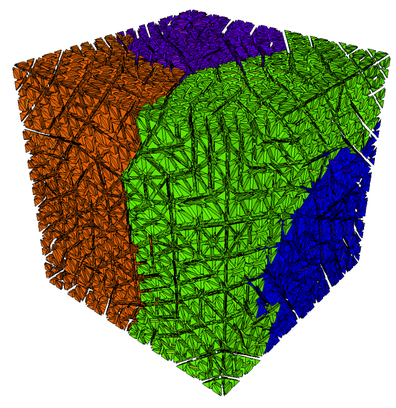

Figure 1: Fine mesh (30K elements)

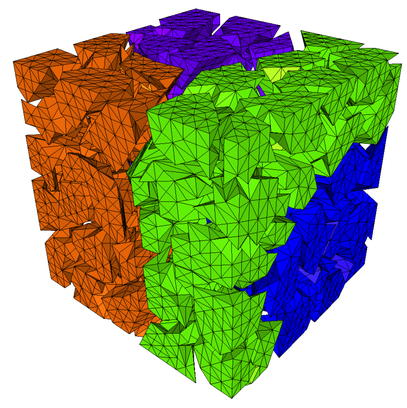

Figure 2: 128 agglomerates

The coarse system is small enough to be solved with a direct sparse solver.

Work in progress

- AMGe solver for the mixed system for pressure and total velocity.
- Multilevel solver for conservation laws.
- Discontinuous Galerkin Finite Element for conservation laws.

Prospects

- Risk assessment and Uncertainty Quantification.
- Optimization (e.g. managing intelligent well systems).
- Large-scale Enhanced Oil Recovery analysis and compositional simulations.

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