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Correlation Functions and Power Spectra

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Preface

The present note is a supplement to the textbook Digital Signal Processing [5, 6] used in the DTU course 02451 (former 04361) Digital Signal Processing (Digital Signalbehandling).

The note addresses correlations functions and power spectra and extends the material in Ch. 12 [5, 6].

Parts of the note are based on material by Peter Koefoed Møller used in the former DTU Course 4232 Digital Signal Processing.

The 6th edition provides an improvement of example 3.2 for which Olaf Peter Strelcyk is acknowledged.

In the 7th edition small errors and references are corrected.

In the 8th edition a few topics are elaborated.

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1 Introduction

The definitions of correlation functions and spectra for discrete-time and continuous-time (analog) signals are pretty similar. Consequently, we confine the discussion mainly to real discrete-time signals. The Appendix contains detailed definitions and properties of correlation functions and spectra for analog as well as discrete-time signals.

It is possible to define correlation functions and associated spectra for aperiodic, periodic and random signals although the interpretation is different. Moreover, we will discuss correlation functions when mixing these basic signal types.

In addition, the note include several examples for the purpose of illustrating the discussed methods.

2 Aperiodic Signals

The *crosscorrelation function* for two aperiodic, real[1], finite energy discrete-time signals $x_a(n)$, $y_a(n)$ is given by:

$$r_{x_a y_a}(m) = \sum_{n=-\infty}^{\infty} x_a(n)y_a(n-m) = x_a(m) * y_a(-m) \tag{1}$$

Note that $r_{x_a y_a}(m)$ is also an aperiodic signal. The autocorrelation function is obtained by setting $x_a(n) = y_a(n)$. The associated *cross-energy spectrum* is given by

$$S_{x_a y_a}(f) = \sum_{m=-\infty}^{\infty} r_{x_a y_a}(m)e^{-j2\pi fm} = X_a(f)Y_a^*(f) \tag{2}$$

The *energy* of $x_a(n)$ is given by

$$E_{x_a} = \int_{-1/2}^{1/2} S_{x_a x_a}(f)\, df = r_{x_a x_a}(0) \tag{3}$$

3 Periodic Signals

The *crosscorrelation function* for two periodic, real, finite power discrete-time signals $x_p(n)$, $y_p(n)$ with a common period N is given by:

$$r_{x_p y_p}(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)y_p((n-m))_N = x_p(m) * y_p(-m) \tag{4}$$

Note that $r_{x_p y_p}(m)$ is a periodic signal with period N. The associated *cross-power spectrum* is given by:

$$S_{x_p y_p}(k) = \frac{1}{N} \sum_{m=0}^{N-1} r_{x_p y_p}(m)e^{-j2\pi \frac{k}{N}m} = X_p(k)Y_p^*(k) \tag{5}$$

where $X_p(k)$, $Y_p(k)$ are the spectra of $x_p(n)$, $y_p(n)$ [2]. The spectrum is discrete with components at frequencies $f = k/N$, $k = 0, 1, \cdots, N-1$, or $F = kF_s/N$ where F_s is the sampling frequency. Further, the spectrum is periodic, $S_{x_p y_p}(k) = S_{x_p y_p}(k + N)$.

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[1]In the case of complex signals the crosscorrelation function is defined by $r_{x_a y_a}(m) = x_a(m) * y_a^*(-m)$ where $y_a^*(m)$ is the complex conjugated.

[2]Note that the definition of the spectrum follows [5, 6, Ch. 4.2] and differs from the definition of the DFT in [5, 6, Ch. 7]. The relation is: $\text{DFT}\{x_p(n)\} = N \cdot X_p(k)$.

The *power* of $x_p(n)$ is given by

$$P_{x_p} = \int_{-1/2}^{1/2} S_{x_p x_p}(f)\, df = r_{x_p x_p}(0) \tag{6}$$

Example 3.1 Determine the autocorrelation function and power spectrum of the tone signal:

$$x_p(n) = a\cos(2\pi f_x n + \theta)$$

with frequency $0 \le f_x \le 1/2$. The necessary requirement for $x_p(n)$ to be periodic is that the fundamental integer period N is chosen according to $N f_x = q$ where q is an integer. That means, f_x has to be a rational number. If $f_x = A/B$ is an irreducible fraction we choose $N_{\min} = B$. Of course any $N = N_{\min}\ell$, $\ell = 1, 2, 3, \cdots$ is a valid choice. Consequently, using Euler's formula with $q = A\ell$ gives:

$$x_p(n) = a\cos\left(2\pi \frac{q}{N} n + \theta\right) = \frac{a}{2}\left(e^{j\theta} e^{j2\pi \frac{q}{N} n} + e^{-j\theta} e^{-j2\pi \frac{q}{N} n}\right)$$

Thus since $x_p(n) = \sum_{k=0}^{N-1} X_p(k) e^{j2\pi \frac{k}{N} n}$, the spectrum is:

$$X_p(k) = \frac{a}{2} e^{j\theta}\delta(k - q) + \frac{a}{2} e^{-j\theta}\delta(k + q)$$

where $\delta(n)$ is the Kronecker delta function. The power spectrum is then found as:

$$S_{x_p x_p}(k) = X_p(k) X_p^*(k) = \frac{a^2}{4}\left(\delta(k - q) + \delta(k + q)\right)$$

Using the inverse Fourier transform,

$$r_{x_p x_p}(m) = \frac{a^2}{2}\cos\left(2\pi \frac{q}{N} m\right)$$

When mixing periodic signals with signals which have continuous spectra, it is necessary to determine the spectrum $S_{x_p x_p}(f)$ where $-1/2 \le f \le 1/2$ is the continuous frequency. Using that the constant (DC) signal $\frac{a}{2} e^{\pm j\theta}$ has the spectrum $\frac{a}{2} e^{\pm j\theta}\delta(f)$, where $\delta(f)$ is the Dirac delta function, and employing the frequency shift property, we get:

$$a\cos(2\pi f_x n + \theta) = \frac{a}{2} e^{j\theta} e^{j2\pi f_x n} + \frac{a}{2} e^{-j\theta} e^{-j2\pi f_x n}$$

That is,

$$X_p(f) = \frac{a}{2} e^{j\theta}\delta(f - f_x) + \frac{a}{2} e^{-j\theta}\delta(f + f_x)$$

and thus

$$S_{x_p x_p}(f) = \frac{a^2}{4}\left(\delta(f - f_x) + \delta(f + f_x)\right)$$

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Example 3.2 Consider two periodic discrete-time signals $x_p(n), y_p(n)$ with fundamental frequencies $0 \leq f_x \leq 1/2$ and $0 \leq f_y \leq 1/2$, respectively. Give conditions for which the cross-power spectrum vanishes.

Let us first consider finding a common period N, i.e., we have the requirements: $Nf_x = p_x$ and $Nf_y = p_y$ where p_x, p_y are integers. It is possible to fulfill these requirements only if both f_x and f_y are rational numbers. Suppose that $f_x = A_x/B_x$ and $f_y = A_y/B_y$ where A_x, B_x, A_y, B_y are integers, then the minimum common period $N_{\min} = \text{lcm}(B_x, B_y)$ where $\text{lcm}(\cdot, \cdot)$ is the least common multiple[3]. If N is chosen as $N = \ell N_{\min}$ where $\ell = 1, 2, 3, \cdots$ the signals will be periodic and $x_p(n)$ has potential components at $k_x = \ell p_x q_x$, where $p_x = N_{\min} f_x$ and $q_x = 0, 1, 2, \cdots, \lfloor 1/f_x \rfloor - 1$. Similarly, $y_p(n)$ has potential components at $k_y = \ell p_y q_y$, where $p_y = N_{\min} f_y$ and $q_y = 0, 1, 2, \cdots, \lfloor 1/f_y \rfloor - 1$. The cross-power spectrum does not vanish if $k_x = k_y$ occurs for some choice of q_x, q_y. Suppose that we choose a common period $N = N_{\min}\ell = B_x B_y$, then $k_x = N f_x q_x = B_y A_x q_x$ and $k_y = N f_y q_y = B_x A_y q_y$. Now, if $x_p(n)$ has a non-zero component at $q_x = B_x A_y$ and $y_p(n)$ has a non-zero component at $q_y = B_y A_x$ then $k_x = k_y$ and the cross-power spectrum does not vanish[4]. Otherwise, the cross-power spectrum will vanish. If N is not chosen as $N = \ell N_{\min}$ the cross-power spectrum does generally not vanish.

Let us illustrate the ideas by considering $x_p(n) = \cos(2\pi f_x n)$ and $y_p(n) = \cos(2\pi f_y n)$.

Case 1: In the first case we choose $f_x = 4/33$ and $f_y = 2/27$. $B_x = 3 \cdot 11$ and $B_y = 3^3$, i.e., $N_{\min} = \text{lcm}(B_x, B_y) = 3^2 \cdot 11 = 297$. Choosing $N = N_{\min}$, $x_p(n)$ has components at $k_x = 36$ and $k_x = 297 - 36 = 261$. $y_p(n)$ has components at $k_y = 22$ and $k_y = 297 - 22 = 275$. The cross-power spectrum thus vanishes.

Case 2: In this case we choose $f_x = 1/3$, $f_y = 1/4$ and $N = 10$. thus $N_{\min} = \text{lcm}(B_x, B_y) = \text{lcm}(3, 4) = 12$. Since N is not $\ell N_{\min}$ the stated result above does not apply. In fact, the cross-power spectrum does not vanish, as shown in Fig. 1.

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4 Random Signals

A random signal or stochastic process $X(n)$ has random amplitude values, i.e., for all time indices $X(n)$ is a random variable. A particular *realization* of the random signal is $x(n)$. The random signal is characterized by its probability density function (PDF)[5] $p(x_n)$, where x_n is a particular value of the signal. As an example, $p(x_n)$ could be Gaussian with zero mean, $\mu = E[x_n] = \int x_n p(x_n) dx_n = 0$ and variance $\sigma_x^2 = E[(x_n - \mu)^2] = \int (x_n - \mu)^2 p(x_n) dx_n$. That is, the PDF is

$$p(x_n) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-x^2(n)}{2\sigma_x^2}} , \forall n \tag{7}$$

Fig. 2 shows three different *realizations*, $x(n, \xi)$, $\xi = 1, 2, 3$ of the random signal. The family of different realizations is denoted the *ensemble*. Note that for e.g., $n = n_1$ that the outcomes of

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[3]In order to find the least common multiple of A, B, we first prime number factorize A and B. Then $\text{lcm}(A, B)$ is the product of these prime factors raised to the greatest power in which they appear.

[4]If $N = N_{\min}$ then this situation happens for $q_x = B_x A_y/\ell$ and $q_y = B_y A_x/\ell$

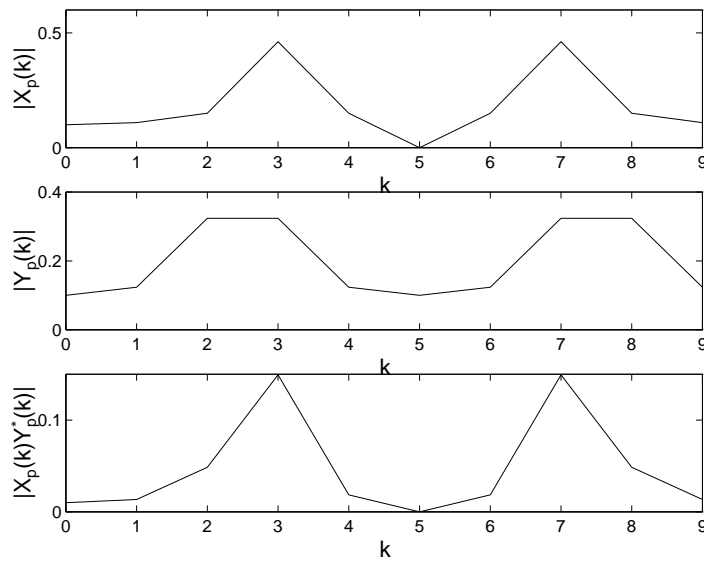[5]Also referred to as the first-order distribution.

Figure 1: Magnitude spectra $|X_p(k)|$, $|Y_p(k)|$ and magnitude cross-power spectrum $|S_{x_p y_p}(k)| = |X_p(k)Y_p^*(k)|$.
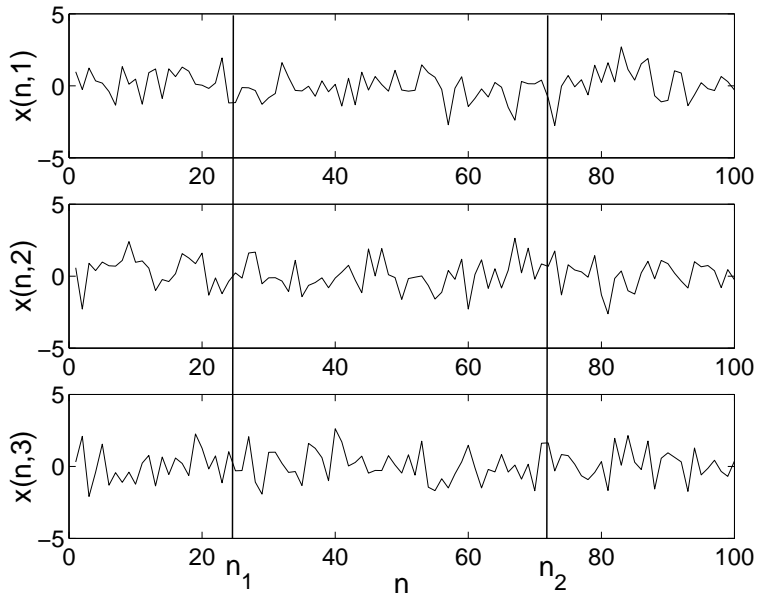

Figure 2: Three different realizations $x(n, \xi)$, $\xi = 1, 2, 3$ of a random signal.

$x(n_1)$ are different for different realizations. If one generated an infinite amount of realizations,

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$\xi = 1, 2, \cdots$, then these will reflect the distribution[6], as shown by

$$P(x_n) = \text{Prob}\{X(n) \le x_n\} = \lim_{K \to \infty} K^{-1} \sum_{\xi=1}^{K} \mu(x - x(n, \xi)) \tag{8}$$

where $P(x; n)$ is the distribution function and $\mu(\cdot)$ is the step function which is zero if the argument is negative, and one otherwise.

Random signals can be classified according to the taxonomy in Fig. 3.
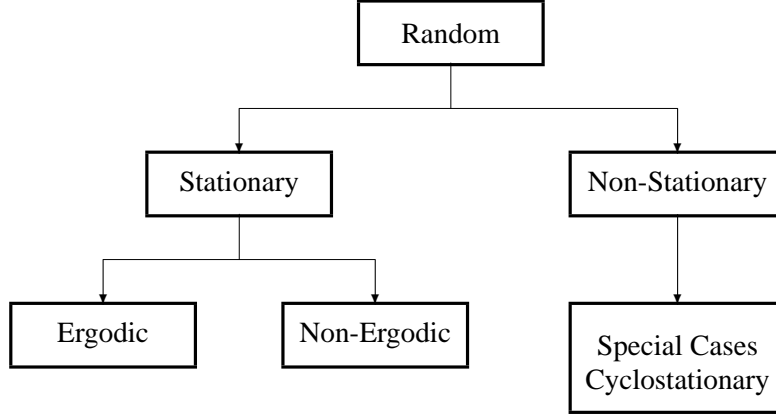

Figure 3: Taxonomy of random signals. Stationary signals are treated in Sec. 4.1, ergodic signals in Sec. 4.2, and cyclostationary signals are briefly mentioned in Sec. 5.2.

4.1 Stationary Signals

In generally we consider k'th order joint probability densities associated with the signal $x(n)$ defined by $p(x_{n_1}, x_{n_2}, \cdots, x_{n_k})$, i.e., the joint probability of $x(n_i)$'s, $i = 1, 2, \cdots, k$.

A signal is *strictly stationary* if

$$p(x_{n_1}, x_{n_2}, \cdots, x_{n_k}) = p(x_{n_1+\ell}, x_{n_2+\ell}, \cdots, x_{n_k+\ell}), \ \forall \ell, k \tag{9}$$

That is for any k, the k'th order probability density does not change over time, i.e., invariant to any time shift, ℓ.

Normally we consider only wide-sense stationary[7] in which random signal is characterized by its time-invariant the *mean value* and the *autocorrelation function*. The mean value is defined by:

$$m_x = E[x(n)] = \int_{-\infty}^{\infty} x_n \cdot p(x_n) \, dx_n \tag{10}$$

where $E[\cdot]$ is the expectation operator. The autocorrelation function is defined by:

$$\gamma_{xx}(m) = E[x(n)x(n-m)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_n x_{n-m} \cdot p(x_n, x_{n-m}) \, dx_n dx_{n-m} \tag{11}$$

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[6]The density function $p(x_n)$ is the derivative of the (cumulative) distribution function $\partial P(x; n)/\partial x$

[7]Also known as second order stationarity or weak stationarity.

Since the 2nd order probability density $p(x_n, x_{n-m})$ is invariant to time shifts for wide-sense stationary processes, the autocorrelation function is a function of m only.

The *covariance function* is closely related to the autocorrelation function and defined by:

$$c_{xx}(n) = E[(x(n) - m_x) \cdot (x(n-m) - m_x)] = \gamma_{xx}(m) - m_x^2 \tag{12}$$

For two signals $x(n), y(n)$ we further define the *crosscorrelation* and *crosscovariance* functions as:

$$\gamma_{xy}(m) = E[x(n)y(n-m)] \tag{13}$$

$$c_{xy}(m) = E[(x(n) - m_x) \cdot (y(n-m) - m_y)] = \gamma_{xy}(m) - m_x m_y \tag{14}$$

If $\gamma_{xy}(m) = m_x \cdot m_y$, i.e. $c_{xy}(m) = 0$ for all m, the signals are said to be *uncorrelated*, and if $\gamma_{xy}(m) = 0$ they are said to be *orthogonal*.

The *power spectrum* and *cross-power spectrum* are defined as the Fourier transforms of the autocorrelation and crosscorrelation functions, respectively, i.e.,

$$\Gamma_{xx}(f) = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m) e^{-j2\pi fm} \tag{15}$$

$$\Gamma_{xy}(f) = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m) e^{-j2\pi fm} \tag{16}$$

The *power* of $x(n)$ is given by

$$P_x = \int_{-1/2}^{1/2} \Gamma_{xx}(f)\, df = \gamma_{xx}(0) = E[x^2(n)] \tag{17}$$

The inverse Fourier transforms read:

$$\gamma_{xx}(m) = \int_{-1/2}^{1/2} \Gamma_{xx}(f) e^{j2\pi fm}\, df \tag{18}$$

$$\gamma_{xy}(m) = \int_{-1/2}^{1/2} \Gamma_{xy}(f) e^{j2\pi fm}\, df \tag{19}$$

Example 4.1 Let $x(n)$ be a *white noise* signal with power P_x, i.e., the power spectrum is flat (white) $\Gamma_{xx}(f) = P_x$. By inverse Fourier transform, the associated autocorrelation function is $\gamma_{xx}(m) = P_x \cdot \delta(m)$. Note that according to properties of autocorrelation functions 21, $\lim_{m\to\infty} \gamma_{xx}(m) = m_x^2$. That is the mean value of a white noise signal is $m_x = 0$.

■

Example 4.2 Evaluate the autocorrelation function and power spectrum for the signal $z(n) = ax(n) + by(n) + c$ where and a, b, c are constants and $x(n), y(n)$ are stationary signals with means m_x, m_y, autocorrelation functions $\gamma_{xx}(m), \gamma_{yy}(m)$, and crosscorrelation function $\gamma_{xy}(m)$. Using the definition Eq. (11), the fact that that the mean value operator is linear, i.e., $E[ax+by] =$

$aE[x] + bE[y]$, and the symmetry property of the crosscorrelation function ($\gamma_{xy}(-m) = \gamma_{yx}(m)$) we get:

$$
\begin{aligned}
\gamma_{zz}(m) &= E[z(n)z(n-m)] \\
&= E[(ax(n) + by(n) + c) \cdot (ax(n-m) + by(n-m) + c)] \\
&= E[a^2 x(n)x(n-m) + b^2 y(n)y(n-m) + abx(n)y(n-m) + aby(n)x(n-m) \\
&\quad + acx(n) + acx(n-m) + bcy(n) + bcy(n-m) + c^2] \\
&= a^2 \gamma_{xx}(m) + b^2 \gamma_{yy}(m) + ab(\gamma_{xy}(m) + \gamma_{xy}(-m)) + 2acm_x + 2bcm_y + c^2 \quad (20)
\end{aligned}
$$

According to Eq. (15) and (16) the power spectrum yields:

$$
\begin{aligned}
\Gamma_{zz}(f) &= a^2 \Gamma_{xx}(f) + b^2 \Gamma_{yy}(f) + ab(\Gamma_{xy}(f) + \Gamma_{xy}^*(f)) + (2acm_x + 2bcm_y + c^2)\delta(f) \\
&= a^2 \Gamma_{xx}(f) + b^2 \Gamma_{yy}(f) + 2ab\mathrm{Re}[\Gamma_{xy}(f)] + (2acm_x + 2bcm_y + c^2)\delta(f) \quad (21)
\end{aligned}
$$

Note that the power spectrum is a sum of a continuous part and a delta function in $f = 0$.

■

4.2 Ergodic Signals

Assuming a wide-sense stationary signal to be ergodic means that expectations - or ensemble averages - involved in determining the mean or correlation functions can be substituted by time averages. For example,

$$
m_x = \langle x(n) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n) \tag{22}
$$

$$
\gamma_{xy}(m) = \langle x(n)y(n-m) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-m) \tag{23}
$$

In the case Eq. (22) holds the signal is said to be *mean ergodic* and if Eq. (23) holds the signals are said to be *correlation ergodic*, see further [4], [5, 6, Ch. 12]. Most physical processes are mean and correlation ergodic and in general, we will tacitly assume ergodicity.

4.2.1 Correlation Function Estimates

Suppose that $r_{xy}(m)$ is an *estimate* of $\gamma_{xy}(m)$ based on N samples of $x(n)$ and $y(n)$. The estimate $r_{xy}(m)$ is recognized as a *random signal* since it is a function of the random signals $x(n)$ and $y(n)$. In order to assess the quality of an estimate we normally consider the *bias*, $B[r_{xy}(m)]$, the *variance*, $V[r_{xy}(m)]$, and the mean square error, $\mathrm{MSE}[r_{xy}(m)]$, defined by:

$$
\begin{aligned}
B[r_{xy}(m)] &= E[r_{xy}(m)] - \gamma_{xy}(m) \tag{24} \\
V[r_{xy}(m)] &= E[(r_{xy}(m) - E[r_{xy}(m)])^2] \tag{25} \\
\mathrm{MSE}[r_{xy}(m)] &= E[(r_{xy}(m) - \gamma_{xy}(m))^2] = B^2[r_{xy}(m)] + V[r_{xy}(m)] \tag{26}
\end{aligned}
$$

Note that the variance and mean square error are positive quantities.

Suppose that $x(n), y(n)$ are correlation ergodic random signals and we have collected N samples of each signal for $n = 0, 1, \cdots, N - 1$. Using a truncated version of Eq. (23) an estimate becomes:

$$
r'_{xy}(m) = \frac{1}{N - m} \sum_{n=m}^{N-1} x(n)y(n-m), \quad \text{for } m = 0, 1, \cdots, N - 1 \tag{27}
$$

For $0 \leq m \leq N - 1$, the bias is assessed by evaluating

$$
\begin{aligned}
E[r'_{xy}(m)] &= E\left[\frac{1}{N-m}\sum_{n=m}^{N-1} x(n)y(n-m)\right] \\
&= \frac{1}{N-m}\sum_{n=m}^{N-1} E[x(n)y(n-m)] \\
&= \frac{1}{N-m}\sum_{n=m}^{N-1} \gamma_{xy}(m) = \gamma_{xy}(m)
\end{aligned}
\tag{28}
$$

That is $B[r'_{xy}(m)] = 0$, and the estimator is said to be *unbiased*. The variance is more complicated to evaluate. An approximate expression is given by (see also [5, Ch. 14])

$$
V[r'_{xy}(m)] = \frac{N}{(N-m)^2}\sum_{n=-\infty}^{\infty} \gamma_{xx}(n)\gamma_{yy}(n) + \gamma_{xy}(n-m)\gamma_{yx}(n+m)
\tag{29}
$$

Provided the sum is finite (which is the case for correlation ergodic signals), the variance vanishes for $N \to \infty$, and consequently $\lim_{N\to\infty} r_{xy}(m) = \gamma_{xy}(m)$. The estimate is thus referred to as a *consistent* estimate. However, notice that $V[r'_{xy}(m)] = O(1/(N-m))$, i.e., for m close to N the variance becomes very large.

An alternative estimator is given by:

$$
r_{xy}(m) = \frac{1}{N}\sum_{n=m}^{N-1} x(n)y(n-m), \quad \text{for } m = 0, 1, \cdots, N-1
\tag{30}
$$

For $0 \leq m \leq N - 1$, the bias is evaluated by considering

$$
\begin{aligned}
E[r_{xy}(m)] &= \frac{1}{N}\sum_{n=m}^{N-1} E[x(n)y(n-m)] \\
&= \frac{N-m}{N}\gamma_{xy}(m) = \left(1 - \frac{m}{N}\right)\gamma_{xy}(m)
\end{aligned}
\tag{31}
$$

That is, the bias is $B[r_{xy}(m)] = E[r_{xy}(m)] - \gamma_{xy}(m) = -m\gamma_{xy}(m)/N$. $r_{xy}(m)$ is thus a biased estimate, but vanishes as $N \to \infty$, for which reason the estimator is referred to as *asymptotically unbiased*. The variance can be approximated by

$$
V[r_{xy}(m)] = \frac{1}{N}\sum_{n=-\infty}^{\infty} \gamma_{xx}(n)\gamma_{yy}(n) + \gamma_{xy}(n-m)\gamma_{yx}(n+m)
\tag{32}
$$

Thus, generally $\lim_{N\to\infty} V[r_{xy}(m)] = 0$. Moreover, $V[r_{xy}(m)] = O(1/N)$, which means that the variance does not increase tremendously when m is close to N, as was the case for $r'_{xy}(m)$. The improvement in variance is achieved at the expense of increased bias. This phenomenon is known as the *bias-variance dilemma* which illustrated in Fig. 4. If the objective is to find an estimator which has minimum mean square error, this is achieved by optimally trading off bias and variance according to Eq. (26)[8].

In most situations the $r_{xy}(m)$ estimator has the smallest MSE, and is therefore preferable.

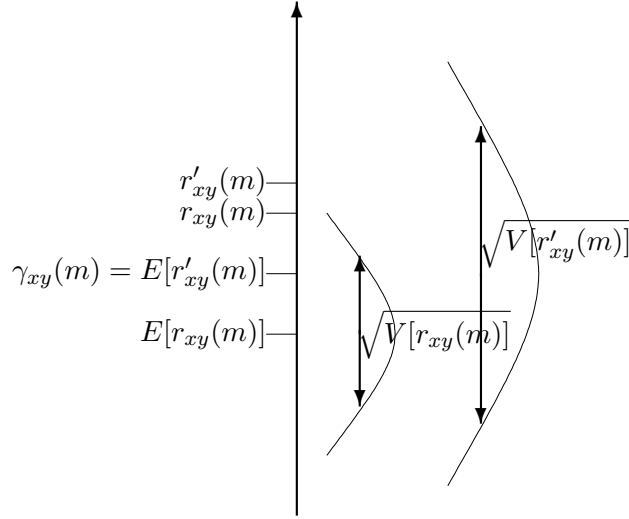[8]MSE is the sum of the variance and the squared bias.

Figure 4: The bias/variance dilemma.

4.3 Sampling of Random Signals

4.3.1 Sampling theorem for random signals

Following [4]: Suppose that $x_a(t)$ is a real stationary random analog signal with power density spectrum $\Gamma_{x_a x_a}(F)$ which is band-limited by B_x, i.e., $\Gamma_{x_a x_a}(F) = 0$, for $|F| > B_x$. By sampling with a frequency $F_s = 1/T > 2B_x$, $x(t)$ can be reconstructed from the samples $x(n) = x_a(nT)$ by the usual reconstruction formula

$$\widehat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin(\pi/T(t - nT))}{\pi/T(t - nT)} \tag{33}$$

The reconstruction $\widehat{x}_a(t)$ equals $x_a(t)$ in the mean square sense[9], i.e.,

$$E[(\widehat{x}_a(t) - x_a(t))^2] = 0 \tag{34}$$

As the autocorrelation function $\gamma_{x_a x_a}(\tau)$ is a non-random function of time, hence it is an ordinary aperiodic continuous-time signal with spectrum $\Gamma_{x_a x_a}(F)$. As a consequence when the sampling theorem is fulfilled then as usual [5, 6, Ch. 6.1]:

$$\gamma_{xx}(m) = \gamma_{x_a x_a}(mT) \tag{35}$$

$$\Gamma_{xx}(f) = F_s \sum_{k=-\infty}^{\infty} \Gamma_{x_a x_a}((f - k)F_s) \tag{36}$$

---

[9]Convergence in mean square sense does not imply convergence everywhere; however, the details are subtle and normally of little practical interest. Further reading on differences between convergence concepts, see [3, Ch. 8-4].

4.3.2 Equivalence of Correlation Functions

In order further to study the equivalence between correlations functions for analog and discrete-time random signals, suppose that $x_a(t)$ and $y_a(t)$ are correlation ergodic random analog with power density spectra $\Gamma_{x_a x_a}(F)$ and $\Gamma_{y_a y_a}(F)$ band-limited by B_x and B_y, respectively[10]. The crosscorrelation function is defined as:

$$\gamma_{x_a y_a}(\tau) = E[x_a(t)y_a(t-\tau)] = \lim_{T_i \to \infty} \frac{1}{T_i} \int_{-T_i/2}^{T_i/2} x_a(t)y_a(t-\tau)\,dt \tag{37}$$

That is, $\gamma_{x_a y_a}(\tau)$ can be interpreted as the integration of the product signal $z_a(t) = x_a(t)y_a(t-\tau)$ for a given fixed τ. The analog integrator is defined as the filter with impulse and frequency responses:

$$h_{\mathrm{int}}(t) = \begin{cases} \dfrac{1}{T_i} & ,|t| < T_i/2 \\[2mm] 0 & ,\text{otherwise} \end{cases} \qquad H_{\mathrm{int}}(F) = \frac{\sin \pi T_i F}{\pi T_i F} \tag{38}$$

Thus $\gamma_{x_a y_a}(\tau) = \lim_{T_i \to \infty} z_a(t) * h_{\mathrm{int}}(t)|_{t=0}$.

The question is: what is the required sampling frequency in order to obtain a discrete-time equivalent $\gamma_{xx}(n)$ of $\gamma_{x_a x_a}(\tau)$?. Suppose that $X(F)$, $Y(F)$ are the Fourier transforms of realizations of $x_a(t)$, $y_a(t)$ for $|t| < T_i/2$. Then, since $z_a(t)$ is a product of the two signals, the corresponding Fourier transform is:

$$Z(F) = X(F) * Y(F)e^{-j2\pi F\tau} \tag{39}$$

Thus, $Z(F)$ will generally have spectral components for $|F| < B_x + B_y$. Sampling $z_a(t)$ in accordance with the sampling theorem thus requires $F_s > 2(B_x + B_y)$. The power spectrum $\Gamma_{zz}(f)$, $f = F/F_s$ of the discrete-time signal $z(n)$ is sketched in Fig. 5. Notice, in principle we
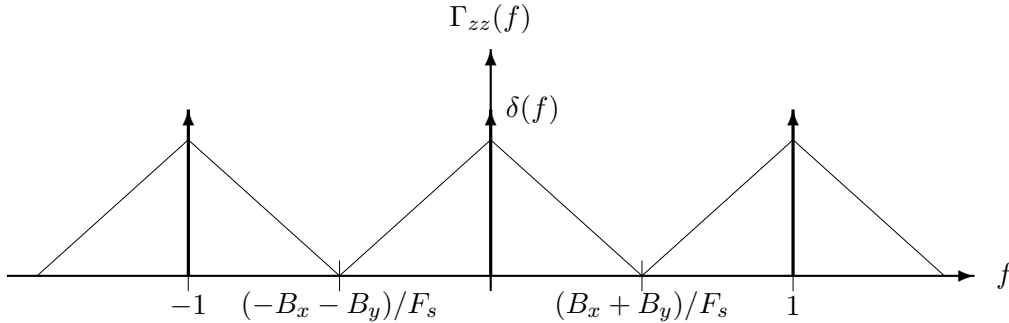
Figure 5: The power spectrum of the sampled $z(n)$, $\Gamma_{zz}(f)$ with possible δ-functions located at $f + k$, $k = 0, \pm 1, \pm 2, \cdots$.

can perform extreme subsampling with F_s arbitrarily close to zero. This causes aliasing; however, since the purpose of the integrator is to pick out the possible DC-component, the aliasing does not introduce error. The drawback is that it is necessary to use a large integration time, T_i, i.e., the

---

[10]That is, $\Gamma_{x_a x_a}(F) = 0$ for $|F| > B_x$ and $\Gamma_{y_a y_a}(F) = 0$ for $|F| > B_y$.

signals need to be observed for a long time. Secondly, we are normally not content with a digital determination of the crosscorrelation for a single lag, τ. Often the goal is to determine spectral properties by Fourier transformation of the discrete-time crosscorrelation function. That is, we want $\gamma_{x_a y_a}(\tau)$ for lags $\tau = m/F_s$ where $F_s \geq 2B_{xy}$ and B_{xy} is the band-limit of $\Gamma_{x_a y_a}(F)$. That is, $x_a(t), y_a(t)$ are sampled with $F_s > 2B_{xy}$. According to the table in Sec. A.6, $|\Gamma_{x_a y_a}(F)|^2 \leq \Gamma_{x_a x_a}(F)\Gamma_{y_a y_a}(F)$, which means that the band-limit $B_{xy} \leq \min(B_x, B_y)$. In consequence, $x_a(t)$ and/or $y_a(t)$ are allowed to be under-sampled when considering the crosscorrelation function[11].

4.4 Discrete-Time Systems and Power/Cross-Power Spectra

4.4.1 Useful Power/Cross-Power Expressions

Suppose that the real random stationary signals $x(n)$ and $y(n)$ are observed in the interval $0 \leq n \leq N - 1$. Now, perform the Fourier transforms of the signals, as shown by:

$$X(f) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi fn} \quad Y(f) = \sum_{n=0}^{N-1} y(n)e^{-j2\pi fn} \tag{40}$$

Note that $X(f)$ and $Y(f)$ also are (complex) random variables since they are the sum of random variables times a deterministic complex exponential function.

The intention is to show that the power and cross-power spectra can be expressed as:

$$\Gamma_{xx}(f) = \lim_{N\to\infty} \frac{1}{N} E\left[|X(f)|^2\right] = \lim_{N\to\infty} \frac{1}{N} E[X(f)X^*(f)] \tag{41}$$

$$\Gamma_{xy}(f) = \lim_{N\to\infty} \frac{1}{N} E[X(f)Y^*(f)] \tag{42}$$

Here we only give the proof of Eq. (42) since the proof of Eq. (41) is similar, see also [1, Ch. 5], [7, Ch. 11]. We start by evaluating

$$X(f)Y^*(f) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} x(n)y(k)e^{-j2\pi f(n-k)} \tag{43}$$

Next performing expectation $E[\cdot]$ gives[12]

$$E[X(f)Y^*(f)] = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E[x(n)y(k)]e^{-j2\pi f(n-k)}$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \gamma_{xy}(n-k)e^{-j2\pi f(n-k)} \tag{44}$$

Let $m = n - k$ and notice $-(N-1) \leq m \leq N - 1$. In the summation w.r.t. n and k it is easy to verify that a particular value of m appears $N - |m|$ times. By changing the summation w.r.t. n and k by a summation w.r.t. m:

$$\frac{1}{N} E[X(f)Y^*(f)] = \frac{1}{N}\sum_{m=-(N-1)}^{N-1} \gamma_{xy}(m)(N - |m|)e^{-j2\pi fm}$$

---

[11]When considering second order correlation functions and spectra, it suffices to study *linear* mixing of random signals. Suppose that $x_a(t) = g_1(t) + g_2(t)$ and $y_a(t) = g_2(t) + g_3(t)$ where the $g_i(t)$ signals all are orthogonal with band-limits B_i. The band-limit $B_x = \max(B_1, B_2)$ and $B_y = \max(B_2, B_3)$. Since $\gamma_{x_a y_a}(\tau) = \gamma_{g_2 g_2}(\tau)$, $B_{xy} = B_2$. Accordingly, $B_{xy} \leq \min(B_x, B_y)$.

[12]Note that the expectation of a sum is the sum of expectations.

$$= \sum_{m=-(N-1)}^{N-1} \gamma_{xy}(m)(1 - \frac{|m|}{N})e^{-j2\pi fm} \qquad (45)$$

By defining the signal $v(m) = 1 - |m|/N$ then $N^{-1}E[X(f)Y^*(f)]$ is seen to be the Fourier transform of the product $\gamma_{xy}(m) \cdot v(m)$. That is,

$$\frac{1}{N}E[X(f)Y^*(f)] = V(f) * \sum_{m=-(N-1)}^{N-1} \gamma_{xy}(m)e^{-j2\pi fm} \qquad (46)$$

where $*$ denotes convolution and $V(f)$ is the spectrum of $v(m)$ given by

$$V(f) = \frac{1}{N}\frac{\sin^2 \pi f N}{\sin^2 \pi f} \qquad (47)$$

which tends to a Dirac delta function $V(f) \to \delta(f)$ as $N \to \infty$. Consequently,

$$\lim_{N\to\infty} \frac{1}{N}E[X(f)Y^*(f)] = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m)e^{-j2\pi fm} = \Gamma_{xy}(f) \qquad (48)$$

Sufficient conditions are that the crosscovariance $c_{xy}(m) = \gamma_{xy}(m) - m_x m_y$ obey

$$\lim_{N\to\infty} \sum_{m=-N}^{N} |c_{xy}(m)| < \infty \quad \text{or} \quad \lim_{m\to\infty} c_{xy}(m) = 0 \qquad (49)$$

These conditions are normally fulfilled and implies that the process is mean ergodic [4].

   Eq. (41) and (42) are very useful for determining various power and cross-power spectra in connection with linear time-invariant systems. The examples below show the methodology.

Example 4.3 Find the power spectrum $\Gamma_{yy}(f)$ and the cross-power spectrum $\Gamma_{xy}(f)$ where $x(n)$ is a random input signal to a LTI system with impulse response $h(n)$ and output $y(n) = h(n) * x(n)$. Suppose that finite realizations of length N of $x(n)$ and $y(n)$ are given, and denote by $X(f)$ and $Y(f)$ the associated Fourier transforms which are related as: $Y(f) = H(f)X(f)$ where $H(f) \leftrightarrow h(n)$ is the frequency response of the filter. In order to find the cross-power spectrum we evaluate

$$X(f)Y^*(f) = X(f)H^*(f)X^*(f) = H^*(f)X(f)X^*(f) \qquad (50)$$

Since $H(f)$ is deterministic, the expectation becomes

$$E[X(f)Y^*(f)] = H^*(f)E[X(f)X^*(f)] \qquad (51)$$

Dividing by N and performing the limit operation yields:

$$\Gamma_{xy}(f) = \lim_{N\to\infty} \frac{1}{N}E[X(f)Y^*(f)] = H^*(f)\Gamma_{xx}(f) \qquad (52)$$

Since $\Gamma_{yx}(f) = \Gamma_{xy}^*(f)$ we further have the relation

$$\Gamma_{yx}(f) = H(f)\Gamma_{xx}(f) \qquad (53)$$

In the time domain, this corresponds to the convolution

$$\gamma_{yx}(m) = h(m) * \gamma_{xx}(m) \tag{54}$$

The output spectrum is found by evaluating

$$Y(f)Y^*(f) = H(f)X(f)H^*(f)X^*(f) = |H(f)|^2|X(f)|^2 \tag{55}$$

Proceeding as above

$$\Gamma_{yy}(f) = |H(f)|^2\Gamma_{xx}(f) \tag{56}$$

In the time domain:

$$\gamma_{yy}(m) = r_{hh}(m) * \gamma_{xx}(m) = h(m) * h(-m) * \gamma_{xx}(m) \tag{57}$$

∎

Example 4.4 Suppose that a signal source $g(n)$ and a noise source $s(n)$ in Fig. 6 are fully orthogonal. Find the power spectra $\Gamma_{x_1x_1}(f)$, $\Gamma_{x_2x_2}(f)$ and the cross-power spectrum $\Gamma_{x_2x_1}(f)$.
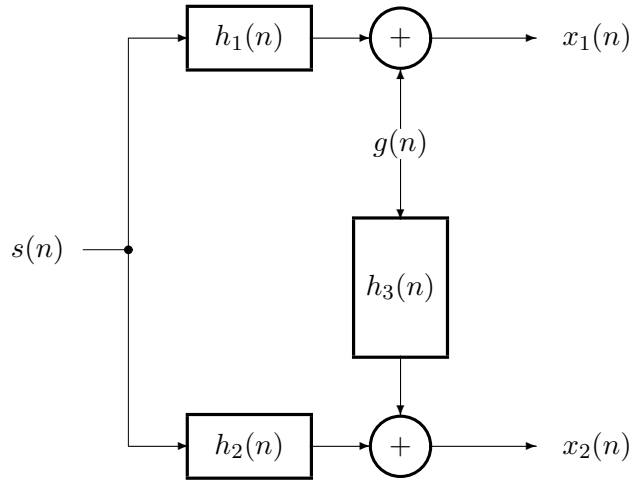

Figure 6: Two microphones $x_1(n)$, $x_2(n)$ record signals from a noise source $s(n)$ and a signal source $g(n)$.

Since $s(n)$ and $g(n)$ are fully orthogonal the superposition principle is applicable. Using the results of Example 4.3 we find:

$$\Gamma_{x_1x_1}(f) = \Gamma_{gg}(f) + \Gamma_{ss}(f)|H_1(f)|^2 \tag{58}$$
$$\Gamma_{x_2x_2}(f) = \Gamma_{gg}(f)|H_3(f)|^2 + \Gamma_{ss}(f)|H_2(f)|^2 \tag{59}$$

In order to determine $\Gamma_{x_2x_1}(f)$ we use Eq. (42), hence, we evaluate

$$
\begin{aligned}
X_2(f)X_1^*(f) &= (H_3(f)G(f) + H_2(f)S(f)) \cdot (G^*(f) + H_1^*(f)S^*(f)) \\
&= |G(f)|^2H_3(f) + G^*(f)S(f)H_2(f) + G(f)S^*(f)H_1^*(f)H_3(f) \\
&\quad + |S(f)|^2H_1^*(f)H_2(f)
\end{aligned} \tag{60}
$$

Performing expectation, dividing by N, and finally carrying out the limit operation gives

$$\Gamma_{x_2 x_1}(f) = \Gamma_{gg}(f)H_3(f) + \Gamma_{ss}(f)H_1^*(f)H_2(f) \tag{61}$$

Here we used $\Gamma_{gs}(f) = 0$ due to the fact that $g(n)$ and $s(n)$ are fully orthogonal.

∎

4.4.2 Some Properties

Following [1, Ch. 5.2.4]: suppose that the cross-power spectrum is expressed by its magnitude and phase, i.e.,

$$\Gamma_{xy}(f) = |\Gamma_{xy}(f)|e^{-j\theta_{xy}(f)} \tag{62}$$

As in Section 4.4.1, consider length N realization of $x(n)$ and $y(n)$ with Fourier transforms $X(f)$, $Y(f)$ and form a signal $z(n)$ via the frequency domain relation

$$Z(f) = aX(f) + Y(f)e^{-j\theta_{xy}(f)} \tag{63}$$

where a is a real constant. Notice, that $X(f)$, $Y(f)$ and $Z(f)$ are random variables whereas $\theta_{xy}(f)$ is not. Using Eq. (63) and the fact that $|Z(f)|^2 \geq 0$ gives:

$$a^2|X(f)|^2 + aX(f)Y^*(f)e^{j\theta_{xy}(f)} + aX^*(f)Y(f)e^{-j\theta_{xy}(f)} + |Y(f)|^2 \geq 0 \tag{64}$$

Proceeding as in Section 4.4.1 – in particular using Eq. (41) – the previous equation leads to

$$a^2\Gamma_{xx}(f) + a\Gamma_{xy}(f)e^{j\theta_{xy}(f)} + a\Gamma_{yx}(f)e^{-j\theta_{xy}(f)} + \Gamma_{yy}(f) \geq 0 \tag{65}$$

$\Gamma_{xy}(f)e^{j\theta_{xy}(f)} = |\Gamma_{xy}(f)|$ and $\Gamma_{yx(f)} = \Gamma_{xy}^*(f)$. Consequently,

$$a^2\Gamma_{xx}(f) + 2a|\Gamma_{xy}(f)| + \Gamma_{yy}(f) \geq 0 \tag{66}$$

Note that Eq. (66) is a quadratic inequality in a, which means that the determinant is negative or zero, i.e.,

$$4|\Gamma_{xy}(f)|^2 - 4\Gamma_{xx}(f)\Gamma_{yy}(f) \leq 0 \tag{67}$$

– or equivalently –

$$|\Gamma_{xy}(f)|^2 \leq \Gamma_{xx}(f)\Gamma_{yy}(f) \tag{68}$$

Evaluating Eq. (66) for $a = -1$ results in

$$2|\Gamma_{xy}(f)| \leq \Gamma_{xx}(f) + \Gamma_{yy}(f) \tag{69}$$

Further properties are listed in the Appendix.

5 Mixing Random and Deterministic Signals

Signals do not normally appear in their basic "pure" types like random, periodic or aperiodic. Often it is necessary to be able to handle the mixing of the basic signal types. It is very important to notice that in general, the mixing of a *stationary* random signal and a deterministic signal will result in a *non-stationary* signal.

In order to handle this problem one can adopt the framework of [2, Ch. 2.3]. Suppose that $x(n)$ is a mixed random/deterministic signal. $x(n)$ is said to be *quasi-stationary* if the following conditions hold:

$$(i) \qquad m_x(n) \;=\; E[x(n)], \quad \forall n, \; |m_x(n)| < \infty \tag{70}$$

$$(ii) \quad \gamma_{xx}(n_1, n_2) \;=\; E[x(n_1)x(n_2)], \quad \forall n_1, n_2, \; |\gamma_{xx}(n_1, n_2)| < \infty$$

$$\gamma_{xx}(m) \;=\; \lim_{N\to\infty} \frac{1}{N} \sum_{n_1=0}^{N-1} \gamma_{xx}(n_1, n_1 - m) = \langle \gamma_{xx}(n_1, n_1 - m) \rangle, \quad \forall m \tag{71}$$

Here $\langle \cdot \rangle$ denotes *time average* and the expectation $E[\cdot]$ is carried out w.r.t. the random components of the signal. If $x(n)$ is a pure random stationary signal then condition (i) and (ii) are trivially fulfilled. If $x(n)$ is a pure deterministic signal condition (ii) gives

$$\gamma_{xx}(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n - m) = \langle x(n)x(n - m) \rangle \tag{72}$$

which coincides with the expression Eq. (4) for periodic signals[13]. When aperiodic components are present, $\gamma_{xx}(m)$ – as defined in Eq. (72) – will normally be equal to zero.

For a general signal $g(n)$, we will introduce the notation

$$\langle E[g(n)] \rangle = \overline{E}[g(n)] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} E[g(n)] \tag{73}$$

to denote that both ensemble and time averages are carried out.

5.1 Erogodicity Result

[2, Theorem 2.3] states that if $x(n)$ is a quasi-stationary signal with mean $m_x(n) = E[x(n)]$ fulfilling

$$x(n) - m_x(n) = \sum_{q=0}^{\infty} h_n(q)e(n - q) \tag{74}$$

where 1) $e(n)$ is a sequence of independent random variables with zero mean, fininte variances $\sigma_e^2(n)$, and bounded fourth order moments. 2) $h_n(q)$, $n = 1, 2, \cdots$ are uniform stable filters. That is, the random part of $x(n)$ can be described as filtered white noise.

With probability one, as $N \to \infty$,

$$\frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n - m) \to \overline{E}[x(n)x(n - m)] = \gamma_{xx}(m) \tag{75}$$

In summary, the result is identical to the standard result for stationary random signals Eq. (22), (23): the time average equals the joint time-ensemble average $\overline{E}(\cdot)$.

The framework addressed autocorrelation functions; however, it can easily be adopted to cross-correlation functions as well.

---

[13]It does not matter whether the average is over one or an infinite number of periods.

5.2 Linear Mixing of Random and Periodic Signals

The above framework is applied to a simple example of linearly mixing a random and a periodic signal. Suppose that $x(n)$ is given by $x(n) = p(n) + s(n)$ where $p(n)$ is a periodic signal with period M, i.e., $p(n) = p(n + M)$, and $s(n)$ is a stationary, correlation ergodic random signal with mean m_s.

Consider first the formal definition of the mean of $x(n)$:

$$E[x(n)] = E[p(n) + s(n)] = p(n) + E[s(n)] \tag{76}$$

Thus the mean is time-varying; hence, the process is non-stationary. However, it is easy to verify that the mean is periodic with M as $E[x(n + M)] = p(n + M) + E[s(n)] = E[x(n)]$. This is known as cyclostationarity [3, Ch. 9]. The autocorrelation function of $x(n)$ is:

$$
\begin{aligned}
\gamma_{xx}(n_1, n_2) &= E[(p(n_1) + s(n_1)) \cdot (p(n_2) + s(n_2))] \\
&= p(n_1)p(n_2) + E[s(n_1)]p(n_2) + E[s(n_2)]p(n_1) + E[s(n_1)s(n_2)] \\
&= p(n_1)p(n_2) + m_s(p(n_1) + p(n_2)) + \gamma_{ss}(n_1 - n_2) \tag{77}
\end{aligned}
$$

Again, it is easy to verify $\gamma_{xx}(n_1 + M, n_2 + M) = \gamma_{xx}(n_1, n_2)$. Thus $x(n)$ is wide-sense cyclostationary. Moreover, $x(n)$ is a quasi-stationary signal, cf. Eq. (70), (71), since $E[x(n)]$ is limited and

$$
\begin{aligned}
\langle \gamma_{xx}(n_1, n_1 - m) \rangle &= \langle p(n_1)p(n_1 - m) \rangle + 2m_s \langle p(n_1) \rangle + \gamma_{ss}(m) \\
&= r_{pp}(m) + 2m_s m_p + \gamma_{ss}(m) \tag{78}
\end{aligned}
$$

is a function of m only. Here m_p is the time average of $p(n)$.

Next we will show that $x(n)$ is ergodic according to the definition in Section 5.1. Consider the ergodic formulation of the autocorrelation function of $x(n)$:

$$r_{xx}(m) = \langle x(n)x(n - m) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n - m) \tag{79}$$

Substituting the expression for $x(n)$ gives

$$
\begin{aligned}
r_{xx}(m) &= \langle (p(n) + s(n)) \cdot (p(n - m) + s(n - m)) \rangle \\
&= \langle p(n)p(n - m) \rangle + \langle p(n)s(n - m) \rangle \langle s(n)p(n - m) \rangle + \langle s(n)s(n - m) \rangle \\
&= r_{pp}(m) + r_{ps}(m) + r_{sp}(m) + \gamma_{ss}(m) \tag{80}
\end{aligned}
$$

$r_{pp}(m)$ and $\gamma_{ss}(m)$ are the usual autocorrelation functions for periodic and random signals, respectively. The crucial object is the crosscorrelation function $r_{ps}(m) = r_{sp}(-m)$. Focus on

$$r_{ps}(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} p(n)s(n - m) \tag{81}$$

If we use the periodicity of $p(n)$ we can rewrite the sum of products by considering first all products which involves $p(0) = p(kM)$, next all products which involves $p(1) = p(kM + 1)$ and so on. Assuming[14] $N = KM$, we can write:

$$r_{ps}(m) = \lim_{N \to \infty} \frac{1}{N} \left[p(0) \sum_{k=0}^{K-1} s(kM - m) + \right.$$

---

[14]It is pretty easy to verify that this restriction is not crucial to the subsequent arguments.

$$p(1) \sum_{k=0}^{K-1} s(kM - m + 1) +$$

$$\cdots +$$

$$\left. p(M-1) \sum_{k=0}^{K-1} s(kM - m + M - 1) \right]$$

$$= \lim_{N \to \infty} \frac{1}{N} \left[\sum_{q=0}^{M-1} p(q) \sum_{k=0}^{K-1} s(kM - m + q) \right]$$

$$= \lim_{N \to \infty} \left[\sum_{q=0}^{M-1} \frac{p(q)}{M} \frac{M}{N} \sum_{k=0}^{K-1} s(kM - m + q) \right] \tag{82}$$

Since M is a constant we can instead perform the limit operation w.r.t. K, that is,

$$r_{ps}(m) = \frac{1}{M} \sum_{q=0}^{M-1} p(q) \lim_{K \to \infty} \left[\frac{1}{K} \sum_{k=0}^{K-1} s(kM - m + q) \right] \tag{83}$$

Define $\widehat{m}_s = K^{-1} \sum_{k=0}^{K-1} s(kM - m + q)$, then $E[\widehat{m}_s] = m_s = E[s(n)]$ and the variance

$$
\begin{aligned}
V[\widehat{m}_s] &= E[\widehat{m}_s^2] - m_s^2 \\
&= \frac{1}{K^2} \sum_{k=0}^{K-1} \sum_{\ell=0}^{K-1} E[s(kM - m + q)s(\ell M - m + q)] - m_s^2 \\
&= \frac{1}{2K-1} \sum_{p=-(K-1)}^{K-1} \left(1 - \frac{|p|}{2K-1} \right) \gamma_{ss}(pM) - m_s^2
\end{aligned}
\tag{84}
$$

Under the standard mean ergodic condition, $\lim_{K \to \infty} V[\widehat{m}_s] = 0$ thus $\lim_{K \to \infty} \widehat{m}_s = m_s$. From Eq. (83) we then conclude that

$$r_{ps}(m) = \langle p(n) \rangle \cdot E[s(n)], \quad \forall 0 \le m \le N - 1 \tag{85}$$

where

$$\langle p(n) \rangle = \lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} p(n) = \frac{1}{M} \sum_{n=0}^{M-1} p(n) \tag{86}$$

In conclusion, a periodic and a random signal are *fully uncorrelated*, and the crosscorrelation function is the product of the time average of the periodic signal and the mean value of the random signal. If either the random signal has zero mean or the periodic signal has zero time average, the signals are *fully orthogonal*, i.e., $r_{ps}(m) = 0$.

Example 5.1 This example considers the determination of a weak periodic signal contaminated by strong noise. Suppose that $s(n)$ is a random noise signal which is generated by a first order AR process, cf. [5, 6, Ch. 12.2]. The autocorrelation function is given by

$$\gamma_{ss}(m) = \sigma^2 \cdot a^{|m|} \tag{87}$$

The power of $s(n)$ is $P_s = \gamma_{ss}(0) = \sigma^2$ and the squared mean value is $m_s^2 = \lim_{m \to \infty} \gamma_{ss}(m) = 0$. Further, assume that $p(n)$ is a periodic signal given by $p(n) = b \cos(2\pi n/N_0)$ with period N_0. The autocorrelation function is

$$r_{pp}(m) = p(m) * p(-m) = \frac{b^2}{2} \cos(2\pi m/N_0) \tag{88}$$

the power is $P_p = b^2/2$, and we assume that $P_p \ll P_s$.

The autocorrelation function of $x(n) = s(n) + p(n)$ is given by Eq. (80). Using the fact that $r_{sp}(m) = 0$ as $E[s(n)] = 0$ we have

$$r_{xx}(m) = \sigma^2 \cdot a^{|m|} + \frac{b^2}{2} \cos(2\pi m/N_0) \tag{89}$$

Thus for $|m|$ large the first term vanishes and we can employ the approximation

$$r_{xx}(m) \approx \frac{b^2}{2} \cos(2\pi m/N_0), \text{ for } |m| \to \infty \tag{90}$$

The maximum of $r_{xx}(m)$ for m large is then $b^2/2 = P_p$ and the period N_0 can be determined as the difference between two maxima.

∎

A Appendix: Properties of Correlation Functions and Power Spectra

A.1 Definitions

$x_a(t)$:　　Aperiodic analog real signal with finite energy.
　　　　$\lim_{t \to \infty} x_a(t) = x_{a\infty}$.

$x_a(n)$:　　Aperiodic discrete-time real signal with finite energy.
　　　　$\lim_{n \to \infty} x_a(n) = x_{a\infty}$.

$X_a(F), X_a(f)$:　　Fourier transform of $x_a(t)$ and $x_a(n)$.

$x_p(t)$:　　Periodic analog real signal with finite power and period T_p, i.e.,
　　　　$x_p(t + T_p) = x(t)$.

$x_p(n)$:　　Periodic discrete-time real signal with finite power and period N, i.e.,
　　　　$x_p(n + N) = x_p(n)$.

$X_p(k)$:　　Fourier transform of $x_p(n)$ or $x_p(t)$.

$x(t)$:　　Stationary, ergodic random real analog signal with mean $E[x(t)] = m_x$.

$x(n)$:　　Stationary, ergodic random real discrete-time signal with mean $E[x(n)] = m_x$.

$X(f)$:　　Fourier transform of $x(n)$ where $n \in [0; N-1]$.

$\gamma_{xx}(\tau), \gamma_{xy}(\tau)$:　　Auto- and crosscorrelation functions of random signals.

$r_{xx}(\tau), r_{xy}(\tau)$:　　Auto- and crosscorrelation functions for aperiodic and periodic signals.

$\Gamma_{xx}(f), \Gamma_{xy}(f)$:　　Power and cross-power spectra for random signals.

$S_{xx}(f), S_{xy}(f)$:　　Power(energy) and cross-power(energy) spectra for aperiodic signals.

$E[x(n)]$:　　Mean value operator, $E[x(n)] = m_x$.

$V[x(n)]$:　　Variance operator, $V[x(n)] = E[x^2(n)] - E^2[x(n)]$.

P:　　Power, $P = P_{\mathrm{AC}} + P_{\mathrm{DC}}$.

E:　　Energy, $E = E_{\mathrm{AC}} + E_{\mathrm{DC}}$.

A.2 Definitions of Correlation Functions and Power Spectra

A.2.1 Analog Random Signals

$$\gamma_{xy}(\tau) = E[x(t)y(t-\tau)] = \lim_{T_p \to \infty} \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t)y(t-\tau)\, dt \tag{91}$$

$$\Gamma_{xy}(F) = \int_{-\infty}^{\infty} \gamma_{xy}(\tau)e^{-j2\pi F\tau}\, d\tau \tag{92}$$

$$\gamma_{xy}(\tau) = \int_{-\infty}^{\infty} \Gamma_{xy}(F)e^{j2\pi F\tau}\, dF \tag{93}$$

Remark: $\Gamma_{xy}(F)$ is a continuous *cross-power density* spectrum.

A.2.2 Discrete-time Random Signals

$$\gamma_{xy}(m) = E[x(n)y(n-m)] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-m) \tag{94}$$

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$$\Gamma_{xy}(f) \quad = \quad \sum_{m=-\infty}^{\infty} \gamma_{xy}(m) e^{-j2\pi fm} \tag{95}$$

$$\gamma_{xy}(m) \quad = \quad \int_{-1/2}^{1/2} \Gamma_{xy}(f) e^{j2\pi fm} \, df \tag{96}$$

Remark: $\Gamma_{xy}(f)$ is a continuous *cross-power* spectrum, replicated periodically with $f = 1$ corresponding to the sampling frequency F_s.

A.2.3 Analog Periodic Signals

$$r_{x_p y_p}(\tau) \quad = \quad \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x_p(t) y_p(t - \tau) \, dt = x_p(\tau) * y_p(-\tau) \tag{97}$$

$$S_{x_p y_p}(k) \quad = \quad \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} r_{x_p y_p}(\tau) e^{-j2\pi \frac{k}{T_p} \tau} \, d\tau \tag{98}$$

$$r_{x_p y_p}(\tau) \quad = \quad \sum_{k=-\infty}^{\infty} S_{x_p y_p}(k) e^{j2\pi \frac{k}{T_p} \tau} \tag{99}$$

Remark: Discrete cross-power spectrum at frequencies $F = k/T_p$.

A.2.4 Discrete-time Periodic Signals

$$r_{x_p y_p}(m) \quad = \quad \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) y_p(n - m) = x_p(m) * y_p(-m) \tag{100}$$

$$S_{x_p y_p}(k) \quad = \quad \frac{1}{N} \sum_{m=0}^{N-1} r_{x_p y_p}(m) e^{-j2\pi \frac{k}{N} m} \tag{101}$$

$$r_{x_p y_p}(m) \quad = \quad \sum_{k=0}^{N-1} S_{x_p y_p}(k) e^{j2\pi \frac{k}{N} m} \tag{102}$$

Remark: Discrete cross-power spectrum at frequencies $f = k/N$ or $F = kF_s/N$ replicated periodically with $k = N$ corresponding to the sampling frequency F_s. Further, note that the pair of Fourier transforms defined in Eq. (101) and (102) corresponds to the definitions in [5, 6, Ch. 4.2]. The Discrete Fourier Transform defined in [5, 6, Ch. 7] is related by: $\text{DFT}\{r_{x_p y_p}(m)\} = N \cdot S_{x_p y_p}(k)$.

A.2.5 Analog Aperiodic Signals

$$r_{x_a y_a}(\tau) \quad = \quad \int_{-\infty}^{\infty} x_a(t) y_a(t - \tau) \, dt = x_a(\tau) * y_a(-\tau) \tag{103}$$

$$S_{x_a y_a}(F) \quad = \quad \int_{-\infty}^{\infty} r_{x_a y_a}(\tau) e^{-j2\pi F \tau} \, d\tau \tag{104}$$

$$r_{x_a y_a}(\tau) \quad = \quad \int_{-\infty}^{\infty} S_{x_a y_a}(F) e^{j2\pi F \tau} \, dF \tag{105}$$

Remark: $S_{x_a y_a}(F)$ is a continuous *cross-energy density* spectrum.

A.2.6 Discrete-time Aperiodic Signals

$$r_{x_a y_a}(m) \;=\; \sum_{-\infty}^{\infty} x_a(n) y_a(n-m) = x_a(m) * y_a(-m) \tag{106}$$

$$S_{x_a y_a}(f) \;=\; \sum_{m=-\infty}^{\infty} r_{x_a y_a}(m) e^{-j2\pi f m} \tag{107}$$

$$r_{x_a y_a}(m) \;=\; \int_{-1/2}^{1/2} S_{x_a y_a}(f) e^{j2\pi f m} \, df \tag{108}$$

Remark: $S_{x_a y_a}(f)$ is a continuous *cross-energy* spectrum, replicated periodically with the sampling frequency F_s.

A.3 Properties of Autocorrelation Functions

Random	Periodic	Aperiodic
$\gamma_{xx}(\tau) = \gamma_{xx}(-\tau)$	$r_{x_p x_p}(\tau) = r_{x_p x_p}(-\tau)$	$r_{x_a x_a}(\tau) = r_{x_a x_a}(-\tau)$
$|\gamma_{xx}(\tau)| \le \gamma_{xx}(0)$	$|r_{x_p x_p}(\tau)| \le r_{x_p x_p}(0)$	$|r_{x_a x_a}(\tau)| \le r_{x_a x_a}(0)$
$\lim_{\tau \to \infty} \gamma_{xx}(\tau) = m_x^2 = P_{\text{DC}}$	$r_{x_p x_p}(\tau) = r_{x_p x_p}(\tau + T_p)$	$\lim_{\tau \to \infty} r_{x_a x_a}(\tau) = x_{a\infty}^2$
$P = \gamma_{xx}(0) = m_x^2 + V[x(n)]$	$P = r_{x_p x_p}(0)$	$E = r_{x_a x_a}(0)$
$\gamma_{yy}(\tau) = r_{hh}(\tau) * \gamma_{xx}(\tau)$	$r_{y_p y_p}(\tau) = r_{hh}(\tau) * r_{x_p x_p}(\tau)$	$r_{y_a y_a}(\tau) = r_{hh}(\tau) * r_{x_a x_a}(\tau)$

Note: Similar properties exists for discrete-time signals.

A.4 Properties of Power Spectra

Random	Periodic	Aperiodic
$\Gamma_{xx}(F) = \Gamma_{xx}(-F)$	$S_{x_p x_p}(k) = S_{x_p x_p}(-k)$	$S_{x_a x_a}(F) = S_{x_a x_a}(-F)$
$P = \int_{-\infty}^{\infty} \Gamma_{xx}(F)\, dF$ $P = \int_{-1/2}^{1/2} \Gamma_{xx}(f)\, df$	$P = \sum_{k=-\infty}^{\infty} S_{x_p x_p}(k)$ $P_{\text{DC}} = S_{x_p x_p}(0)$	$E = \int_{-\infty}^{\infty} S_{x_a x_a}(F)\, dF$
$\Gamma_{xx}(F) = \lim_{T_p \to \infty} \frac{1}{T_p} E\left[|X(F)|^2\right] \geq 0$ $X(F) = \int_0^{T_p} x(t) e^{-j2\pi F t}\, dt$	$S_{x_p x_p}(k) = |X_p(k)|^2 \geq 0$	$S_{x_a x_a}(F) = |X_a(F)|^2 \geq 0$
Cont. real spect. + $m_x^2 \delta(F)$	Real discrete spect. at $F = k/T_p$	Cont. real spect. + $x_{a\infty}^2 \delta(F)$
$\Gamma_{yy}(F) = |H(F)|^2 \Gamma_{xx}(F)$	$S_{y_p y_p}(k) = |H(k/T_p)|^2 S_{x_p x_p}(k)$	$S_{y_a y_a}(F) = |H(F)|^2 S_{x_a x_a}(F)$

A.5 Properties of Crosscorrelation Functions

Random	Periodic	Aperiodic
$\gamma_{xy}(\tau) = \gamma_{yx}(-\tau)$	$r_{x_p y_p}(\tau) = r_{y_p x_p}(-\tau)$	$r_{x_a y_a}(\tau) = r_{y_a x_a}(-\tau)$
$\gamma_{xy}^2(\tau) \leq \gamma_{xx}(0)\gamma_{yy}(0)$	$r_{x_p y_p}^2(\tau) \leq r_{x_p x_p}(0) r_{y_p y_p}(0)$	$r_{x_a y_a}^2(\tau) \leq r_{x_a x_a}(0) r_{y_a y_a}(0)$
$2|\gamma_{xy}(\tau)| \leq \gamma_{xx}(0) + \gamma_{yy}(0)$	$2|r_{x_p y_p}(\tau)| \leq r_{x_p x_p}(0) + r_{y_p y_p}(0)$	$2|r_{x_a y_a}(\tau)| \leq r_{x_a x_a}(0) + r_{y_a y_a}(0)$
$\lim_{\tau\to\infty} \gamma_{xy}(\tau) = m_x m_y$	$r_{x_p y_p}(\tau) = r_{x_p y_p}(\tau + T_p)$	$\lim_{\tau\to\infty} r_{x_a y_a}(\tau) = x_{a\infty} y_{a\infty}$
$\gamma_{xy}(\tau) = m_x m_y$ uncorrelated	$r_{x_p y_p}(\tau) = 0, \forall \tau$ if no common frequencies	$r_{x_a y_a}(\tau) = x_{a\infty} y_{a\infty}$ uncorrelated
$\gamma_{yx}(\tau) = h(\tau) * \gamma_{xx}(\tau)$	$r_{y_p x_p}(\tau) = h(\tau) * r_{x_p x_p}(\tau)$	$r_{y_a x_a}(\tau) = h(\tau) * r_{x_a x_a}(\tau)$

Random	Periodic	Aperiodic
$\Gamma_{xy}(F) = \Gamma_{yx}^*(F)$	$S_{x_p y_p}(k) = S_{y_p x_p}^*(k)$	$S_{x_a y_a}(F) = S_{y_a x_a}^*(F)$
$\Gamma_{xy}(F) = \Gamma_{xy}^*(-F)$	$S_{x_p y_p}(k) = S_{x_p y_p}^*(-k)$	$S_{x_a y_a}(F) = S_{x_a y_a}^*(-F)$
$2|\Gamma_{xy}(F)| \le \Gamma_{xx}(F) + \Gamma_{yy}(F)$	$2|S_{x_p y_p}(k)| \le S_{x_p x_p}(k) + S_{y_p y_p}(k)$	$2|S_{x_a y_a}(F)| \le S_{x_a x_a}(F) + S_{y_a y_a}(F)$
$|\Gamma_{xy}(F)|^2 \le \Gamma_{xx}(F)\Gamma_{yy}(F)$	$S_{x_p y_p}(k) = 0, \forall k$ if no common frequencies	$|S_{x_a y_a}(F)|^2 = S_{x_a x_a}(F)S_{y_a y_a}(F)$
$\Gamma_{xy}(F) = \lim_{T_p \to \infty} \dfrac{1}{T_p} E\left[X(F)Y^*(F)\right]$ $X(F) = \int_0^{T_p} x(t)e^{-j2\pi Ft}\,dt$	$S_{x_p y_p}(k) = X_p(k)Y_p^*(k)$	$S_{x_a y_a}(F) = X_a(F)Y_a^*(F)$
Compl. cont. spectrum $+ m_x m_y \delta(F)$	Compl. discrete spectrum at $F = k/T$	Compl. cont. spectrum $+ x_{a\infty}y_{a\infty}\delta(F)$
$\Gamma_{yx}(F) = H(f)\Gamma_{xx}(f)$	$S_{y_p x_p}(k) = H(k/T)S_{x_p x_p}(k)$	$S_{y_a x_a}(F) = H(f)S_{x_a x_a}(f)$

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