The meaning of curvature
A distance geometric approach

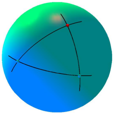
Manifold Learning

On the island of Hven

August 17-21, 2009

Steen Markvorsen

DTU Mathematics



Synopsis

1. Curvature sensitive geodesic sprays

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2. Structural results

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3. Curvature controlled comparison theory

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1. Curvature sensitive geodesic sprays
2. Structural results
3. Curvature controlled comparison theory
4. Length space analysis

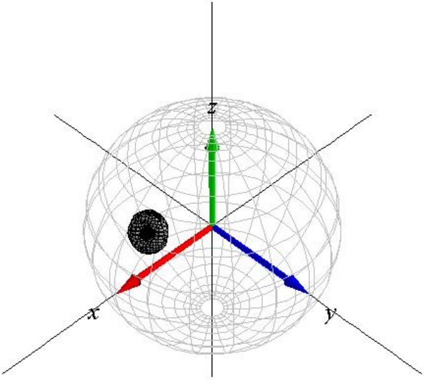
General case

Definition (Geodesics in a Riemannian manifold (M, g))

With a given starting point p and a unit initial direction $\dot{\gamma}(0)$ in the tangent space to M at p :

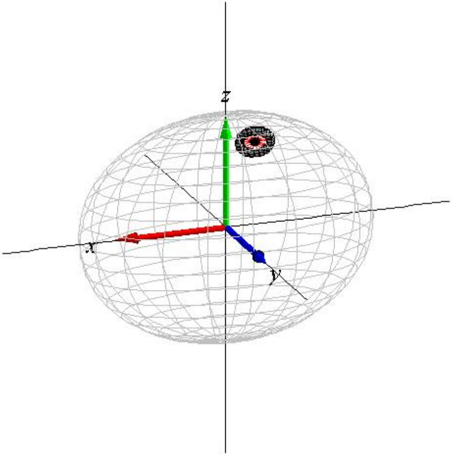
$$\frac{D\dot{\gamma}(t)}{dt} = 0 \quad .$$

Sphere case



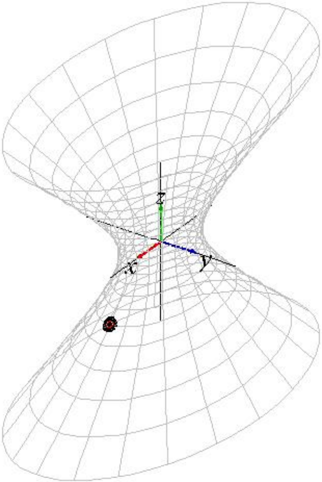
Geodesic spray on the sphere

Ellipsoid case, positive curvature



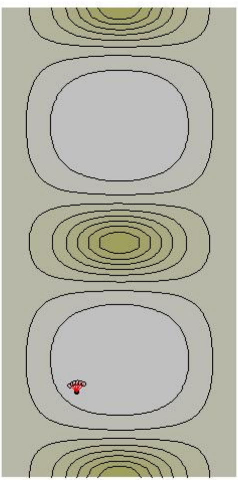
Geodesic spray on an ellipsoid

Hyperboloid of one sheet, negative curvature



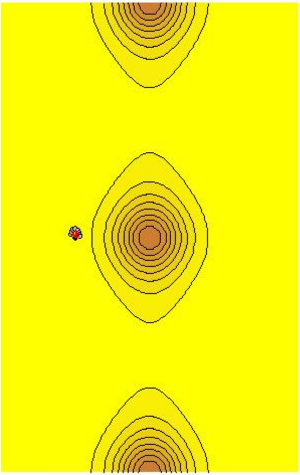
Geodesic spray on an elliptic hyperboloid of one sheet

Geodesic sprays converge when the curvature is positive



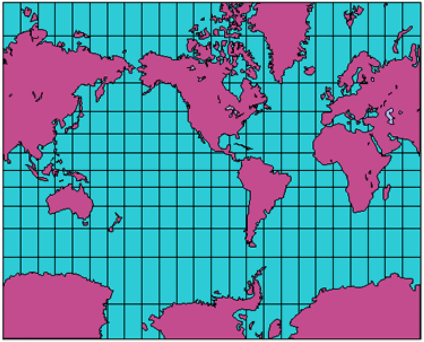
Geodesic spray in a curvature-colored map of the ellipsoid

Geodesic sprays diverge when the curvature is negative



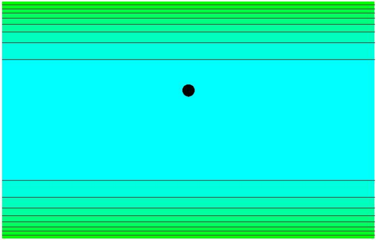
Geodesic spray in a curvature-colored map of the hyperboloid

Special maps: Mercator map of the globe



The well known Mercator map from any atlas

Conformally flat Mercator map of the sphere



The Mercator map with conformal factor coloring

Conformal curvature example

Proposition

A conformally flat metric

$$g(u,v) = e^{-2\psi(u,v)} g_0(u,v)$$

has the Gaussian curvature

$$K(u,v) = e^{2\psi(u,v)} \Delta\psi(u,v)$$

Conformal positive curvature example

Example (Constant curvature $K = 1$)

With conformal factor

$$e^{-2\psi(u,v)} = \cosh^{-2}(v)$$

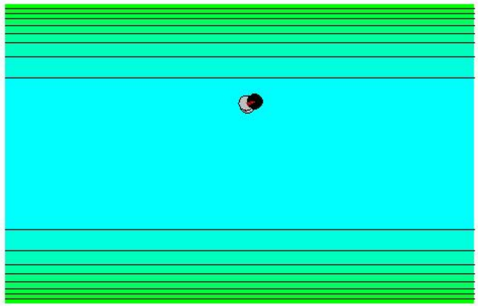
we have

$$\begin{aligned} \psi(u,v) &= \log(\cosh(v)) \\ \Delta\psi(u,v) &= 1 - \tanh^2(v) \end{aligned}$$

so that

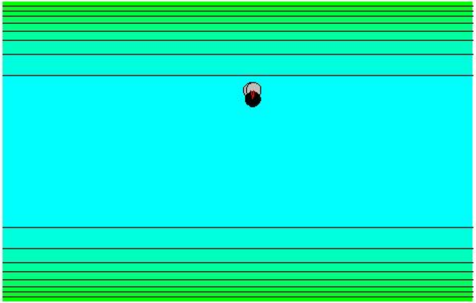
$$K(u,v) = e^{2\psi(u,v)}\Delta\psi(u,v) = \cosh^2(v)\left(1 - \tanh^2(v)\right) = 1 \quad .$$

Geodesics in the conformal Mercator map projection of the sphere



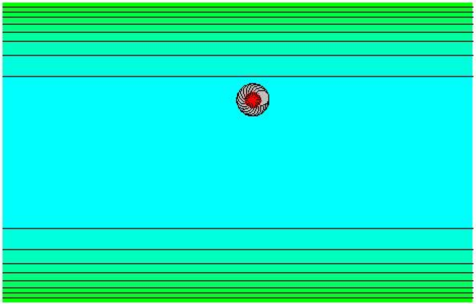
Two geodesics in conformally colored map of the sphere

Geodesics in the conformal Mercator map projection of the sphere



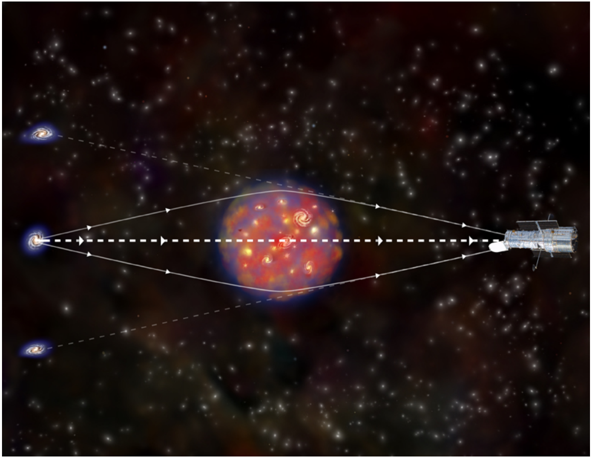
Two geodesics seemingly diverging?

Geodesics in the conformal Mercator map projection of the sphere



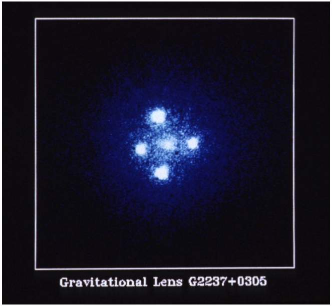
Geodesic spray in the Mercator map

Gravitational lensing



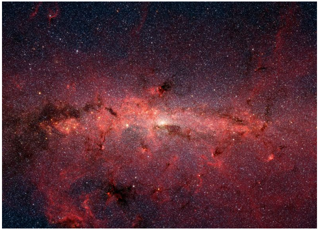
Gravitational lens principle

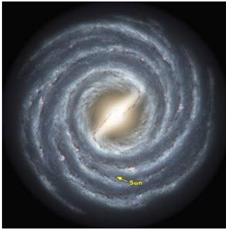
Gravitational lensing



A specific gravitational lens as seen by the Hubble telescope

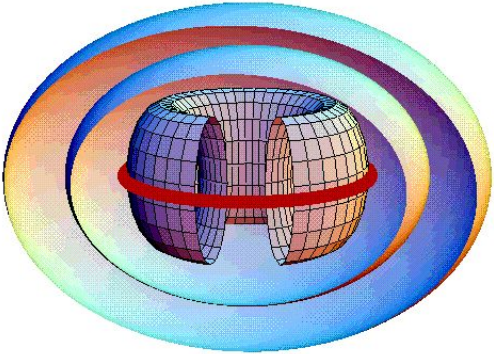
Black holes everywhere





A black hole resides at the center of every galaxy

Rotating black holes



The structure of a Kerr solution

Equations for gravity

Field equations (A. Einstein, 1915)

$$\mathrm{Ric} - \frac{1}{2}\,\mathsf{S}\,g \;=\; 8\pi\kappa\,T$$

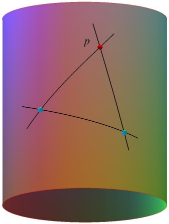
Lines and nonnegative curvature

Theorem (Cohn-Vossen, 1935)

Let F be a surface which satisfies the following conditions:

- *F is geodesically complete.*
- *F has nonnegative Gauss curvature everywhere.*
- *F contains a geodesic line.*

Then F is a generalized CYLINDER.



Flat standard cylinder $\mathbb{S}^1 \times \mathbb{R}^1$

Cosmologies

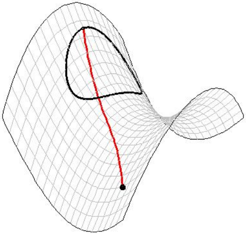
Theorem (Cheeger–Gromoll 1971, Yau 1982, —, Newman 1990)

Let M be a space time which satisfies the following conditions:

- M *is* timelike geodesically complete.
- M *has nonnegative timelike Ricci curvature everywhere.*
- M *contains a timelike line.*

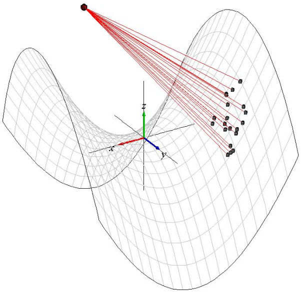
Then M is a generalized CYLINDER.

Distance Geometric Analysis



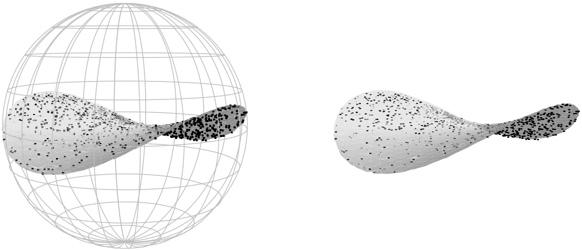
Geodesic distance contact to 1D submanifold in a 2D 'ambient' surface

Distance Geometric Analysis



Geodesic distance contact to a 2D submanifold in 3D flat space

Extrinsic disk of submanifold



Extrinsic disk of a surface

Distance Geometric Analysis

Proposition (Laplacian comparison technique)

$$
\begin{aligned}
\Delta^P \psi(r(x)) &\leq \left(\psi''(r(x)) - \psi'(r(x))\eta_w(r(x))\right) \|\nabla^P r\|^2 \\
&\quad + m\psi'(r(x))\left(\eta_w(r(x)) - h(r(x))\right) \\
&\leq \mathrm{L}\,\psi(r(x)) = -1 = \Delta^P E(x) \quad ,
\end{aligned}
$$

where

$$
\mathrm{L}\, f(r) = f''(r)\, g^2(r) + f'(r)\left(\left(m - g^2(r)\right)\eta_w(r) - m\, h(r)\right)
$$

is a special tailor made rotationally symmetric Poisson solution in a suitably chosen warped product comparison space.

Solutions to Laplacian processes on manifolds

$$H(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y)$$

$$G(x, y) = \int_0^{\infty} H(x, y, t)\, dt$$

$$E(x) = \int_P G(x, y)\, dy$$

$$\mathcal{A} = \int_P E(x)\, dx$$

Equations of Laplacian processes on manifolds

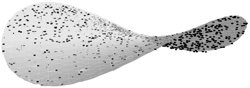
$$
\begin{aligned}
\left(\Delta_x^P - \frac{\partial}{\partial t}\right) H(x,y,t) &= 0 \\
\Delta_x^P G(x,y) &= 0 \\
\Delta_x^P E(x) &= -1
\end{aligned}
$$

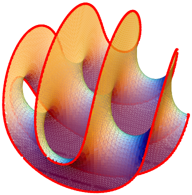
Theorem (SM and V. Palmer, *GAFA, 2003*)

Let P^m be a complete minimally immersed submanifold of an Hadamard–Cartan manifold N^n with sectional curvatures bounded from above by $b \leq 0$. Suppose that either $(b < 0 \;\; and \;\; m \geq 2)$ *or* $(b = 0 \;\; and \;\; m \geq 3)$.

Then P^m is transient.

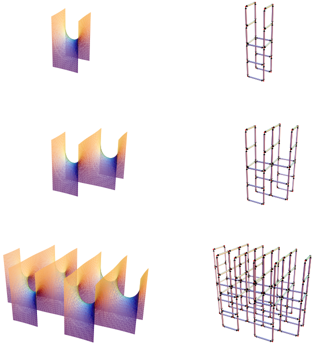
Minimality





Extrinsic disks of minimal surfaces in $\mathbb{R}^3$

Minimality



Scherk's doubly periodic minimal surface in $\mathbb{R}^3$ and a corresponding minimal web

Intrinsic mean exit time expansion

Theorem (A. Gray and M. Pinsky, 1983)

Let $B_r^m(p)$ denote an intrinsic geodesic ball of small radius r and center p in a Riemannian manifold (M^m, g) which has scalar curvature $\tau(p)$ at the center point p.

Then the mean exit time from $B_r(p)$ for Brownian particles starting at p is

$$E_r(p) = \frac{r^2}{2m} + \frac{\tau(p)\, r^4}{12m^2(m+2)} + r^5\, \varepsilon(r) \quad ,$$

where $\varepsilon(r) \to 0$ when $r \to 0$.

Extrinsic mean exit time expansion

Theorem (A. Gray, L. Karp, and M. Pinsky, 1986)

Let P^2 be a 2D surface in $\mathbb{R}^3$. For a point p in P we let $D_r(p)$ denote the extrinsic geodesic disk of small radius r and center p.

Then the mean exit time from $D_r(p)$ for Brownian particles starting at p is

$$E_r(p) = \frac{r^2}{4} + \frac{r^4}{6}(H^2 - K) + r^5 \varepsilon(r) \quad ,$$

where $\varepsilon(r) \to 0$ when $r \to 0$.

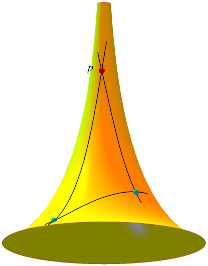
Slim, normal, and fat triangles

Theorem (Alexandrov, Toponogov, 60)

The (sectional) curvatures of a Riemannian manifold M^n satisfy $\mathrm{curv}(M) \geq 1$ if and only if every geodesic triangle Δ in M^n and comparison triangle Δ^* (with same edge lengths as Δ) in the unit sphere $\mathbb{S}^2_1$ satisfy the fatness condition:

$$\alpha_i \geq \alpha_i^*, \qquad i = 1, 2, 3 \quad .$$

Sign of Gaussian Curvature



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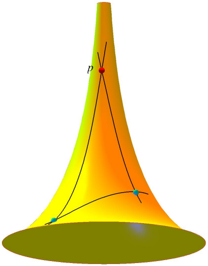
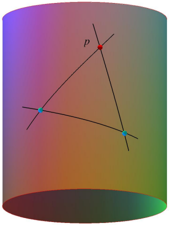
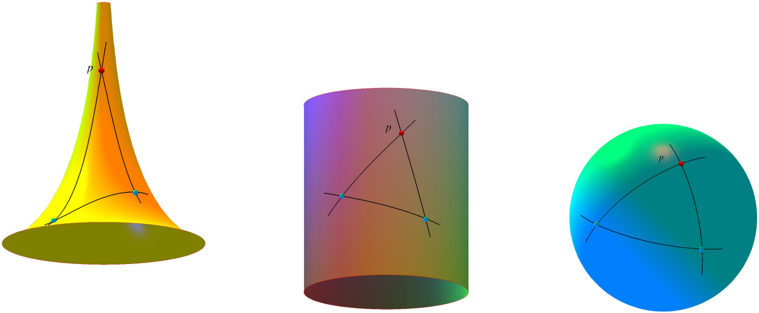
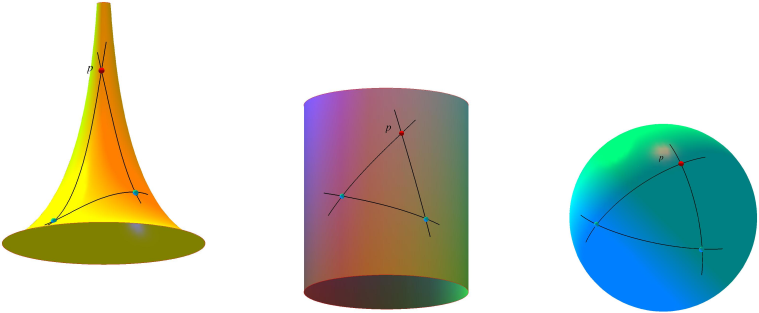




Sign of Gaussian Curvature

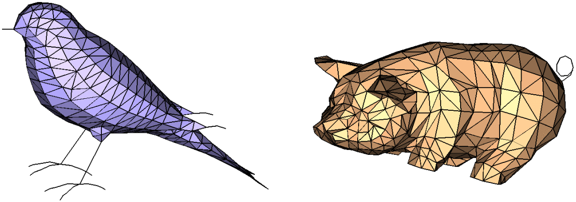


Sign of Gaussian Curvature



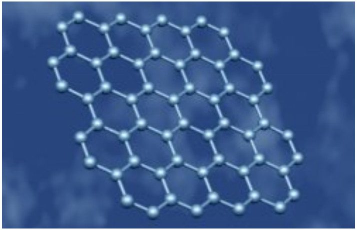
Negative, zero, and positive curvature

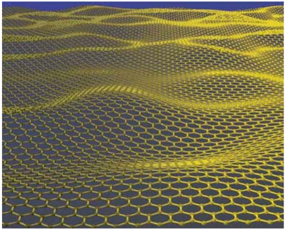
Objects admitting geodesic distances



Length spaces

Objects admitting geodesic distances





Graphene landscape

Other measures of size and shape

Definition

Let X denote a compact metric space.

For any q-tuple $\{x_1, \ldots, x_q\}$ of points in X
we let xt_q denote the average total distance

$$\mathrm{xt}_q(x_1, \ldots, x_q) = \binom{q}{2}^{-1} \sum_{i<j}^{n} \mathrm{dist}(x_i, x_j) \quad .$$

Consider the maximum, the q- extent of X:

$$\mathrm{xt}_q(X) = \max_{x_1, \ldots, x_q} \mathrm{xt}_q(x_1, \ldots, x_q) \quad .$$

Other measures of *size*

Theorem (O. Gross, 1964)

Let X be a compact connected metric space.

Then there is a unique positive real number $\mathrm{rv}(X)$ – the rendez vous value of X – with the following property:

For each finite collection of points $x_1, \ldots, x_q$ in X there exists a point y in X such that

$$(1/q)\sum_{i=1}^{q} \mathrm{dist}(x_i, y) = \mathrm{rv}(X) \ .$$

Large scale results for extents and rendez vous values

Theorem (K. Grove and SM, 1997)

Let X^n be an Alexandrov space with

$$\mathrm{curv}(X) \ \geq \ 1 \ .$$

Then

$$\mathrm{xt}_{\infty}(X) \leq \pi/2 \qquad \textit{and}$$

$$\mathrm{rv}(X) \leq \pi/2 \ .$$

One (and thence both) equality occurs if and only if X^n is a spherical suspension over an "equatorial" Alexandrov space Θ^{n-1} with $\mathrm{curv}(\Theta) \ \geq \ 1$ *.*

Large scale recognition stability

Theorem (G. Perelman and T. Yamaguchi, 1991)

Let X^n *be a compact Alexandrov space with* $\mathrm{curv}(X) \geq k$.

Then there exists a positive real number $\varepsilon = \varepsilon(X)$ *such that every other compact Alexandrov space* Y^n *with* $\mathrm{curv}(Y) \geq k$ *and Gromov–Hausdorff distance* $\mathrm{d}_{GH}(X, Y) \leq \varepsilon$ *is homeomorphic to the given space* X^n*.*

Reference: F. Memoli, *Gromov–Hausdorff distances in Euclidean spaces*.

B. Springborn, P. Schröder, and U. Pinkall

ACM Transactions on Graphics, Vol. 27, Article 77, No. 3, August 2008:

Definition

Two discrete metrics $\mathcal{L}$ and $\bar{\mathcal{L}}$ on M are (discretely) *conformally equivalent* if, for some assignment of numbers ψ_i to the vertices v_i, the metrics are related by

$$\mathcal{L}_{ij} = e^{-(\psi(i)+\psi(j))}\bar{\mathcal{L}}_{ij}$$

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Compare with the smooth definition of conformal maps

$$g(u,v) = e^{-2\psi(u,v)} g_0(u,v)$$

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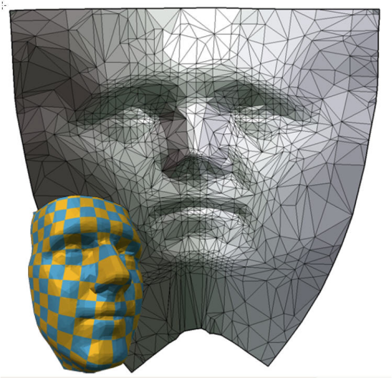

Conformal representation

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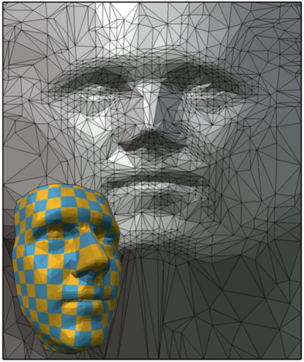
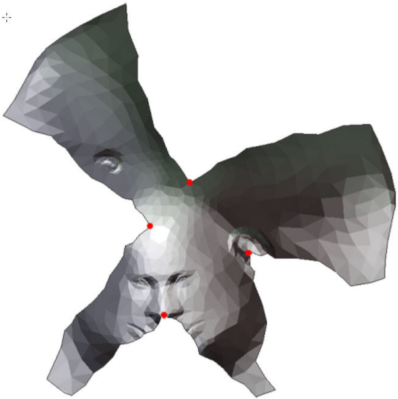


Conformal representation

Relaxing curvature along the image boundary



Conformal representation with cone singularities

Conclusion

Curvature matters on all scales:

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Curvature matters on all scales:

1. Globally, locally, and micro-locally

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Curvature matters on all scales:

1. Globally, locally, and micro-locally
2. In smooth and in discrete geometry

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Thank you for your attention!