Basic Concepts in LP

Terminology, the Simplex method, duality, and optimality conditions

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Basic concepts - LP I

We consider an LP-problem **LP** on standard form:

$$\max \; cx$$
$$Ax = b$$
$$x \in \mathbb{R}^n_+$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{n \times m}$.

All LP-problems can be transformed to this form.

Basic concepts - LP II

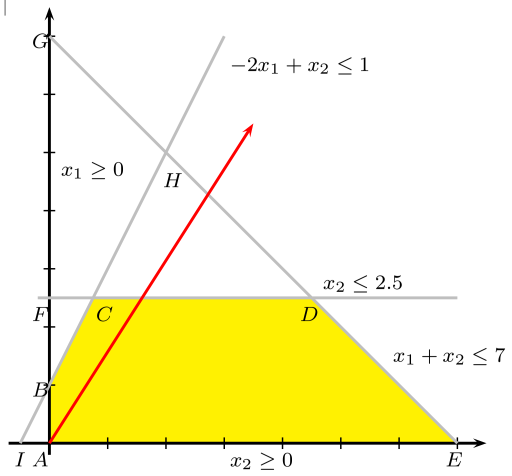
Stated in detail:

$$\begin{aligned} \max \quad z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ x_i \geq 0 \;\; , \;\; i \; = \; 1 \, , \dots \, n \end{aligned}$$

Basic concepts - LP III

- A **solution** to LP satisfies $Ax = b$.
- A **feasible solution** to LP satisfies $Ax = b \,\wedge\, x \geq 0$.
- An **optimal solution** to LP, x^* is a feasible solution satisfying that for any other feasible solution $\overline{x}$
  $cx^* \geq c\overline{x}$
- A **basis** for A is a set of m linearly independent columns from A.

Basis and corner points



$$\begin{array}{lrrl} \max & 7x_1 + 11x_2 & & \\ \text{s.t.} & -2x_1 + & x_2 \leq & 1 \\ & x_1 + & x_2 \leq & 7 \\ & & x_2 \leq & 5/2 \\ & & x_1, x_2 \geq & 0 \end{array}$$

A	$(0,0,1,7,\frac{5}{2})$	0
B	$(0,1,0,6,\frac{3}{2})$	11
G	$(0,7,-6,0,-\frac{9}{2})$	77
F	$(0,\frac{5}{2},-\frac{3}{2},\frac{9}{2},0)$	$27\frac{1}{2}$
I	$(-\frac{1}{2},0,0,\frac{15}{2},\frac{5}{2})$	$-3\frac{1}{2}$
E	$(7,0,15,0,\frac{5}{2})$	49
	$(-,0,-,-,0)$	$-$
H	$(2,5,0,0,-\frac{5}{2})$	69
C	$(\frac{3}{4},\frac{5}{2},0,\frac{15}{4},0)$	$32\frac{3}{4}$
D	$(\frac{9}{2},\frac{5}{2},\frac{15}{2},0,0)$	59

Basic concepts - LP IV

- **The basic solution** corresponding to the basis

$$B = A_{:B} = \{A_{j1}, ..., A_{jm}\}$$

  is the solution obtained from $Ax = b$ by setting $x_j = 0 \,, j \notin \{j_1, ..., j_m\}$. This is unique.
- **A basic solution** $\tilde{x}$ to LP is a solution, for which a basis B exists such that $\tilde{x}$ is the basic solution corresponding to B.

Basic concepts - LP V

Consider now the basis $B = \{A_{j1}, ..., A_{jm}\}$
The variables $x_{j_1}, ..., x_{j_m}$ are called **basic variables**, the other variables ($x_j \,, j \notin \{j_1, ..., j_m\}$) are **non-basic variables**.

The basic solution corresponding to B is found by

1. set all non-basic variables to 0 in $Ax = b$.
2. solve the “remaining system”:

$$Bx = b \Leftrightarrow x = B^{-1}b$$

3. Find value $z = cx$

Solving LP-problems - Algebra I

Consider the problem

$$\begin{array}{rrcl} \max & \text{c}x & & \\ & \text{A}x & = & \text{b} \\ & x & \geq & 0 \end{array}$$

Suppose that we have a basis B, a partitioning of A in a basis-part and a non-basis part $A = (B\ N)$, and a corresponding partitioning of the vector of variables x into $(x_B\ x_N)$.

Solving LP-problems - Algebra II

The basic solution corresponding to B is algebraically found as follows:

$$\begin{aligned} \max \quad & cx \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & c_B x_B + c_N x_N \\ B x_B + N x_N &= b \\ x_B,\ x_N &\geq 0 \end{aligned}$$

Solving LP-problems - Algebra III

Left-multiply with B^{-1}

$$\begin{aligned} \max \quad & c_B x_B + c_N x_N \\ I x_B + B^{-1} N x_N &= B^{-1} b \\ x_B,\ x_N &\geq 0 \end{aligned}$$

Isolate x_B at left side

$$\begin{aligned} \max \quad & c_B x_B + c_N x_N \\ x_B &= B^{-1} b - B^{-1} N x_N \\ x_B,\ x_N &\geq 0 \end{aligned}$$

Solving LP-problems - Algebra IV

Insert the expression for x_B into the objective function

$$\begin{aligned} \max \quad & c_B(B^{-1}b \; - B^{-1}N x_N) \; + c_N x_N \\ x_B &= B^{-1}b - B^{-1}N x_N \\ x_B,\ x_N &\geq 0 \end{aligned}$$

Collect terms:

$$\begin{aligned} \max \quad 0x_B + (c_N - c_B B^{-1} N) x_N &+ c_B B^{-1} b \\ I x_B + B^{-1} N x_N &= B^{-1} b \\ x_B,\ x_N &\geq 0 \end{aligned}$$

The j'th **reduced cost**: $\boxed{\overline{c_j} = c_j - (c_B B^{-1} N)_j}$.

Solving LP-problems - Algebra V

$$\begin{aligned} \max \quad cx &= c_B x_B + c_N x_N \\ (B\ N)x &= b \\ x &\geq 0 \end{aligned}$$

c_B	c_N	0
B	N	b

Solving LP-problems - Algebra VI

$$\max \quad cx = (c_N - c_B B^{-1} N) x_N \quad + \quad c_B B^{-1} b$$
$$x_B = B^{-1} b - B^{-1} N x_N$$
$$x_B,\ x_N \ \geq \ 0$$

0	$c_N - c_B B^{-1} N$	$-c_B B^{-1} b$
$\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}$	$B^{-1} N$	$B^{-1} b$

Canonical form

Solving an LP - I

Consider the problem

$$\begin{array}{ll} \max & 7p + 11q \\ \text{s.t.} & 1 \leq p \leq 8 \\ & 1 \leq q \leq 3.5 \\ & 2p - q \geq 0 \\ & p + q \leq 9 \\ & p, q \geq 0 \end{array}$$

$$\begin{array}{ll} \max & 7x_1 + 11x_2 \quad + 18 \\ \text{s.t.} & 0 \leq x_1 \leq 7 \\ & 0 \leq x_2 \leq 2.5 \\ & 2x_1 - x_2 \geq -1 \\ & x_1 + x_2 \leq 7 \\ & x_1, x_2 \geq -1 \end{array}$$

Transform: $x_1 = p - 1, \quad x_2 = q - 1 \Leftrightarrow p = x_1 + 1, \quad q = x_2 + 1$

Notice:

$x_1 \geq -1$ and $x_1 \geq 0$

$x_2 \geq -1$ and $x_2 \geq 0$

$x_1 \leq 7$ and $x_1 + x_2 \leq 7$

Solving an LP - III

In standard form:

$$\begin{array}{lrrrrrl} \max & 7x_1 + 11x_2 & & & & & \\ \text{s.t.} & -2x_1 + & x_2 + x_3 & & & = & 1 \\ & x_1 + & x_2 & + x_4 & & = & 7 \\ & & x_2 & & + x_5 & = & 5/2 \\ & & x_1, x_2, x_3, x_4, x_5 & \geq & 0 & & \end{array}$$

The variables x_3, x_4, x_5 are called **slack variables** and are introduced to obtain a system in standard form.

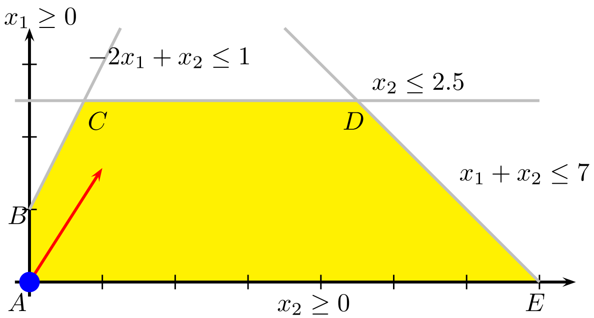
Constant +18 is ignored in the following

Solving an LP - IV

Canonical Simplex tableau wrt. the basis {3,4,5}:

	x_1	x_2	x_3	x_4	x_5	0
Red. Costs	7	11	0	0	0	0
x_3	-2	1	1	0	0	1
x_4	1	1	0	1	0	7
x_5	0	1	0	0	1	5/2

Solving an LP - V



BS: (0,0,1,7,2.5). Value: 0 (+18). Optimal: No - increase x_1 or x_2.

Interplay with x_3, x_4, x_5 ?

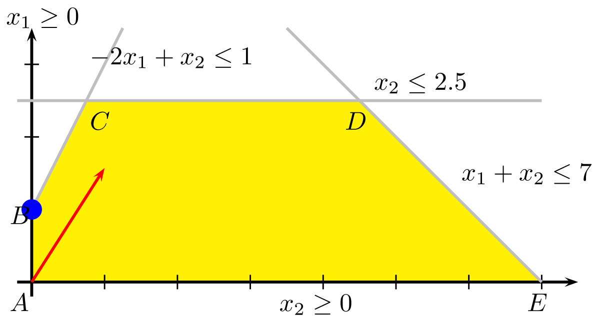
Solving an LP - VI

Fix x_1 to 0. Then the equation system is

$$\max \quad 11x_2$$
$$x_3 = 1 - x_2$$
$$x_4 = 7 - x_2$$
$$x_5 = 5/2 - x_2$$
$$x_2, ..., x_5 \geq 0$$

x_3, x_4, x_5 all decrease when x_2 increases. Increase x_2 as much as possible, all variables must stay non-negative. x_3 **sets the bound**, i.e. x_2 can be increased to 1. Simplex tableau wrt. the basis {2,4,5} ?

Solving an LP - VII

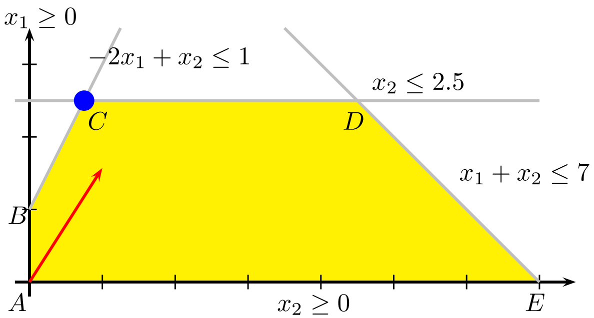


Solving an LP - VIII

	x_1	x_2	x_3	x_4	x_5	0
Red. Costs	7	11	0	0	0	0
x_3	-2	[1]	1	0	0	1
x_4	1	1	0	1	0	7
x_5	0	1	0	0	1	5/2

Red. Costs	29	0	-11	0	0	-11
x_2	-2	1	1	0	0	1
x_4	3	0	-1	1	0	6
x_5	2	0	-1	0	1	3/2

Solving an LP - IX

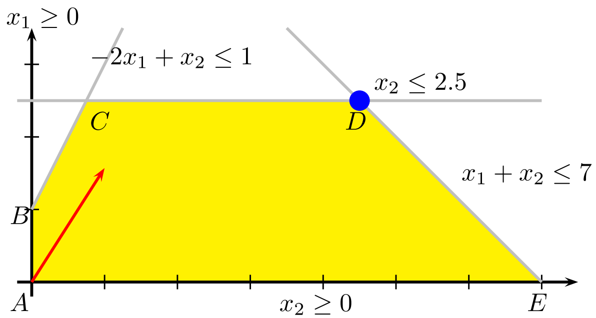


Solving an LP - X

Red. Costs	29	0	-11	0	0	-11
x_2	-2	1	1	0	0	1
x_4	3	0	-1	1	0	6
x_5	2	0	-1	0	1	3/2

Red. Costs	0	0	7/2	0	-29/2	-131/4
x_2	0	1	0	0	1	5/2
x_4	0	0	1/2	1	-3/2	15/4
x_1	1	0	-1/2	0	1/2	3/4

Solving an LP - XI



Solving an LP XII

Red. Costs	0	0	7/2	0	-29/2	-131/4
x_3	0	1	0	0	1	5/2
x_4	0	0	1/2	1	-3/2	15/4
x_1	1	0	-1/2	0	1/2	3/4

Red. Costs	0	0	0	-7	-4	-59
x_2	0	1	0	0	1	5/2
x_3	0	0	1	2	-3	15/2
x_1	1	0	0	1	-1	9/2

The Simplex Algorithm for LP I

Input: A maximization-LP-problem in canonical form w.r.t. a basis (the columns of the basic variables are unit vectors - each basic variable has 0 as coefficient in the objective function).

- repeat
  1. if $\overline{c_j} \leq 0$ for all j then optimal := "yes"; stop
  2. choose s with $\overline{c_s} > 0$ (often largest positive);
     j is the pivot column
  3. if $\overline{a_{is}} \leq 0$ for all i then unbounded := "yes"; stop
  4. find $q = min_{i:\overline{a_{is}}>0} \; \{\overline{b_i}/\overline{a_{is}}\} \;\; = \;\; \overline{b_r}/\overline{a_{rs}}$
     r is the pivot row, q is the increase in the non-basic variable x_s to enter the basis in the new solution.
  5. pivot on $\overline{a_{rs}}$

Line 2 for minimization problem: "$\overline{c_j} < 0$ (often smallest negative)"

The Simplex Algorithm for LP - pivoting

The Pivot operation on $\overline{a_{rs}}$:

1. Divide row r with $\overline{a_{rs}}$ to produce tableau with a 1 in row r, column s
   $\overline{a_{rj}} := \overline{a_{rj}}/\overline{a_{rs}}, \quad j \in \{1, ..., n\}$
2. For each other row p (including the "objective function row"), subtract a multiplum of row r such that $\overline{a_{ps}}$ becomes 0 in the new tableau:
   - $\overline{a_{pj}} := \overline{a_{pj}} - (\overline{a_{rj}}/\overline{a_{rs}}) \cdot \overline{a_{ps}} \quad j \in \{1, ..., n\}$
   - $\overline{b_p} := \overline{b_p} - (\overline{b_r}/\overline{a_{rs}}) \cdot \overline{a_{ps}}$

The Simplex Algorithm - end result

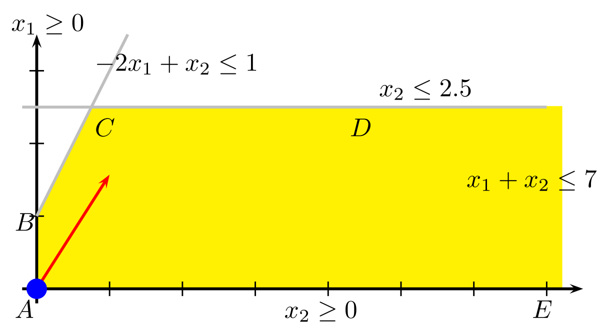
When Simplex terminates we have

x	x_s	RHS
c	0	0
A	I	b

$c - c_B B^{-1} A$	$-c_B B^{-1}$	$-c_B B^{-1} b$
$B^{-1} A$	B^{-1}	$B^{-1} b$

- reduced costs $\overline{c} = c - c_B B^{-1} A \leq 0$
- dual $-y = -c_B B^{-1} \leq 0$
- righthand side $B^{-1} b \geq 0$
- inverse B^{-1}
- $-z = -c_B B^{-1} b$

Removing a constraint I



The problem is **unbounded** - x_1 can be increased infinitely, increasing the objective function all the way ...

Removing a constraint II

Red. Costs	7	11	0	0	0
x_3	-2	[1]	1	0	1
x_4	0	1	0	1	2.5
Red. Costs	29	0	-11	0	-11
x_2	-2	1	1	0	1
x_4	[2]	0	-1	1	1.5
Red. Costs	0	0	7/2	0	-131/4
x_2	0	1	0	0	5/2
x_1	1	0	-1/2	0	3/4

Removing a constraint III

How to recognize an unbounded LP from a Simplex tableau

- a non-basic variable with positive reduced cost and all entires ≤ 0

Explanation: If this variable is increased, all basic variables will either stay at their value (non-basic column contains a 0) or increase (non-basic column contains a negative number). Hence there is no bound for the possible increase, and increasing the non-basic variable also increases value of the solution.

Optimality argument

From Albegra IV slide:

- The Simplex tableau for a basis B is **another representation** of the original problem.
- For any feasible $x : x \in \{x | Ax = b \, , x \geq 0\}$ it holds that

$$\boxed{z = cx = \bar{c}x + c_B B^{-1} b}$$

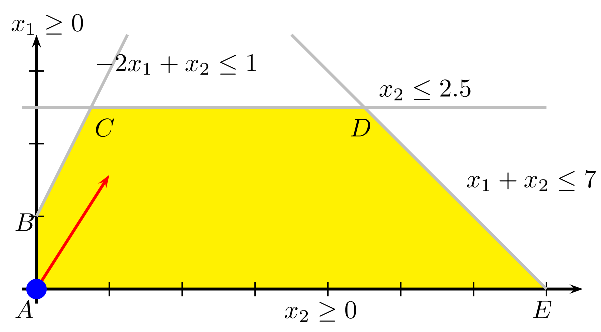
  where $\bar{c}$ are the reduced costs w.r.t. the basis B.
- If for a basis B we have $\bar{c} \leq 0$ then $z_B = c_B B^{-1} b$, and for all $x \in S$

$$z = cx = \bar{c}x + c_B B^{-1} b \leq 0 + z_B$$

  hence z_B is best possible solution value

Convergence

Can be shown easily if all basic solutions are different (the problem is **non-degenerate**) - otherwise an “anti-cycling” mechanism is necessary.



Pairs of primal and Dual LPs

LP-problems come in “pairs” – an LP-problem and its dual. The structure of such pairs is illustrated below:

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \qquad (P)$$

$$\begin{array}{lrcl} \min & yb & & \\ \text{s.t.} & yA & \geq & c \\ & y & \geq & 0 \end{array} \qquad (D)$$

Duality, motivation

Upper bound for

$$\begin{array}{lrrcl} \max & 7x_1 + 11x_2 & & & \\ \text{s.t.} & -2x_1 + & x_2 & \leq & 1 \\ & x_1 + & x_2 & \leq & 7 \\ & & x_2 & \leq & 5/2 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

11 $\times$ second constraint $11x_1 + 11x_2 \leq 77$
7 $\times$ second constraint + 4 $\times$ third constraint $7x_1 + 11x_2 \leq 59$
$y_1 \times$ first constraint $y_2 \times$ second constraint, $y_3 \times$ third constraint

$$y_1(-2x_1 + x_2) + y_2(x_1 + x_2) + y_3(x_2) \leq 1y_1 + 7y_2 + 5/2y_3$$

$$(-2y_1 + y_2)x_1 + (y_1 + y_2 + y_3)x_2 \leq 1y_1 + 7y_2 + 5/2y_3$$

where $y_1, y_2, y_3 \geq 0$ and $-2y_1 + y_2 \geq 7$ and $y_1 + y_2 + y_3 \geq 11$

Duality, motivation

Dual problem seeks tightest upper bound

$$\begin{array}{lrrrcl} \min & 1y_1 + 7y_2 + 5/2y_3 & & & & \\ \text{s.t.} & -2y_1 + & y_2 & & \geq & 7 \\ & y_1 + & y_2 + & y_3 & \geq & 11 \\ & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

Duality, motivation

Seeking upper bound for (P)

$$\begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad (P)$$

linear combination of constraints, using multipliers $y \geq 0$

$$yAx \leq yb$$

If coefficients $yA \geq c$ then upper bound by $cx \leq yAx \leq yb$
Tightest upper bound (D)

$$\begin{array}{ll} \min & yb \\ \text{s.t.} & yA \geq c \\ & y \geq 0 \end{array} \qquad (D)$$

Duality theorems

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \qquad (P)$$

$$\begin{array}{lrcl} \min & yb & & \\ \text{s.t.} & yA & \geq & c \\ & y & \geq & 0 \end{array} \qquad (D)$$

The Weak Duality Theorem: For any feasible solution x to (P) and any feasible solution y to (D) it holds that $\boxed{cx \leq yb}$

The Strong Duality Theorem: If (P) and (D) both have **feasible** solutions, then both have **optimal** solutions $\overline{x}$ resp. $\overline{y}$, and the optimum values are equal: $\boxed{c\overline{x} = \overline{y}b}$

minimization problem **The Weak Duality Theorem:** $\boxed{cx \geq yb}$

Dual variables in final Simplex tableau

When Simplex terminates we have

x	x_s	RHS
c	0	0
A	I	b

$c - c_B B^{-1}A$	$-c_B B^{-1}$	$-c_B B^{-1}b$
$B^{-1}A$	B^{-1}	$B^{-1}b$

1) reduced costs $\overline{c} = c - c_B B^{-1}A \leq 0$
2) dual $-y = -c_B B^{-1} \leq 0$
3) righthand side $B^{-1}b \geq 0$
4) inverse B^{-1}
5) $-z = -c_B B^{-1}b$

Guess $y = c_B B^{-1}$

1) $yA \geq c$
2) $y \geq 0$
5) $cx = z = yb$

Interpretation of dual variables

y_i, dual variables, “shadow prices”, “marginal prices”

Dual variables y_i denote price per unit of resource b_i

If primal problem

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

has optimal value z^*. Then

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b + \epsilon \\ & x & \geq & 0 \end{array}$$

has optimal vaule $z^* + y\epsilon$

Pairs of Dual LPs - a recipe I

maximization problem

Primal: Max		Dual: Min	
a) i'th constraint	$\geq$	i'te variable	≤ 0
b) ”	$\leq$	”	≥ 0
c) ”	$=$	”	free
d) j'th variable	≥ 0	j'th constraint	$\geq$
e) ”	≤ 0	”	$\leq$
d) ”	free	”	$=$

Pairs of Dual LPs - a recipe II

minimization problem

Primal: Min		Dual: Max	
a) i'th constraint	$\geq$	i'th variable	≥ 0
b) "	$\leq$	"	≤ 0
c) "	$=$	"	free
d) j'th variable	≥ 0	j'th constraint	$\leq$
e) "	≤ 0	"	$\geq$
d) "	free	"	$=$

Example

$$\begin{array}{ll} \max & 7x_1 + 3x_2 - 4x_3 \\ \text{s.t.} & -2x_1 + 5x_2 + x_3 \leq 1 \\ & x_1 + x_2 - 2x_3 = -2 \\ & x_2 + x_3 \geq 5 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \in \mathbb{R} \end{array}$$

$$\begin{array}{ll} \min & 1y_1 - 2y_2 + 5y_3 \\ \text{s.t.} & -2y_1 + y_2 \geq 7 \\ & 5y_1 + y_2 + y_3 \leq 3 \\ & y_1 - 2y_2 + y_3 = -4 \\ & y_1 \geq 0, y_2 \in \mathbb{R}, y_3 \leq 0 \end{array}$$

Complementary Slackness Theorem

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \quad (P) \qquad \begin{array}{lrcl} \min & yb & & \\ \text{s.t.} & yA & \geq & c \\ & y & \geq & 0 \end{array} \quad (D)$$

and a pair $\overline{x}, \overline{y}$ of *feasible* solutions to P resp. D.

$\overline{x}$ and $\overline{y}$ are **optimal** solution to P resp. D *if and only if*:

$$\boxed{(\mathbf{b} - \mathbf{A}\overline{\mathbf{x}}) \cdot \overline{\mathbf{y}} = 0 \ \wedge \ (\overline{\mathbf{y}}\mathbf{A} - \mathbf{c}) \cdot \overline{\mathbf{x}} = \mathbf{0}}$$

The conditions spelled out are:

$$\forall i : \overline{y}_i = 0 \quad \vee \quad \sum_{j=1}^{n} a_{ij}\overline{x}_j = b_i$$

$$\forall j : \overline{x}_j = 0 \quad \vee \quad \sum_{i=1}^{m} a_{ij}\overline{y}_i = c_i$$

Complementary Slackness Theorem - proof

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \quad (P) \qquad \begin{array}{lrcl} \min & yb & & \\ \text{s.t.} & yA & \geq & c \\ & y & \geq & 0 \end{array} \quad (D)$$

Assume that x feasible to (P) and y feasible to (D)

$$cx = x^T c \leq x^T A^T y = y^T A x \leq y^T b = by$$

By strong duality, x, y are optimal iff $cx = by$ i.e. iff

$$x^T c = x^T A^T y \quad \Leftrightarrow \quad (yA - c)x = 0$$

and

$$y^T A x = y^T b \quad \Leftrightarrow \quad (b - Ax)y = 0$$

Optimality conditions - Geometry I

We consider now an LP-problem (P)

$$\begin{array}{rcl} \text{max} & cx & \\ Ax & \leq & b \\ x & \text{free} & \end{array}$$

A given x^* is an optimal solution to P *if and only if*:
1) x^* is a feasible solution to P, and
2) there exists a vector y satisfying the following:

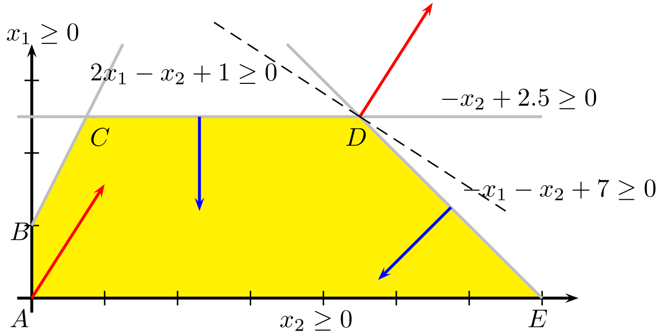
$$\begin{array}{rcll} yA & = & c & (1) \\ y(b - Ax^*) & = & 0 & (2) \\ y & \geq & 0 & (3) \end{array}$$

Optimality conditions - Geometry II

What is the geometric interpretation of (1), (2), and (3) ?
The vector c is the **gradient** of the objective function cx, i.e. that direction for a move in $\mathbb{R}^n$, for which the objective function increases most rapidly.
The feasible region S for P, $\{x | Ax \leq b\}$, is defined by the functions $a_{11}x_1 + ... + a_{1n}x_n - b_1 \leq 0$, ... , $a_{m1}x_1 + ... + a_{mn}x_n - b_m \leq 0$. The gradients $(a_{11}$, ... , $a_{1n})$, ... , $(a_{m1}$, ... , $a_{mn})$ for these functions point *into* S. Condition (1) and (3) state, that at the optimal point of S - an extreme point, Q - the gradient of the objective function must be a **non-positive linear combination** of the gradients of the constraints.

Optimality conditions - Geometry III

For Q (which corresponds to a basic feasible solution), (2) states that the linear combination must be constructed using only the gradients of those constraints, which are **binding** at Q:



The Dual Simplex, motivation

x	x_s	RHS
c	0	0
A	I	b
$c - yA$	$-y$	$-yb$
$B^{-1}A$	B^{-1}	$B^{-1}b$

- $c - yA \leq 0$ satisfied
- $-y \leq 0$ satisfied
- $B^{-1}b = x^*_B \geq 0$ searching

The Dual Simplex I

Input: A minimization LP-problem in canonical form w.r.t. a basis (the columns of the basic variables are unit vectors - basic variables have 0 as coefficient in the obj. function), such that **all cost are non-negative**. The basic solution is "optimal, but maybe infeasible".

- repeat
  1. if $\overline{b_i} \geq 0$ for all i then optimal := "yes"; stop
  2. choose r with $\overline{b_r} < 0$ (often "largest" negative); *)
     r is the pivot row
  3. if $\overline{a_{rj}} \geq 0$ for all j then infeasible := "yes"; stop
  4. find $\max_{j:\overline{a_{rj}}<0} \{\overline{c_j}/\overline{a_{rj}}\} = \overline{c_s}/\overline{a_{rs}}$
     r is the pivot column. Note that all ratios are negative, so the column with the ratio "closest to 0" is sought for.
  5. pivot on $\overline{a_{rs}}$

The Dual Simplex II

If maximization problem, then all costs must initially be non-positive, and one has to search for the column with the smallest ratio (again the one "closest to 0").

Pivot ? - Exact description ? - as for the primal Simplex

Another LP - suitable for dual Simplex

Primal LP

$$\begin{aligned} \min \quad & 4x_1 + 4x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\ & x_1 + 2x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 6y_1 + 6y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \leq 4 \\ & y_1 + 2y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Solving an LP - dual simplex I

	x_1	x_2	x_3	x_4	0
Red. Costs	4	4	0	0	0
x_3	[-2]	-1	1	0	-6
x_4	-1	-2	0	1	-6

	x_1	x_2	x_3	x_4	0
Red. Costs	0	2	2	0	-12
x_1	1	1/2	-1/2	0	3
x_4	0	-3/2	-1/2	1	-3

Solving an LP - dual simplex I

	x_1	x_2	x_3	x_4	0
Red. Costs	0	2	2	0	-12
x_1	1	1/2	-1/2	0	3
x_4	0	[-3/2]	-1/2	1	-3

	x_1	x_2	x_3	x_4	0
Red. Costs	0	0	4/3	4/3	-16
x_1	1	0	-2/3	-1/3	2
x_2	0	1	-1/3	-2/3	2

Implicit handling of Upper Bounds

$$\begin{array}{llllll} \max & 2x_1 & +x_2 & +2x_3 & & \\ \text{s.t.} & 4x_1 & +x_2 & & = & 12 \\ & -2x_1 & & +x_3 & = & 4 \end{array}$$

$$0 \le x_1 \le 4,\ 0 \le x_2 \le 15,\ 0 \le x_3 \le 6$$

Upper bounds can be handled as constraint
Expensive since A_B is $m \times m$ matrix (to be inverted)
Easy to handle implicicly
Quotient test (leaving varible) tests upper bounds
Quotient test returns variable which is 0 (leaving basis)

$x_3 = 6 \iff y_3 = 6 - x_3 = 0$

x_3 leaves basis on upper bound

Upper bounds: Iterations

First bring on canonical form

Z	x_1	x_2	x_3	
1	-2	-1	-2	0
	4	$\to 1$		12
	-2		$\to 1$	4
1	-2			20
	4	1		12
	-2		$\to 1$	4

Entering variable x_1

$x_1 \le 4$ since $u_1 = 4$

$x_1 \le \frac{12}{4} = 3$

$x_1 \le \frac{6-4}{2} = 1$ since $u_3 = 6$

x_3 hits u_3 in last inequality (smallest)

x_3 leaves basis on upper bound, y_3 enters basis
In row (2) substitute $x_3 = 6 - y_3$:

$-2x_1 + x_3 = 4$ becomes $2x_1 + y_3 = 2$

Upper bounds: Quotient test

Basis equations

$$4x_1 + x_2 = 12 \quad \Leftrightarrow \quad 12 - 4x_1 = x_2 \ge 0 \quad \Leftrightarrow \quad x_1 \le \frac{12}{4}$$
$$-2x_1 + x_3 = 4 \quad \Leftrightarrow \quad 4 + 2x_1 = x_3 \ge 0 \quad \Leftrightarrow \quad x_1 \le \infty$$

But, $x_3 \le 6$ so last equation becomes

$$-2x_1 + x_3 = 4 \quad \Leftrightarrow \quad 4 + 2x_1 = x_3 \le 6 \quad \Leftrightarrow \quad x_1 \le \frac{6-4}{2} = 1$$

If positive coefficient in column, normal quotient test

If negative coefficient in column, upper bound test

Upper bounds: Last iteration

Now, we can make Simplex iteration

Z	x_1	x_2	y_3	$(y_3 = 6 - x_3)$
1	-2			20
	4	1		12
	$\Rightarrow$ 2		1	2
1			1	22
		1	-2	8
	1		$1/2$	1

$x_1^* = 1,\ x_2^* = 8,\ x_3^* = 6 - y_3^* = 6$

$Z^* = 22$

Upper bounds

- faster solution times since smaller A_B
- implicit handling saves space
- will be used in min-cost-flow problem