Static and Dynamic Optimization

Course Introduction

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L1

Dynamic Optimization

What is Dynamic Optimization?

Dynamic Optimization has 3 ingredients:

- A performance index (cost function, objective function) depending on the states and decisions.

  In our case it is a summation (or integral) of contribution over a period of time of fixed or free length (might be a part of the optimization).

- Eventually some constraints

  on the decisions or on the states.

- Some dynamics.

  Here (in this course) described by a state space model.

Lets have a look at some examples:

Ex1: Optimal Pricing (simplified)

We are producing a product (brand A) and have to determine its price in order to maximize our income.

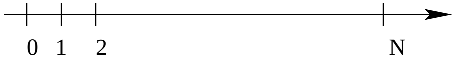
There is a competitor product B and a problem.

If we are to modest we might have almost all the costumers but we will not earn that much.

If we are to greedy then the bulk majority of the costumers will bye the other brand B.

Optimal Pricing - the performance index

We have to decide the price of the product $u_i \sim \underline{u}$ ($\underline{u}$ being the production cost) in each interval.



Let M be the size of the marked and let x_i $(0 \leq x_i \leq 1)$ be the (A) share of the marked in the i'th interval.

Objective: to make some money - i.e. to maximize

$$Max\ J \quad \text{where} \quad J = \sum_{i=0}^{N-1} M \bar{x}_i \big(u_i - \underline{u} \big) \qquad\qquad \bar{x}_i = \frac{1}{2}(x_i + x_{i+1})$$

More precisely, x_i is the marked share at the beginning of interval i and $\bar{x}_i$ is the average share of the marked in interval i.

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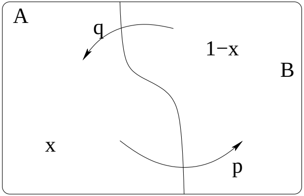
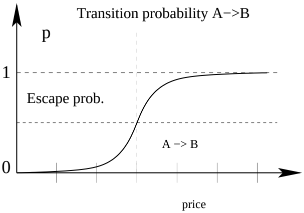
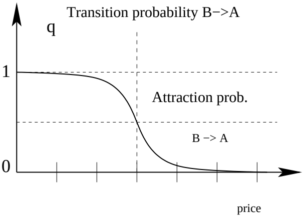
Optimal Pricing - the dynamics







Dynamics: A→A B→A

$$x_{i+1} = \big(1 - p[u_i]\big)x_i + q[u_i]\big(1 - x_i\big) \qquad x_0 = \underline{x}_0$$

Recap Optimal Pricing

Dynamics:

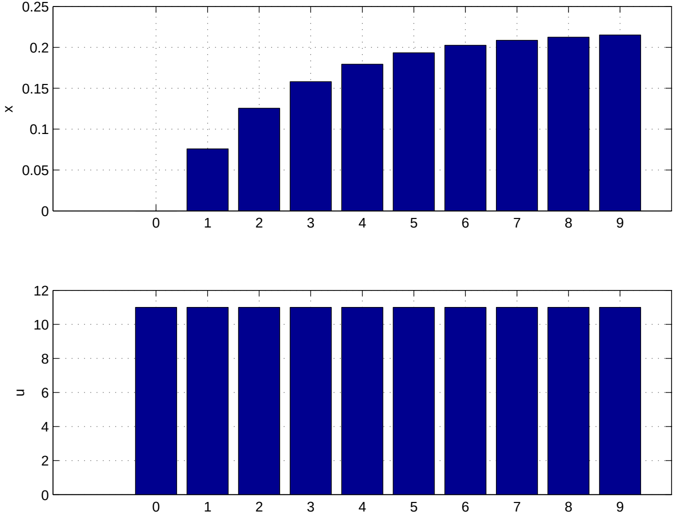
$$x_{i+1} = \big(1 - p(u_i)\big)x_i + q(u_i)\big(1 - x_i\big) \qquad\qquad x_0 = \underline{x}_0$$

Objective:

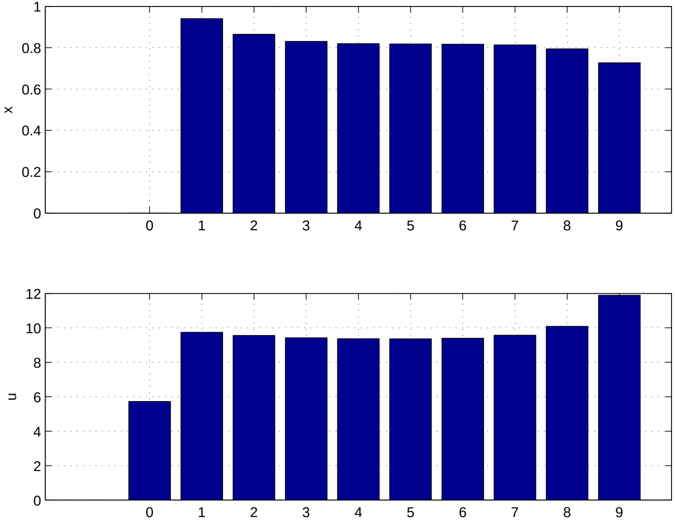
$$Max\ J \quad \text{where} \quad J = \sum_{i=0}^{N-1} M\bar{x}_i\big(u_i - \underline{u}\big)$$

Notice: This is a discrete time model. No constraints. The length of the period (the horizon, N) is fixed.

If $u_i = \underline{u} + 5$ ($\underline{u} = 6$, $N = 10$) we get $J = 8$ (rounded to integer).



Optimal pricing (given correct model): $J = 27$ (rounded to integer).



Notice different axis for x.

Free Dynamic Optimization

Dynamics (described by a state space model):

$$x_{i+1} = f_i(x_i, u_i) \qquad x_0 = \underline{x}_0$$

Objective (to optimize the index):

$$J = \phi_N(x_N) + \sum_{i=0}^{N-1} L_i(x_i, u_i)$$

Here N and $\underline{x}_0$ are fixed (given), J, ϕ and L are scalars. x_i and f_i are n-dimensional vector and vector function and u_i is a vector of decisions.

Notice: no constraints (except given by the dynamics).

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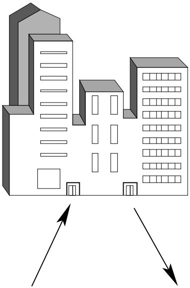
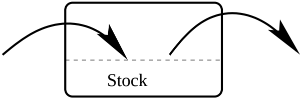
Ex2: Inventory control - A classical OR problem





Dynamics:

$$x_{i+1} = x_i + u_i - s_i \qquad x_0 = \underline{x}_0$$

Stock :	x_i	$0 \leq x_i \leq \bar{x}$
Production:	u_i	$0 \leq u_i \leq \bar{u}$
Sale:	s_i	$0 \leq s_i \leq \min(x_i, w_i)$
Order:	w_i	

Notice: constraints on decisions and states. Stochastics involved.

Goals:

- to earn some money
- to avoid situation with no stock
- to reduce stock charge
- to obtain an even production.

Objective (index to be maximized):

$$J = \sum_{i=0}^{N-1} \; p\, s_i - c\, u_i - k\, x_i - h\, Max\big(w_i - s_i, 0\big)$$

where p, c, k and h are constants (prices).

Constrained Dynamic Optimization

Dynamics (described by a state space model):

$$x_{i+1} = f_i(x_i, u_i) \qquad x_0 = \underline{x}_0$$

Objective (to optimize the index):

$$J = \phi_N(x_N) + \sum_{i=0}^{N-1} L_i(x_i, u_i)$$

Constraints:

$$g(x_i, u_i) \leq C_i$$

Variations of the problem

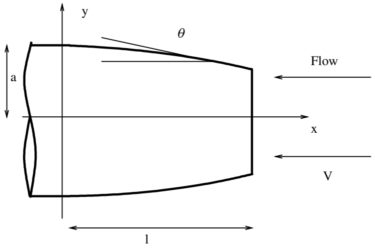
Dynamic optimization with:

- Terminal constraints (take the system from one place to another).
- Constraints (on u_i and x_i within the horizon).
- Continuous time problems
- Open final time (Minimum time problems).
- Stochastic elements (orders in the inventory problem).

2 examples.

Minimum drag nose shape (Newton 1686)

Find the shape i.e. $r(x)$ of a axial symmetric nose, such that the drag is minimized.



The decision $u(x)$ is the slope of the profile:

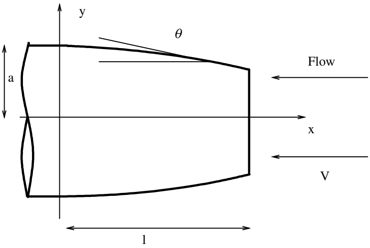
$$\frac{\partial r}{\partial x} = -u = -tan(\theta) \qquad r(0) = a$$

Minimum drag nose shape (Newton)

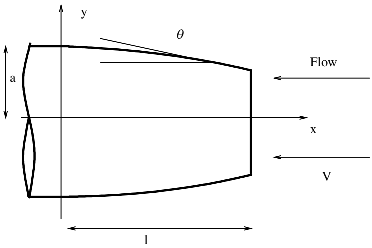
Find the shape i.e. $r(x)$ of a axial symmetric nose, such that the drag is minimized.

$$D = q \int_0^a C_p(\theta) 2\pi r dr$$

$$q = \frac{1}{2}\rho V^2 \qquad \text{(Dynamic pressure)} \qquad C_p(\theta) = 2sin(\theta)^2 \text{ for } \theta \geq 0$$



Minimum drag nose shape (Newton)



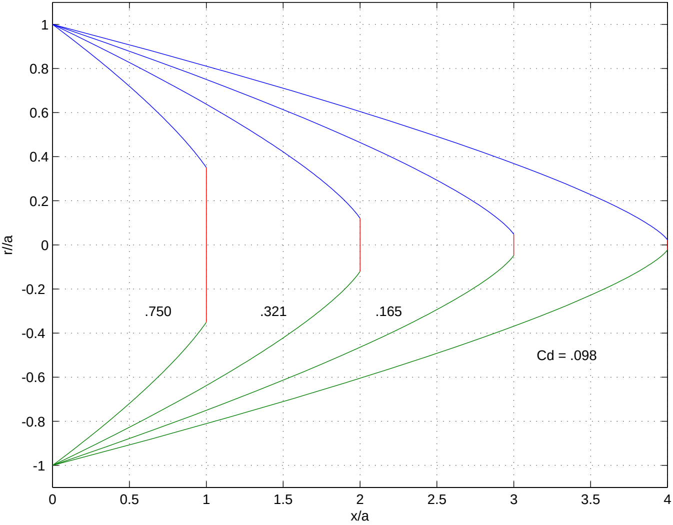
Dynamic:

$$\frac{\partial r}{\partial x} = -u \qquad r_0 = a \qquad\qquad tan(\theta) = u$$

Cost function (drag coefficient, including a blunt nose):

$$C_d = \frac{D}{q\pi a^2} = 2r_l^2 + 4\int_0^l \frac{ru^3}{1+u^2}dx \qquad\qquad \leq 1$$

Minimum drag nose shape (Newton)



Free Dynamic Optimization (C)

Find a function u_t $t \in [0;\ T]$ which takes the system system

$$\dot{x} = f_t(x_t, u_t)$$

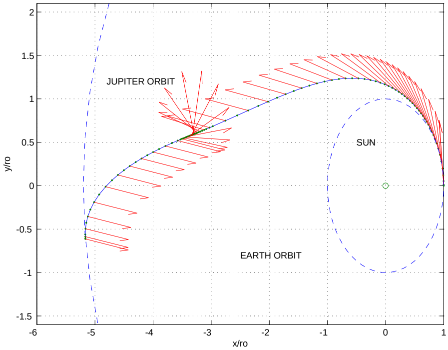
from its initial state $\underline{x}_0$ along trajectories such that the performance index

$$J = \phi_T[x_T] + \int_0^T L_t(x_t, u_t)\, dt$$

is minimized.

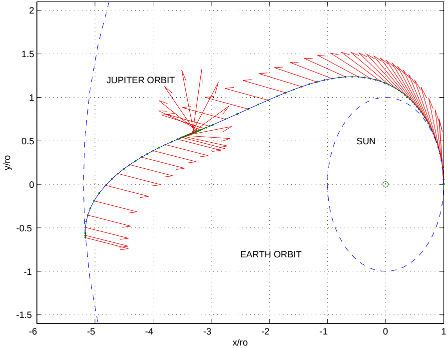
Min. Time Orbit Transfer

Thrust direction program for minimum time transfer from Earth orbit to Jupiter orbit.



$$\dot{r} = u \qquad \dot{u} = \frac{v^2}{r} - \frac{1}{r^2} + a\sin(\theta) \qquad \dot{v} = -\frac{uv}{r} + a\cos(\theta)$$

Min. Time Orbit Transfer



$$\frac{d}{dt}\begin{bmatrix} r \\ u \\ v \end{bmatrix} = \begin{bmatrix} u \\ \frac{v^2}{r} - \frac{1}{r^2} + a\sin(\theta) \\ -\frac{uv}{r} + a\cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \text{Initial conditions} \\ \text{Terminal conditions} \\ \\ J = T \end{bmatrix}$$

42111/02711 Static and Dynamic Optimization

- Course description (in Danish)
- Course description (in English)

Lecture slides for Static Optimization are found on CampusNet in the folder Static Slides

Lecture slides for Dynamic Optimization:

- L1: Introduction/NKP (pdf).
- L7: Free dynamics optimization-D (pdf).
- L8: Free dynamics optimization-(D+C) (pdf).
- L9: DO with end point constraints (pdf).
- L10: DO with control constraints (pdf).
- L11: Dynamic Programming (pdf).
- L12: Stochastic Dynamic Programming (pdf).
- L13: Time Optimal Problems (pdf).

Dynamic exercise slides:

- Exercise DOex1 (lecture 7) (pdf). Solutions (pdf). m-files (zip.).
- Exercise DOex2 (lecture 8) (pdf). Solutions (pdf). m-files (zip).
- Exercise DOex3 (lecture 11) (pdf). Solutions (pdf).

The note "Dynamic Optimization" is found here in pdf.
Mark Gockenback: A Practical Introduction to MATLAB (as ps) or (as html)

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Concluding remarks

- have 42111 and your study nomber in the subject field when emailing us
- Matlab available on Gbar download site