Exercise 8: Free Dynamic Optimization
in
Continuous time

Static and Dynamic Optimization

Background

Matlab files (i.e. m files) for this exercise (including solutions) are available on the course webpage.

1 ODE solver

Consider the Van der Pool oscillator represented by the ODE:

$$\ddot{z} - a(1 - z^2)\dot{z} + z = 0$$

where $a = 1$ is a parameter. This is an example on a second order, non-linear differential equation. If we choose the two (since it is a second order system) as:

$$x_1 = z \qquad x_2 = \dot{z}$$

we have

$$\dot{x}_1 = \dot{z} = x_2 \qquad \dot{x}_2 = \ddot{z} = a(1 - z^2)\dot{z} - z = a(1 - x_1^2)x_2 - x_1$$

This is a state space description of the Van der Pool oscillator and can be written in a compact form as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ a(1 - x_1^2)x_2 - x_1 \end{bmatrix}$$

Write a short m-file (e.g. with the name: vdp.m and) with the following contents:

```
function dx=vdp(t,x,a)
x1=x(1); x2=x(2);
dx=[x2; a*(1-x1^2)*x2-x1];
```

Assume the initial state is characterized by the fact $z = 1$ and $\dot{z} = 0$ (i.e. $\underline{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$) and we want se the solution in the interval $[0;\ 30]$ with a resolution equals 0.1.

Execute the followin matlab commands:

```
Tspan=0:0.1:30;
x0=[1;0];
a=1;

[t,x]=ode45(@vdp,Tspan,x0,[],a);

z=x(:,1);
plot(t,z); grid; title('Van der Pool oscillator');
xlabel('time'); ylabel('position');
```

Change the value of a in the interval $[1;\ 10]$ and study (i.e. plot) the effect on the solution.

2 Dynamic Optimization

Consider the continuous time version of the problem of down payment of a (study) loan which at the start of the period is 50.000 Dkr. Let us focus on the situation over a period of 10 years. We are going to determine the optimal down payment strategy for this loan, ie. to determine how much we have to pay. Assume that the rate of interests is 5 % per year ($\alpha = 0.05$), then the dynamics of the problem can be described by:

$$\dot{x} = ax + bu \qquad a = \alpha \;\; b = -1$$

where x_t is the actual size of the loan (including interests) and u_t is the (continuous) downpayment.

On one hand, we are interested in minimizing the amount we have to pay to the bank. On the other hand, we are not interested in paying to much. The objective function, which we might use in the minimization could be

$$J = \frac{1}{2}px_T^2 + \int_0^T \frac{1}{2}qx_t^2 + \frac{1}{2}ru_t^2 \;\; dt$$

where $q = \alpha^2$. The weights r and p are at our disposal. Let for a start choose $r = q$ and $p = q$ (but let the parameters be variable in your program in order to change them easily).

Question: 1 Identify the quantities (T, $\underline{x}_0$, f, ϕ and L) that specifies the problem. □

Question: 2 Write down the Hamiltonian function. □

Question: 3 Find the derivatives that enters into the Euler-Lagrange equation. □

Question: 4 Write down the Euler-Lagrange equations (KKT conditions) for this problem and verify they are:

$$\begin{aligned} \dot{x} &= ax_t + bu_t & x_0 = \underline{x}_0 \\ -\dot{\lambda} &= qx_t + a\lambda_t & \lambda_T = px_T \\ 0 &= ru_t + b\lambda_t \end{aligned}$$

□

Question: 5 Solve the stationarity condition with respect to u_t, i.e. express u_t as function of λ_t. □

Question: 6 Rearrange the costate equation such that it is forward in time, i.e. express $\dot{\lambda}$ as a function of x_t and λ_t. □

Question: 7 Consider the initial values x_0 and λ_0. Verify that the Euler-Lagrange equations are equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} ax_t + bu_t \\ -qx_t - a\lambda_t \end{bmatrix}$$

where

$$u_t = -\frac{b}{r}\lambda_t$$

This set of differential equations can easily be solved by means of an ODE solver (as e.g. ode45 from the previous exercise). □

The solution to the following question can be found in dlq.m.

Notice, the above set of equations can brouht to form

$$\dot{z} = f(z_t)$$

where $z = \begin{bmatrix} x & \lambda \end{bmatrix}^T$. In order to solve the set of differential equation with an ODE solver we must (in a m-file) specify the relation between z and $\dot{z}$ (as in vdp.m in the previous exercise).

Question: 8 Write a piece of code (e.g. with name dlq.m) with realize the mentioned relation between z and $\dot{z}$. □

If λ_0 (or a guess of λ_0) is known then we can solve the ODE and check if the terminal condition:

$$\lambda_T - px_T = 0$$

is fulfilled.

The solution to the following question can be found in loss.m.

Question: 9 Write a piece of code that given an initial guess on λ_0 (and the values of x_0 and other parameters) solves the ODE and returns the error in the terminal condition on λ. □

The solution to the next two questions can be found in runex2.m.

Question: 10 Use eg. fsolve (in matlab) to find the correct initial costate value. □

Question: 11 plot the variation of x_i and u_i. Study the effect of the parameters r and s by changing their values. Try eg. $r = 10q$ and $r = 0.1q$ and $p = 0$ and $p = 100 * q$. □