Static and Dynamic Optimization (42111)

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Lecture 13: Time Optimal Problems

Outline of lecture

- Time optimal problems
- Motion control
- Bang-Bang control
- Reading: 63-72

Time Optimal Problems (Final-time-free problems)

Now let T be a part of the optimization. Find a function $\underline{u_t \in \mathcal{U}_t}$ $t \in [0;\ T]$ which takes the system system

$$\dot{x} = f_t(x_t, u_t)$$

from its initial state $\underline{x}_0$ to a terminal situation where

$$\psi_T(x_T) = 0$$

such that the performance index

$$J = \phi_T(x_T) + \int_0^T \ L_t(x_t, u_t)\ dt$$

is minimized.

DTU

Time Optimal Problems (Final-time-free problems)

Define the Hamilton function as:

$$H(x, u, \lambda, t) = L_t(x_t, u_t) + \lambda_t^T f_t(x_t, u_t)$$

Then the Maximum princple for this problem is

$$\dot{x} = f_t(x_t, u_t) \qquad -\dot{\lambda}^T = \frac{\partial}{\partial x} H_t \qquad u_t = arg \min_{u_t \in \mathcal{U}_t} H_t$$

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \lambda_T^T = \nu^T \frac{\partial}{\partial x} \psi_T(x_T) + \frac{\partial}{\partial x} \phi_T(x_T)$$

Transversality condition:

$$\frac{\partial \phi_T}{\partial T} + \nu^T \frac{\partial \psi_T}{\partial T} + H_T = 0$$

Motion control

The problem is to bring the system

$$\dot{x} = u_t \qquad x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T$$

from the initial position, $\underline{x}_0$, to the final position, $\underline{x}_T$, in minimum time,

while the control action (or the decision function) is bounded to

$$|u_t| \leq c$$

The performance index is

$$J = T \qquad\qquad = T + \int_0^T 0 \; dt = 0 + \int_0^T 1 \; dt$$

Notice, either is ($\phi_T = T,\ L = 0$) or ($\phi = 0,\ L = 1$). Let us chose $\phi_T = T,\ L = 0$.

The Hamiltonian function is:

$$H = \lambda_t u_t$$

The Hamiltonian function is:

$$H = \lambda_t u_t$$

and the optimality conditions are simply

$$\begin{aligned} \dot{x} &= u_t \\ -\dot{\lambda} &= 0 \\ u_t &= arg \min_u H \qquad |u_t| \leq c \end{aligned}$$

with the boundary conditions:

$$x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T \qquad \lambda_T = \nu$$

The **Transversality condition** is

$$1 + \lambda_T u_T = 0$$

Motion control (Costate)

The Hamiltonian function is:

$$H = \lambda_t u_t$$

and the optimality conditions are simply

$$\begin{aligned} \dot{x} &= u_t \\ \lambda &= \nu \\ u_t &= arg \min_u H \qquad |u_t| \leq c \end{aligned}$$

with the boundary conditions:

$$x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T \qquad \lambda_T = \nu$$

The **Transversality condition** is

$$1 + \lambda_T u_T = 0$$

The Hamiltonian function is:

$$H = \lambda_t u_t$$

and the optimality conditions are simply

$$\begin{aligned} \dot{x} &= u_t \\ \lambda &= \nu \\ u_t &= -c \; sign(\lambda_t) = -c \; sign(\nu) \end{aligned}$$

with the boundary conditions:

$$x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T \qquad \lambda_T = \nu$$

The **Transversality condition** is

$$1 + \lambda_T u_T = 0$$

Motion control (Transversality)

The Hamiltonian function is:

$$H = \lambda_t u_t$$

and the optimality conditions are simply

$$\begin{aligned} \dot{x} &= u_t \\ \lambda &= \nu \\ u_t &= -c \; sign(\nu) \end{aligned}$$

with the boundary conditions:

$$x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T \qquad \lambda_T = \nu$$

The **Transversality condition** is

$$1 - \nu c \; sign(\nu) = 0 \qquad \rightarrow \qquad \nu = \pm \frac{1}{c}$$

DTU

Motion control (State equation)

The Hamiltonian function is:

$$H = \lambda_t u_t$$

and the optimality conditions are simply

$$\begin{aligned} x_t &= \underline{x}_0 - c\ sign(\nu)\ t \qquad \text{and} \qquad x_T = \underline{x}_0 - c\ sign(\nu)\ T \\ \lambda &= \nu \\ u_t &= -c\ sign(\nu) \end{aligned}$$

with the boundary conditions:

$$x_0 = \underline{x}_0 \qquad x_T = \underline{x}_T \qquad \rightarrow \qquad T = \frac{|\underline{x}_T - \underline{x}_0|}{c} \qquad sign(\nu) = -sign(\underline{x}_T - \underline{x}_0) \qquad \lambda_T = \nu$$

The **Transversality condition** is

$$\nu = \pm\frac{1}{c} \qquad \rightarrow \qquad \nu = -\frac{1}{c} sign(x_T - x_0)$$

Motion control (Conclusion)

This results in the control strategy

$$u_t = c \; sign(\underline{x}_T - \underline{x}_0)$$

and

$$x_t = \underline{x}_0 + c \; sign(\underline{x}_T - \underline{x}_0) \; t$$

Bang Bang Control (in 1D).

Consider a mass affected by a force. This is a second order system given by

$$\frac{d}{dt}\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix} \qquad \begin{bmatrix} z \\ v \end{bmatrix}_0 = \begin{bmatrix} \underline{z}_0 \\ \underline{v}_0 \end{bmatrix}$$

z position $\quad v$ velocity $\quad a$ control action, specific force, decision variable

Assume the control action, i.e. the specific force is limited to

$$|a| \leq 1 \qquad \text{(just to simplify)}$$

while the objective is to take the system from its original state to the origin

$$\underline{x}_T = \begin{bmatrix} \underline{z}_T \\ \underline{v}_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in minimum time. The performance index is accordingly

$$J = T \qquad \phi = T \quad L = 0$$

The Hamilton function is

$$H = \lambda^z v + \lambda^v a$$

♠

Bang Bang Control.

The Hamilton function and the terminal contribution are

$$H = \lambda^z v + \lambda^v a \qquad\qquad \phi = T \qquad L = 0$$

We can now write the conditions as the state equation,

$$\frac{d}{dt}\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix}$$

the costate equations

$$-\frac{d}{dt}\begin{bmatrix} \lambda^z & \lambda^v \end{bmatrix} = \begin{bmatrix} 0 & \lambda^z \end{bmatrix} \qquad \Leftarrow$$

the optimality condition (Pontryagins maximum principle)

$$a_t = -sign(\lambda_t^v) \qquad \bowtie$$

and the boundary conditions

$$\begin{bmatrix} z_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \underline{z}_0 \\ \underline{v}_0 \end{bmatrix} \qquad \begin{bmatrix} z_T \\ v_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_T^z \\ \lambda_T^v \end{bmatrix} = \begin{bmatrix} \nu^z \\ \nu^v \end{bmatrix}$$

The transversality condition is:

$$1 + H_T = 1 + \lambda_T^z v_T + \lambda_T^v a_T = 1 + \lambda_T^v a_T = 0 \qquad \bowtie \spadesuit$$

DTU

Bang Bang Control.

The Hamilton function is

$$H = \lambda^z v + \lambda^v a$$

We can now write the conditions as the state equation,

$$\frac{d}{dt} \begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix}$$

the costate equations

$$\lambda_t^z = \nu^z \qquad \lambda_t^v = \nu^v + \nu^z (T - t) \qquad \bowtie$$

the optimality condition (Pontryagins maximum principle)

$$a_t = -sign(\lambda_t^v) = \pm 1 \qquad \Leftarrow$$

and the boundary conditions

$$\begin{bmatrix} z_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \underline{z}_0 \\ \underline{v}_0 \end{bmatrix} \qquad \begin{bmatrix} \underline{z}_T \\ \underline{v}_T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \lambda_T^z \\ \lambda_T^v \end{bmatrix} = \begin{bmatrix} \nu^z \\ \nu^v \end{bmatrix}$$

The transversality condition is:

$$1 + \lambda_T^v a_T = 0 \qquad \rightarrow \qquad \lambda_T^v = \nu^v = \pm 1 \qquad \bowtie$$

DTU

Bang Bang Control.

From the Costate equation we can conclude that λ^z is constant and that λ^v is linear. More precisely we have

$$\lambda_t^z = \nu^z \qquad \lambda_t^v = \nu^v + \nu^z(T - t)$$

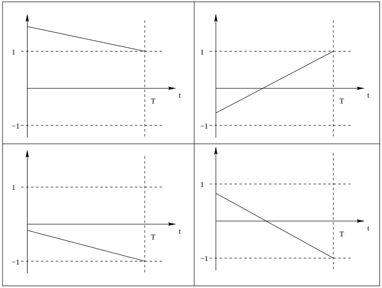
Since $v_T = 0$ the transversality conditions gives us

$$\lambda_T^v a_T = -1$$

but since u_t is saturated at ± 1 we only have two possible values for u_T (and λ_T^v), i.e.

- $a_T = \ \ 1$ and $\lambda_T^v = -1$
- $a_T = -1$ and $\lambda_T^v = \ \ 1$

The linear switching function, λ_t^v, can have one zero crossing. That leaves us with 4 possible situations as indicated in the figure.



To summarize:

we have 3 unknown quantities, ν^z, ν^v and T and 3 conditions to meet,

$z_T = 0$ $v_T = 0$ and $\nu^v = \pm 1$.

If the control (in a period) has a constant values a $(= \pm 1)$, then the solution to the state equation

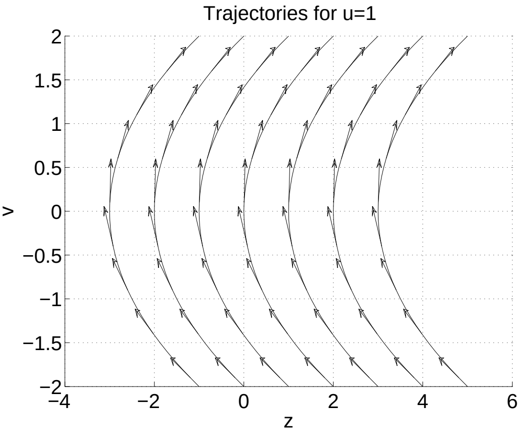
$$\frac{d}{dt}\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix}$$

is simply

$$\begin{aligned} v_t &= v_0 + at \\ z_t &= z_0 + v_0 t + \frac{1}{2}at^2 \end{aligned}$$

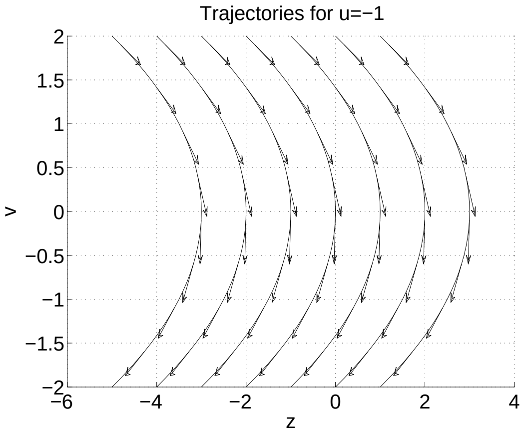
$a = 1$ (Acceleration)

$$v_t = v_0 + t$$
$$z_t = z_0 + v_0 t + \frac{1}{2}t^2$$



$a = -1$ (Deacceleration)

$$\begin{aligned} v_t &= v_0 - t \\ z_t &= z_0 + v_0 t - \frac{1}{2} t^2 \end{aligned}$$



Bang Bang Control - On the switching curve.

If the terminal point (origin) and the starting point is on the same parabola (and in the directionis of arrows), i.e. satisfy the equations where $T_f = T$.

Deacceleration ($v_0 > 0, \;\; a = -1$)

$$\begin{aligned} 0 &= v_0 - T_f \\ 0 &= z_0 + v_0 T_f - \frac{1}{2} T_f^2 \end{aligned}$$

$$T_f = v_0$$

$$z_0 = -\frac{1}{2} v_0^2$$

Accelertion ($v_0 < 0, \;\; a = 1$)

$$\begin{aligned} 0 &= v_0 + T_f \\ 0 &= z_0 + v_0 T_f + \frac{1}{2} T_f^2 \end{aligned}$$

$$T_f = -v_0$$

$$z_0 = \frac{1}{2} v_0^2$$

Bang Bang Control - On the switching curve.

No switching.

If the terminal point (origin) and the starting point is on the same parabola (and in the directionis of arrows), i.e. satisfy the equations

$$\begin{aligned} 0 &= v_0 + aT_f & a &= \pm 1 \\ 0 &= z_0 + v_0 T_f + \frac{1}{2} a T_f^2 \end{aligned} \tag{1}$$

where $T_f = T$. Solution:

$$T_f = -\frac{v_0}{a} \quad \geq 0 \qquad\qquad z_0 = \frac{1}{2}\frac{v_0^2}{a} \qquad \text{(must be satisfied)} \tag{2}$$

for either $a = 1$ or $a = -1$. Since $T_f \geq 0$:

$$a = -sign(v_0) \qquad\qquad T_f = |v_0|$$

$$z0 = -\frac{1}{2} sign(v_0)\ v_0^2 \qquad\qquad \textit{(The Switching curve)}$$

If	$v_0 > 0$	then	$a = -1$	and	$z_0 = -\frac{1}{2}v_0^2$.
If	$v_0 < 0$	then	$a = 1$	and	$z_0 = \frac{1}{2}v_0^2$.

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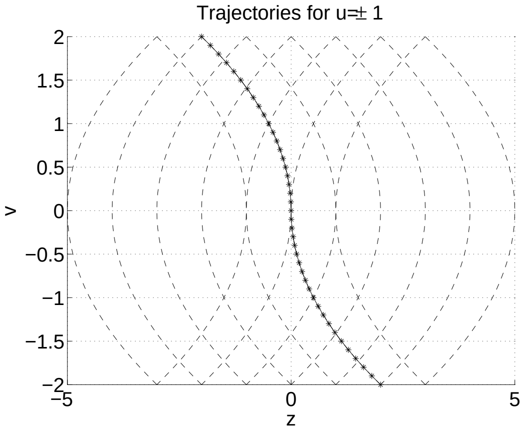
Bang Bang Control.

Along the switching (or the acceleration/de-acceleration) curve:

$$z_0 = \begin{cases} -\frac{1}{2}v_0^2 & \text{deacc.} \quad a=-1 \quad \text{for} \quad v_0 > 0 \\ \frac{1}{2}v_0^2 & \text{acc.} \quad a=1 \quad \text{for} \quad v_0 < 0 \end{cases} \quad = \quad -\frac{1}{2}v_0^2 sign(v_0)$$



Let us return to the general case.

From Pontryagin and the solution to the costate equation:

$$a_t = -sign\Big(\nu^v + \nu^z(T-t)\Big)$$

There will be a switching unless the initial point lies on switching curve. We will have 2 possibilities.

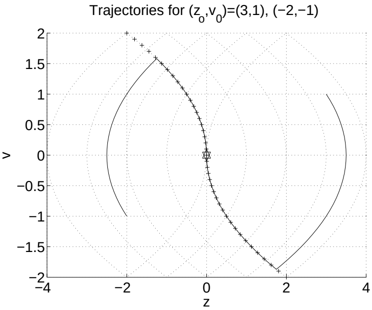
$$a_t = \begin{cases} 1 & \text{for } 0 \le t \le T_s \\ -1 & \text{for } T_s < t \le T \end{cases} \qquad z_0 < -\frac{1}{2}v_0^2 sign(v_0) \qquad \text{Below the switching curve.}$$

$$a_t = \begin{cases} -1 & \text{for } 0 \le t \le T_s \\ 1 & \text{for } T_s < t \le T \end{cases} \qquad z_0 > -\frac{1}{2}v_0^2 sign(v_0) \qquad \text{Above the switching curve.}$$

$$a_t = 1 \qquad z_0 = \frac{1}{2}v_0^2 \qquad v_0 < 0 \qquad \text{On the switching curve}$$

$$a_t = -1 \qquad z_0 = -\frac{1}{2}v_0^2 \qquad v_0 > 0 \qquad \text{On the switching curve}$$

Bang Bang Control.

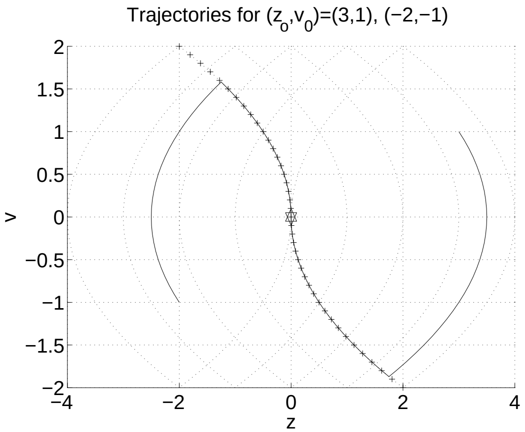


As a state feedback law

$$a_t = \begin{cases} 1 & \text{for} \quad z < -\frac{1}{2}v^2 sign(v) & \text{(Below switching curve)} \\ -1 & \text{for} \quad z = -\frac{1}{2}v^2 \quad \text{and} \quad v > 0 & \text{(Down along switching curve)} \\ -1 & \text{for} \quad z > -\frac{1}{2}v^2 sign(v) & \text{(Above switching curve)} \\ 1 & \text{for} \quad z = \frac{1}{2}v^2 \quad \text{and} \quad v < 0 & \text{(Up along switching curve)} \end{cases}$$

Searching for T and T_s.

Assume $z_0 > -\frac{1}{2}v_0^2 sign(v_0)$ (above (to the right of) the switching curve)



For $t < T_s \; a_t = -1$ and $T_s < t \leq T \; a_t = 1$

For $t < T_s$ $a_t = -1$ and

$$v_t = v_0 - t \qquad\qquad z_t = z_0 + v_0 t - \frac{1}{2}t^2 \tag{1}$$

which is valid until we reach the switching curve given by

$$z = -\frac{1}{2}v^2 sign(v) = \frac{1}{2}v^2 \qquad \text{(because } v < 0\text{)} \tag{2}$$

This happens at T_s where (solution meets switching curve)

$$z_0 + v_0 T_s - \frac{1}{2}T_s^2 = \frac{1}{2}(v_0 - T_s)^2$$

or

$$T_s = v_0 + \sqrt{z_0 + \frac{1}{2}v_0^2}$$

Bang Bang Control.

The velocity and position at the switching point is

$$v_s = v_0 - T_s \qquad (\leq 0) \qquad\qquad z_s = \frac{1}{2}(v_0 - T_s)^2 \qquad (\geq 0)$$

For $T_s \leq t \leq T$ we have $a_t = 1$ and

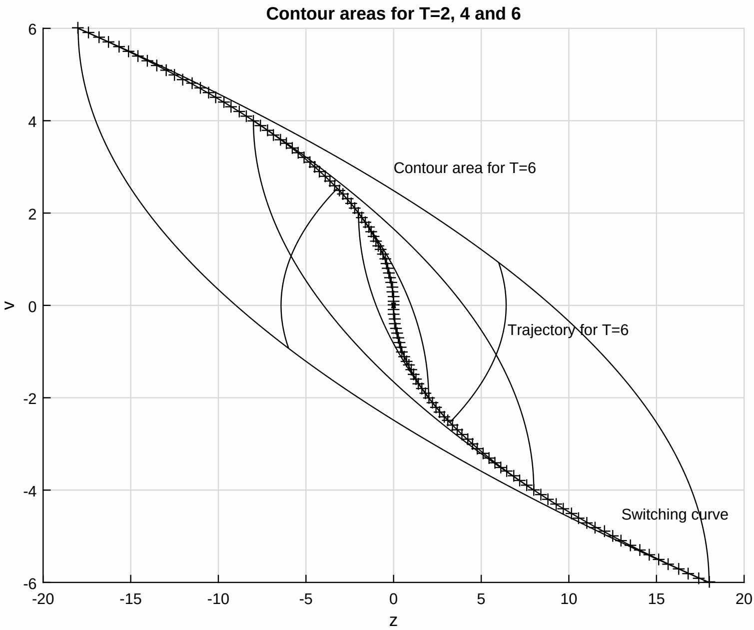
$$v_t = v_s + \tilde{t} \qquad\qquad z_t = z_s + v_s\tilde{t} + \frac{1}{2}\tilde{t}^2 \qquad\qquad \tilde{t} = t - T_s$$

The time left for the origin is

$$\tilde{T} = -v_s = T_s - v_0$$

For $t > T_s$ we accelerate $(u_t = 1)$ and in total

$$T = \tilde{T} + T_s = T_s - v_0 + T_s = v_0 + 2\sqrt{z_0 + \frac{1}{2}v_0^2}$$


Contour areas for T=2, 4 and 6
Contour area for T=6
Trajectory for T=6
Switching curve
v
z
6
4
2
0
-2
-4
-6
-20
-15
-10
-5
0
5
10
15
20

Project 2

- One person - one report in pdf (essential m-files eventually listed in appendix or uploaded in a zipped file).
- Name and ID on the middle on front.
- Name(s) and ID(s) of group mates on lowest part of front (separated from ovn id).
- Danish or English.
- Only pdf files are uploaded to CampusNet with name s123456.pdf.
- Deadline 20.12.2019 23:59
- Helpdesk: nkp (303/048, nkpo@dtu.dk, +45 4525 3356).
- Include 42111 and your study ID in subject field, when you are sending me a mail.