Static and Dynamic Optimization (42111)

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Lecture 10: Pontryagins principle

Outline of lecture

- Recap F9 (End Point Constraints)
  - Free Dynamic Optimization (D+C)
  - End Point Constraints
- Constrained Control - Decisions
- Pontryagins Principle (D)
- Investment planning
- Pontryagins Principle (C)
- Orbit injection (II)
- Reading guidance (DO: chapter 4)

Dynamic Optimization (D, free)

Find a sequence u_i, $i = 0, \ldots, N-1$ which takes the system

$$x_{i+1} = f_i(x_i, u_i) \qquad\qquad x_0 = \underline{x}_0$$

from its initial state $\underline{x}_0$ along a trajectory such that the performance index

$$J = \phi_N[x_N] + \sum_{i=0}^{N-1} L_i(x_i, u_i)$$

is optimized. Define the Hamiltonian function as:

$$H_i = L_i(x_i, u_i) + \lambda_{i+1}^T f_i(x_i, u_i)$$

The Euler-Lagrange equations:

$$x_{i+1} = f_i(x_i, u_i) \qquad \lambda_i^T = \frac{\partial}{\partial x_i} H_i$$

$$0 = \frac{\partial}{\partial u_i} H_i$$

Dynamic Optimization (C, free)

Find a function u_t $t \in [0;\ T]$ which takes the system system

$$\dot{x} = f_t(x_t, u_t) \qquad x_0 = \underline{x}_0$$

from its initial state $\underline{x}_0$ along a trajectory such that the performance index

$$J = \phi_T[x_T] + \int_0^T L_t(x_t, u_t)\ dt$$

is optimized. Define the Hamilton function as:

$$H(x, u, \lambda, t) = L_t(x_t, u_t) + \lambda_t^T f_t(x_t, u_t)$$

The Euler-Lagrange equations:

$$\dot{x} = f_t(x_t, u_t) \qquad -\dot{\lambda}^T = \frac{\partial}{\partial x_t} H_t$$

$$0 = \frac{\partial}{\partial u_t} H_t$$

Free DDO

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \lambda_N^T = \frac{\partial}{\partial x_N}\phi_N(x_N)$$

Free CDO

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \lambda_T^T = \frac{\partial}{\partial x_T}\phi_T(x_T)$$

DDO, End points constraints (EPC)

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \psi_N(x_N) = 0$$

$$\lambda_N^T = \nu^T \frac{\partial}{\partial x_N}\psi_N(x_N) + \frac{\partial}{\partial x_N}\phi_N(x_N)$$

CDO, End point constraints (EPC)

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \psi_T(x_T) = 0$$

$$\lambda_T^T = \nu^T \frac{\partial}{\partial x_T}\psi_T(x_T) + \frac{\partial}{\partial x_T}\phi_T(x_T)$$

End point constraints (EPC)

$$x_0 = \underline{x}_0 \qquad \psi_T(x_T) = 0 \qquad \lambda_T^T = \nu^T \frac{\partial}{\partial x_T} \psi_T(x_T) + \frac{\partial}{\partial x_T} \phi_T(x_T)$$

Simple EPC

$$x_T = \underline{x}_T \qquad \lambda_T^T = \nu^T + \frac{\partial}{\partial x_T} \phi_T(x_T)$$

Partial simple EPC

$$x_T = \begin{bmatrix} \tilde{x}_T \\ \bar{x}_T \end{bmatrix}$$

$$\tilde{x}_T = \underline{\tilde{x}}_T \qquad \bar{x}_T \text{ is free}$$

The boundary conditions becomes:

$$\tilde{x}_T = \underline{\tilde{x}}_T \qquad \tilde{\lambda}_T^T = \nu^T + \frac{\partial}{\partial \tilde{x}_T} \phi_T(x_T)$$

$$\bar{x}_T \text{ is free} \qquad \bar{\lambda}_T = \frac{\partial}{\partial \bar{x}_T} \phi_T(x_T)$$

Linear EPC

$$C x_T = \underline{r} \qquad C: \; p \times n \; \text{ matrix}$$

The boundary conditions are:

$$C x_T = \underline{r}$$

$$\lambda_T^T = \nu^T C + \frac{\partial}{\partial x_T} \phi_T(x_T)$$

DTU

Pontryagins Maximum principle

Constrained decisions:

$$u_i \in \mathcal{U}_i$$

Example:

$$|u_i| \leq \bar{u}$$

Example:

$$\underline{u} \leq u_i \leq \bar{u}$$

Example:

$$\underline{u}_i \leq u_i \leq \bar{u}_i$$

Example:

$$u_i \in \{-1,\ 0,\ 1\}$$

Pontryagin

Lev Semenovich Pontryagin (3 September 1908 - 3 May 1988) was a Soviet Russian mathematician. He was born in Moscow and lost his eyesight in a primus stove explosion when he was 14.

He made major discoveries in a number of fields of mathematics, including the geometric parts of topology. Later in his career he worked in optimal control theory. His maximum principle is fundamental to the modern theory of optimization.

Pontryagin was quite a controversial personality.

Source: Wikipedia

Pontryagin (D)

Find a sequence u_i, $i = 0, \dots, N-1$ where

$$u_i \in \mathcal{U}_i$$

which takes the system

$$x_{i+1} = f_i(x_i, u_i) \qquad\qquad x_0 = \underline{x}_0$$

from its initial state $\underline{x}_0$ along a trajectory such that the performance index

$$J = \phi[x_N] + \sum_{i=0}^{N-1} L_i(x_i, u_i)$$

is optimized (minimized or maximized). Defining the Hamiltonian function

$$H_i = L_i(x_i, u_i) + \lambda_{i+1}^T f_i(x_i, u_i)$$

The necessary equations:

$$x_{i+1} = f_i(x_i, u_i) \qquad \lambda_i^T = \frac{\partial}{\partial x_i} H_i \qquad u_i = arg \min_{u_i \in \mathcal{U}_i} [H_i]$$

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \lambda_N^T = \frac{\partial}{\partial x_N} \phi_N(x_N)$$

If EPC present the last is as given in Chapter 3.

Plan: During a period of time (N) to invest a amount of money u_i (limitted to max 600 Dkr) each interval to obtain a specified sum (x_N).

Dynamics:

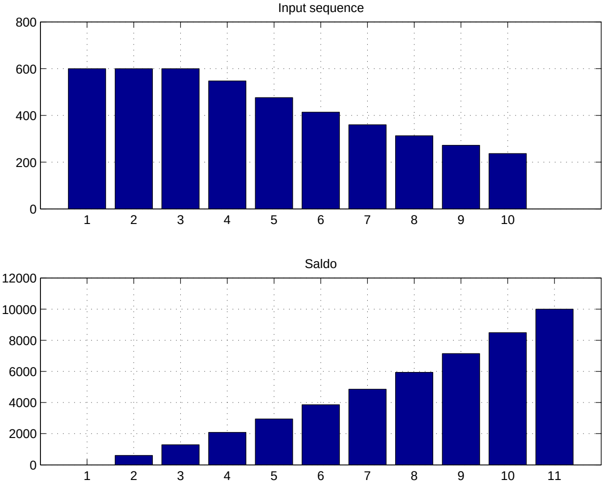
$$x_{i+1} = (1+\alpha)x_i + u_i \qquad x_0 = 0 \quad x_N = 10.000 \ Dkr$$

Objective:

$$Min \ J \qquad J = \sum_{i=0}^{N-1} \frac{1}{2} u_i^2$$

subject to:

$$0 \leq u_i \leq 600 Dkr$$


Input sequence
800
600
400
200
0
1
2
3
4
5
6
7
8
9
10
Saldo
12000
10000
8000
6000
4000
2000
0
1
2
3
4
5
6
7
8
9
10
11

The Hamiltonian function

$$H_i = \frac{1}{2}u_i^2 + \lambda_{i+1}\left[ax_i + u_i\right] \qquad\qquad a = 1 + \alpha$$

EL (or KKT) conditions:

$$\begin{aligned} x_{i+1} &= ax_i + u_i & x_0 = 0 \qquad x_N = 10000 \\ \lambda_i &= a\lambda_{i+1} & \lambda_N = \nu \\ u_i &= arg \min_{u_i \in \mathcal{U}_i} (H_i) \end{aligned}$$

$$\lambda_i = \nu a^{N-i}$$

$$u_i = -\lambda_{i+1} \quad \text{for} \quad \underline{u} \le u_i \le \bar{u} \qquad\qquad (-\underline{u} \ge \lambda_{i+1} \ge -\bar{u})$$

or

$$u_i = max(\underline{u}, min(\bar{u}, -\nu a^{N-i-1}))$$

For a given ν solve the state equation with the control inserted.

Ajusting ν such that EPC is met

$$x_N = \underline{x}_N = 10000 \ Dkr$$

DTU

Investment planning with economical (linear) cost

What happens (?) if the objective is changed into:

$$Min\ J \qquad J = \sum_{i=0}^{N-1} u_i$$

In that case:

$$H = u_i + \lambda_{i+1}(ax_i + u_i) \qquad = (1 + \lambda_{i+1})u_i + \lambda_{i+1}ax_i$$

and Pontryagins principle yields:

$$\begin{aligned} x_{i+1} &= ax_i + u_i \\ \lambda_i &= a\lambda_{i+1} \\ u_i &= arg \min_{u_i \in \mathcal{U}_i} (H_i) \end{aligned}$$

As previuos we have the costate evolution (ν is a constant or a Lagrange multiplier)

$$\lambda_i = \nu a^{N-i}$$

The optimization gives:

$$u_i = \begin{cases} \underline{u} & (1 + \lambda_{i+1}) > 0 \qquad \lambda_{i+1} > -1 \\ \bar{u} & (1 + \lambda_{i+1}) < 0 \qquad \lambda_{i+1} < -1 \end{cases}$$

Find a function u_t $t \in [0;\ T]$ where

$$u_t \in \mathcal{U}_t$$

which takes the system system

$$\dot{x} = f_t(x_t, u_t)$$

from its initial state $\underline{x}_0$ along trajectories such that the performance index

$$J = \phi_T[x_T] + \int_0^T L_t(x_t, u_t)\ dt$$

is optimized. Define the Hamilton function as:

$$H_t(x, u, \lambda) = L_t(x_t, u_t) + \lambda_t^T f_t(x_t, u_t)$$

Then the necessary conditions for this problem can be written as:

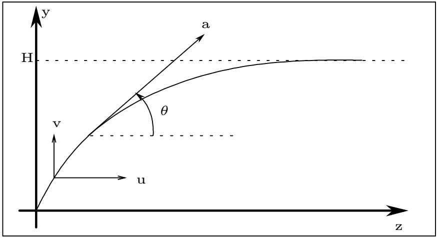
$$\dot{x} = f_t(x_t, u_t) \qquad -\dot{\lambda}^T = \frac{\partial}{\partial x_t} H_t \qquad u_t = arg \min_{u_t \in \mathcal{U}_t} [H_t]$$

with boundary conditions:

$$x_0 = \underline{x}_0 \qquad \lambda_T = \frac{\partial}{\partial x_T} \phi_T(x_T) \qquad = \frac{\partial}{\partial x} \phi_T$$

or as in Chapter 3 for EPC.

Orbit injection problem II



The problem is to find the specific thrust force with components, a_t^z and a_t^y, satisfying

$$(a_t^z)^2 + (a_t^y)^2 = a^2$$

such that the terminal horizontal velocity, u_T, is maximized subject to the dynamics

$$\frac{d}{dt}\begin{bmatrix} u_t \\ v_t \\ z \\ y \end{bmatrix} = \begin{bmatrix} a_t^z \\ a_t^y \\ u_t \\ v_t \end{bmatrix} \qquad \begin{bmatrix} u_0 \\ v_0 \\ z_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the terminal constraints

$$v_T = 0 \qquad y_T = H$$

$$J = u_T \qquad\qquad (\ \phi_T = u_T \qquad L_t = 0 \ )$$

DTU

The Hamilton functions (and others) are

$$H_t = \lambda_t^u a_t^z + \lambda_t^v a_t^y + \lambda_t^z u_t + \lambda_t^y v_t \qquad \phi_T = u_T \qquad \psi_T = \begin{bmatrix} v_T \\ y_T \end{bmatrix} = \begin{bmatrix} 0 \\ H \end{bmatrix}$$

The costate equation:

$$-\frac{d}{dt}\begin{bmatrix} \lambda_t^u & \lambda_t^v & \lambda_t^z & \lambda_t^y \end{bmatrix} = \begin{bmatrix} \lambda_t^z & \lambda_t^y & 0 & 0 \end{bmatrix}$$

has the boundary conditions

$$\lambda_T^v = \nu_v \qquad \lambda_T^y = \nu_y \qquad \text{(fixed state, free costate)}$$

$$\lambda_T^u = 1 \qquad \lambda_T^z = 0 \qquad \text{(free state, fixed costate)}$$

resulting in

$$\lambda_t^z = 0 \qquad \lambda_t^y = \nu_y$$

$$\lambda_t^u = 1 \qquad \lambda_t^v = \nu_v + \nu_y(T - t)$$

The maximization of

$$\begin{bmatrix} a_t^z \\ a_t^y \end{bmatrix} = arg\max\left(\lambda_t^u a_t^z + \lambda_t^v a_t^y + \lambda_t^z u_t + \lambda_t^y v_t\right)$$

subject to

$$(a_t^z)^2 + (a_t^y)^2 = a^2$$

has the solution:

$$\boxed{\begin{bmatrix} a_t^z \\ a_t^y \end{bmatrix} = \begin{bmatrix} \lambda_t^u \\ \lambda_t^v \end{bmatrix} \frac{a}{\sqrt{(\lambda_t^u)^2 + (\lambda_t^v)^2}}}$$

DTU

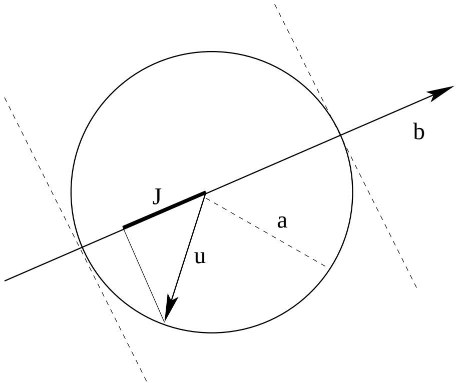
The MP problem

$$Min\Big(b^T u\Big) \qquad st. \qquad u^T u \leq a^2 \qquad a \geq 0$$

has the solution:

$$u^* = -\frac{a}{||b||} b$$

Geometric approach



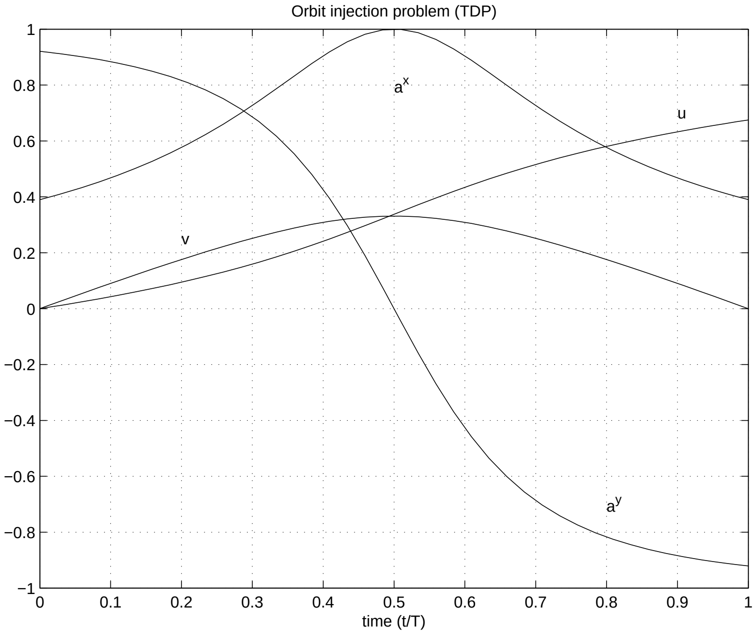
Analytic approach

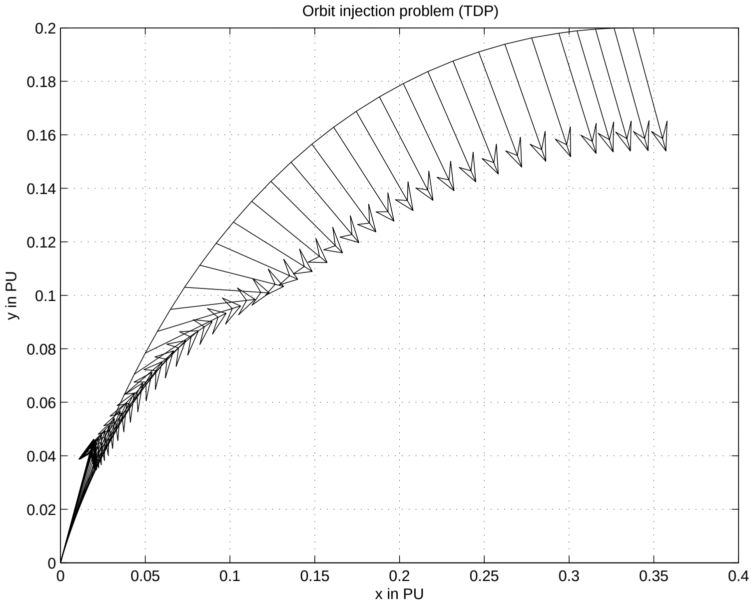
$$J_L = b^T u + \lambda(u^T u - a^2)$$

KKT:

$$u^T u \leq a^2$$
$$b^T + 2\lambda u^T = 0$$

$$u = -\frac{b}{2\lambda} \qquad \lambda^2 = \frac{b^T b}{4a^2} \qquad \boxed{u = -b\frac{a}{\sqrt{b^T b}}}$$


Orbit injection problem (TDP)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (t/T)
a^x
u
v
a^y


Orbit injection problem (TDP)
y in PU
x in PU
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

```
% ---------------------------------------------
function main2
% ---------------------------------------------
T=1;                    % parameters
a=1;
H=0.2;

parm=[-2.4 4.7]';       % Initial guess on parm
x0=zeros(4,1);          % Initial state variable

opt=optimset;                 % Options for fsolve
opt=optimset(opt,'Display','iter');
parm=fsolve(@erf,parm,opt,T,a,x0,H);  % Call fsolve for finding parameters

[err,time,xt]=erf(parm,T,a,x0,H);     % Call erf ones more for getting the
avt=[];                               % state trajectories.
for i=1:length(time),
 t=time(i);
 la=[1; parm(1)+parm(2)*(T-t)];
 av=la/sqrt(la'*la)*a;                 % Thrust force as vector
 avt=[avt; av'];                       % and stored in a matrix
end

% Here goes the plotting commands. (file: ~nkpo/02711/dist3/main2.m)
plot(time,[xt(:,1:2) avt]); grid minor; % Plot
```

```
% ---------------------------------------------
function [err,time,xt]=erf(parm,T,a,x0,H)
% ---------------------------------------------
% Determine the end point error (err) given the EPC Lagrange multipliers
% in parm (and the constants that specifies the problem).
Tspan=0:T;
[time,xt]=ode45(@tdpc,Tspan,x0,[],parm,T,a);
xT=xt(end,:)';
err=[xT(2)-0;
     xT(3)-H];

% ---------------------------------------------
function dx=tdpc(t,x,parm,T,a)
% ---------------------------------------------
% System model. Determine the (time) derivative of the state vector
% given the time, state (x) and the EPC Lagrange multipliers.
u=x(1); v=x(2); z=x(3); y=x(4);
p1=parm(1); p2=parm(2);
la=[1; p1+p2*(T-t)];
av=la/sqrt(la'*la)*a;                % Specific thrust force as a vector
dx=[av;                              % remember - a vector
    u;
    v];
```

Free production

Consider a production

$$\dot{x}_t = \alpha x_t \qquad x_0 = \underline{x}_0 \geq 0$$

where $\alpha > 0$.

Resource Allocation

Let $0 \leq u_t \leq 1$ be the fraction kept for production (reinvestment).

Then $1 - u_t$ will be the fraction to be harvested.

The DO problem is:

$$\dot{x}_t = \alpha u_t x_t \qquad x_0 = \underline{x}_0 \qquad x_t \geq 0$$

and

$$J = \int_0^T (1 - u_t) x_t dt$$

Maximize J subject to $0 \leq u_t \leq 1$.

Pontryagin

$$H = L_t + \lambda_t^T f_t = (1 - u_t) x_t + \lambda_t \alpha u_t x_t$$
$$= x_t + (\alpha \lambda_t - 1) x_t u_t$$

$$\dot{x}_t = \alpha u_t x_t \qquad x_0 = \underline{x}_0 > 0$$

$$-\dot{\lambda}_t = 1 + (\alpha \lambda_t - 1) u_t \qquad \lambda_T = 0$$

$$u_t = \begin{cases} 1 & (\alpha \lambda_t - 1) x_t > 0 \\ 0 & (\alpha \lambda_t - 1) x_t < 0 \end{cases}$$

---

since $x_t \geq 0$:

$$u_t = \begin{cases} 1 & \lambda_t > \frac{1}{\alpha} \quad (Production) \\ 0 & \lambda_t < \frac{1}{\alpha} \quad (Harvest) \end{cases}$$

Harvest

Since

$$\lambda_T = 0$$

there exist an interval $[T_1; T] \quad (T_1 < T)$ where

$$\lambda_t < \frac{1}{\alpha}$$

Here (in this interval):

$$u_t = 0$$

$$\dot{x}_t = 0 \qquad x_t = x_T$$

$$\dot{\lambda}_t = -1 \qquad \lambda_t = (T - t)$$

From this we have $(\lambda_{T_1} = \frac{1}{\alpha} = T - T_1)$

$$T_1 = T - \frac{1}{\alpha}$$

Production

For $0 \le t < T_1$

$$u_t = 1$$

$$\dot{x} = \alpha x_t \qquad x_0 = \underline{x}_0$$

$$\dot{\lambda}_t = -\alpha \lambda_t \qquad \lambda_{T_1} = \frac{1}{\alpha}$$

$$x_t = x_0 \; e^{\alpha t} \qquad x_{T_1} = x_0 e^{\alpha T_1}$$

$$\lambda_t = \frac{1}{\alpha} \;\; e^{\alpha(T_1 - t))}$$

Solution summary

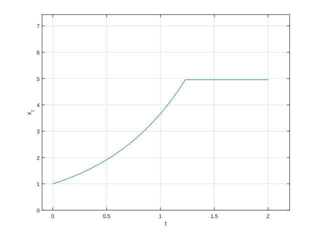
$$T_1 = T - \frac{1}{\alpha}$$

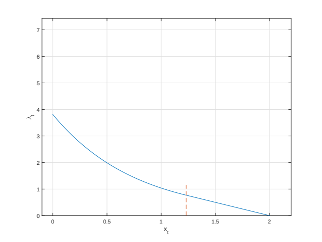
Then:

$$u_t = \begin{cases} 1 & \text{for} \quad 0 \leq t < T_1 \\ 0 & \text{for} \quad T_1 < t \leq T \end{cases}$$

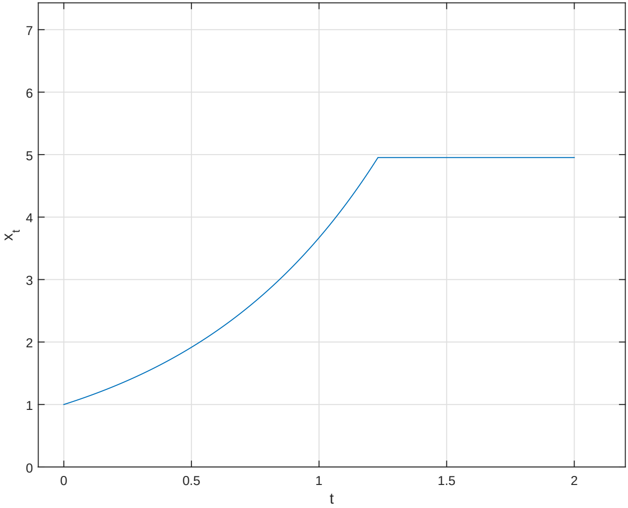
$$x_t = \begin{cases} x_0 \; e^{\alpha t} & \text{for} \quad 0 \leq t \leq T_1 \\ x_0 \; e^{\alpha T_1} & \text{for} \quad T_1 \leq t \leq T \end{cases}$$

$$\lambda_t = \begin{cases} \frac{1}{\alpha} \quad e^{\alpha(T_1 - t))} & \text{for} \;\; 0 \leq t \leq T_1 \\ T - t & \text{for} \;\; T_1 \leq t \leq T \end{cases}$$

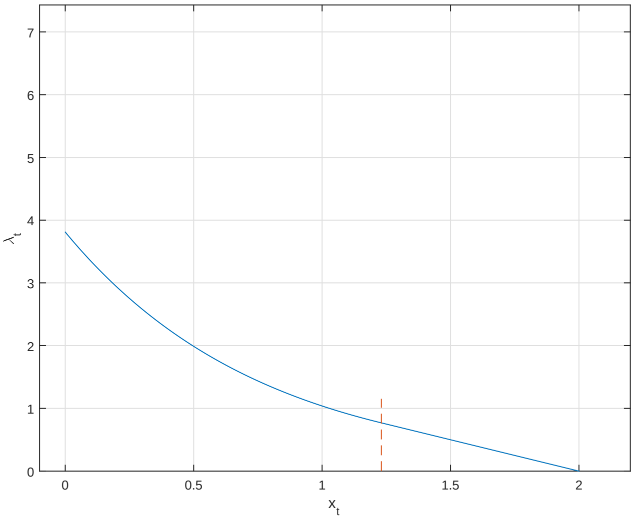




Resource allocation

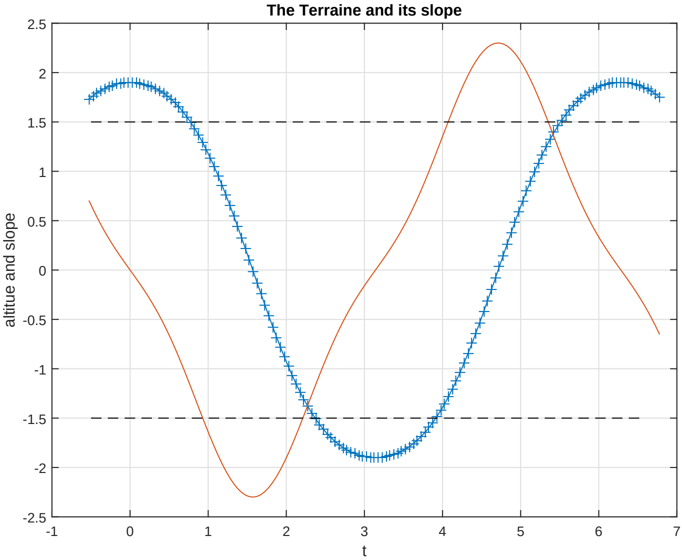


Resource allocation



Road construction

Road construction



Road construction

Objective: Find road level, x_t, such that

$$J = \int_{t=0}^{T} \frac{1}{2}(x_t - z_t)^2 dt$$

is minimized. Here z_t is the level of terrain.

The dynamic is:

$$\dot{x}_t = u \qquad x_0 = \underline{x}_0$$

where

$$|u_t| \leq a$$

Road construction

$$H_t = \frac{1}{2}(x_t - z_t)^2 + \lambda_t u_t \qquad \phi(x_T) = 0$$

$$\dot{x}_t = u \qquad x_0 = \underline{x}_0$$

$$-\dot{\lambda}_t = x_t - z_t \qquad \lambda_T = 0$$

$$u_t = arg \min_{|u_t| \leq a} \left\{ \frac{1}{2}(x_t - z_t)^2 + \lambda_t u_t \right\}$$

$$\lambda_t = \int_0^t (z_t - x_t)dt$$

Notice: $\lambda_t = 0$ for $x_t = z_t$.

$$u_t = \begin{cases} a & \text{for } \lambda < 0 \\ ? & \text{for } \lambda = 0 \\ -a & \text{for } \lambda > 0 \end{cases}$$

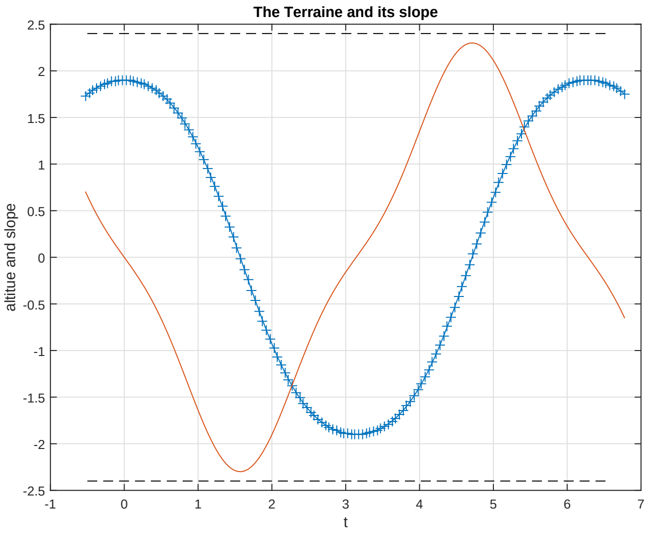
Optimal trajectories are obtained by concatenation of three types of arcs

- Regular arcs where $\lambda_t > 0$ and $u_t = -a$ (maximum downhill slope ars).
- Regular arcs where $\lambda_t < 0$ and $u_t = a$ (maximum uphill slope ars).
- Singular arcs where $\lambda_t = 0$ and where $|u_t| < a$ can take any value.

  In that interval $\dot{\lambda}_t = 0$ and then $x_t = z_t$. Since $\dot{x} = u$ we have $u = \dot{z}$.

DTU

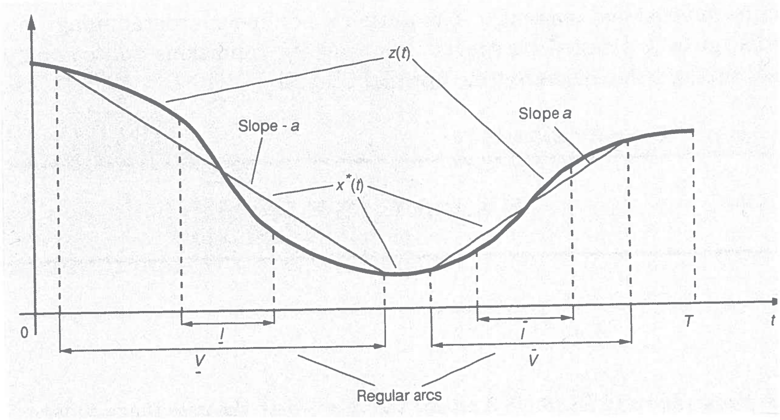
Assume $|\dot{z}| < a$



then

$$x_t = z_t \qquad u_t = \dot{z}_t \qquad \lambda_t = 0$$

Road construction



Road construction

$\lambda > 0$

$$u_t = -a$$
$$x_t = z_{t_1} - a(t - t_1)$$
$$\lambda_t = \int_{t_1}^{t} (z_t - x_t)dt$$

$t_1 < t < t_2$

$$u_t = -a$$
$$x_{t_2} = z_{t_1} - a(t_2 - t_1)$$
$$\lambda_{t_2} = \int_{t_1}^{t_2} (z_t - x_t)dt$$

$\lambda_t = 0$

$$u_t = \dot{z}_t$$
$$x_t = z_t$$
$$\lambda_t = 0$$

For determination of t_1 and t_2:

$$\int_{t_1}^{t_2} (z_t - x_t)dt = 0 \qquad z_{t_2} = z_{t_1} - a(t_2 - t_1)$$

Reading guidance

DO: Chapter 4