



# Topics

- Lagrange Multipliers
- Lagrangian Relaxation
- Lagrangian Duality

## Example: Economic Order Quantity Model

- Parameters
  - Demand rate:  $d$
  - Order cost:  $K$
  - Unit cost:  $c$
  - Holding cost:  $h$
- Decision Variable: The optimal order quantity  $Q$
- Objective Function:

$$\text{minimize } \frac{dK}{Q} + dc + \frac{hQ}{2}$$

- Optimal Order Quantity:

$$Q^* = \sqrt{\frac{2dK}{h}}$$

# EOQ Model

Suppose we have several items with a space constraint  $q$

## The problem

$$\begin{aligned} & \text{minimize} && \sum_j \left( \frac{d_j K_j}{Q_j} + d_j c_j + \frac{h Q_j}{2} \right) \\ & \text{subject to:} && \sum_j Q_j \leq q \end{aligned}$$

## We have the following possibilities

- ① The constraint is **non-binding/slack**, i.e

$$\sum_j \sqrt{\frac{2d_j K_j}{h}} < q$$

- ② The constraint is **binding/active**

# Lagrange Multiplier

## The problem

$$\begin{aligned} \text{minimize} \quad & T(Q_1, Q_2, \dots, Q_n) = \sum_j \left( \frac{d_j K_j}{Q_j} + d_j c_j + \frac{h Q_j}{2} \right) \\ \text{subject to:} \quad & \sum_j Q_j = q \end{aligned}$$

- Lagrange multiplier  $\lambda$

$$\text{minimize } T(Q_1, Q_2, \dots, Q_n) + \lambda \left( \sum_j Q_j - q \right)$$

- Differentiate with respect to  $Q_j$ :

$$\frac{\partial T}{\partial Q_j} = -\frac{d_j K_j}{Q_j^2} + \frac{h}{2} + \lambda = 0 \quad \forall j$$

- Solving for  $Q_j$

$$Q_j = \sqrt{\frac{2d_j K_j}{h + 2\lambda}} \quad \forall j$$

## Lagrange multiplier

- Substituting this into the constraint we have

$$\sum_j \sqrt{\frac{2d_j K_j}{h + 2\lambda}} = q$$

- Solving for  $\lambda$  and  $Q_j$  gives:

$$\lambda = \lambda^* = \frac{\left( \frac{\sum_j \sqrt{2d_j K_j}}{q} \right)^2 - h}{2}$$

$$Q_j^* = \sqrt{\frac{2d_j K_j}{h + 2\lambda^*}} \quad \forall j$$

## Interpretation of $\lambda$

- In linear programming a dual variable is a shadow price:

$$y_i^* = \frac{\partial Z^*}{\partial b_i}$$

- Similarly, in the EOQ model, the Lagrange multiplier measures the marginal change in the total cost resulting from a change in the available space

$$\lambda^* = \frac{\partial T^*}{\partial q}$$

## Example

### Problem

$$\begin{array}{ll}\text{minimize:} & x^2 + y^2 + 2z^2 \\ \text{subject to:} & 2x + 2y - 4z \geq 8\end{array}$$

- The Lagrangian is:

$$L(x, y, z, \mu) = x^2 + y^2 + 2z^2 + \mu(8 - 2x - 2y + 4z)$$

- Note that the unconstrained minimum  $x = y = z = 0$  is **not** feasible



## Example Continued

- Differentiating with respect to  $x, y, z$

$$\frac{\partial L}{\partial x} = 2x - 2\mu = 0$$

$$\frac{\partial L}{\partial y} = 2y - 2\mu = 0$$

$$\frac{\partial L}{\partial z} = 4z + 4\mu = 0$$

- We can conclude that  $z = -\mu, x = y = \mu$
- Substituting this into  $2x + 2y - 4z = 8$  gives  $x = 1, y = 1, z = -1$
- Optimal objective function value = 4

## Checking the value of $\mu$

- $\mu = 1 \rightarrow$  states that we can expect an increase (decrease) of one unit for a unit change in the right hand side of the constraint
- Resolve the problem with a righthandside on the constraint of 9
- $\mu^* = \frac{9}{8}, x^* = \frac{9}{8}, y^* = \frac{9}{8}, z^* = -\frac{9}{8}$
- New objective function value:

$$\left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2 + 2\left(-\frac{9}{8}\right)^2 = \frac{324}{64}$$

- This is an increase of  $\approx 1$  unit

# Lagrange relaxation

## Problem $\mathcal{P}$

$$\begin{array}{ll}\text{minimize:} & f(\mathbf{x}) \\ \text{subject to:} & \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\end{array}$$

- Choose non negative multipliers  $\mathbf{u}$
- Solve the Lagrangian: minimize  $f(\mathbf{x}) + \mathbf{u}\mathbf{g}(\mathbf{x})$ ,
- Optimal solution  $\mathbf{x}^*$

# Lagrange relaxation

- $f(x^*) + u g(x^*)$  provides a lower bound  $\mathcal{P}$
- If  $g(x^*) \leq 0$ ,  $u g(x^*) = 0$ ,  $x^*$  is an optimal solution to problem  $\mathcal{P}$
- $x^*$  is an optimal solution to:

$$\begin{array}{ll} \text{minimize:} & f(x) \\ \text{subject to:} & g(x) \leq g(x^*) \end{array}$$

## Example from last time ...

### Example

$$\begin{array}{ll}\text{minimize} & 2x_1^2 + x_2^2 \\ \text{subject to:} & x_1 + x_2 = 1\end{array}$$

$$L(x_1, x_2, \lambda_1) = 2x_1^2 + x_2^2 + \lambda_1(1 - x_1 - x_2)$$

## Different values of $\lambda_1$

$\lambda_1 = 0 \rightarrow$  get solution  $x_1 = x_2 = 0, 1 - x_1 - x_2 = 1$

$$L(x_1, x_2, \lambda_1) = 0$$

$\lambda_1 = 1 \rightarrow$  get solution  $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, 1 - x_1 - x_2 = \frac{1}{4}$

$$L(x_1, x_2, \lambda_1) = \frac{5}{8}$$

$\lambda_1 = 2 \rightarrow$  get solution  $x_1 = \frac{1}{2}, x_2 = 1, 1 - x_1 - x_2 = -\frac{1}{2}$

$$L(x_1, x_2, \lambda_1) = \frac{1}{2}$$

$\lambda_1 = \frac{4}{3} \rightarrow$  get solution  $x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, 1 - x_1 - x_2 = 0$

$$L(x_1, x_2, \lambda_1) = \frac{2}{3}$$

# Lagrangian Dual

## Primal

$$\begin{aligned} \text{minimize:} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned}$$

## Lagrangian Dual

$$\begin{aligned} \text{maximize:} \quad & \theta(\mathbf{u}, \mathbf{v}) \\ \text{subject to:} \quad & \mathbf{u} \geq \mathbf{0} \end{aligned}$$

$$\theta(\mathbf{u}, \mathbf{v}) = \min_{\mathbf{x}} \{f(\mathbf{x}) + \mathbf{u}\mathbf{g}(\mathbf{x}) + \mathbf{v}\mathbf{h}(\mathbf{x})\}$$

# Lagrangian Dual

## Weak Duality: For Feasible Points

$$\theta(u, v) \leq f(x)$$

## Strong Duality: Under Constraint Qualification

If  $f$  and  $g$  are convex and  $h$  is affine, the optimal objective function values are equal

- Often there is a duality gap



## Example 1

### The problem

$$\begin{array}{ll}\text{minimize:} & x_1^2 + x_2^2 \\ \text{subject to:} & x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0\end{array}$$

- Let  $X := \{x \in \mathbb{R}^2 | x_1, x_2 \geq 0\} = \mathbb{R}_+^2$
- The Lagrangian is:

$$L(\mathbf{x}, \lambda) = x_1^2 + x_2^2 + \lambda(4 - x_1 - x_2)$$

## Example 1

The Lagrangian dual function:

$$\begin{aligned}\theta(\lambda) &= \min_{\mathbf{x} \in X} \{x_1^2 + x_2^2 + \lambda(4 - x_1 - x_2)\} \\ &= 4\lambda + \min_{\mathbf{x} \in X} \{x_1^2 + x_2^2 - \lambda x_1 - \lambda x_2\} \\ &= 4\lambda + \min_{x_1 \geq 0} \{x_1^2 - \lambda x_1\} + \min_{x_2 \geq 0} \{x_2^2 - \lambda x_2\}\end{aligned}$$

- For a fixed value of  $\lambda \geq 0$ , the minimum of  $L(\mathbf{x}, \lambda)$  over  $\mathbf{x} \in X$  is attained at  $x_1(\lambda) = \frac{\lambda}{2}, x_2(\lambda) = \frac{\lambda}{2}$

$$\Rightarrow L(\mathbf{x}(\lambda), \lambda) = 4\lambda - \frac{\lambda^2}{2} \quad \forall \lambda \geq 0$$

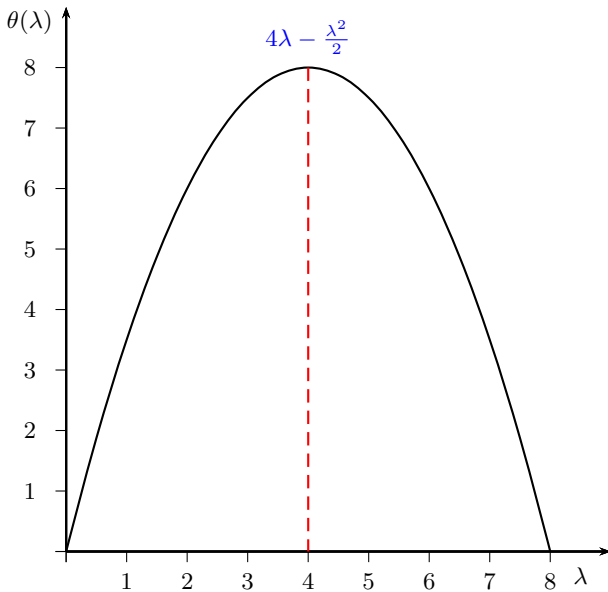
## Example 1

- The dual function is concave and differentiable
- We want to maximize the value of the dual function

$$\frac{\partial L}{\partial \lambda} = 4 - \lambda = 0$$

- This implies  $\lambda^* = 4, \theta(\lambda^*) = 8$
- $\mathbf{x}(\lambda^*) = \mathbf{x}^* = (2, 2)$

## Example 1



## Example 2

### The problem

$$\begin{array}{ll}\text{minimize:} & 3x_1 + 7x_2 + 10x_3 \\ \text{subject to:} & x_1 + 3x_2 + 5x_3 \geq 7 \\ & x_1, x_2, x_3 \in \{0, 1\}\end{array}$$

- Let  $X := \{x \in \mathbb{R}^3 | x_j \in \{0, 1\}, j = 1, 2, 3\}$
- The Lagrangian is:

$$L(\mathbf{x}, \lambda) = 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)$$

## Example 2

The Lagrangian dual function:

$$\begin{aligned}\theta(\lambda) &= \min_{x \in X} \{3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)\} \\ &= 7\lambda + \min_{x_1 \in \{0,1\}} \{(3 - \lambda)x_1\} + \min_{x_2 \in \{0,1\}} \{(7 - 3\lambda)x_2\} \\ &\quad \min_{x_3 \in \{0,1\}} \{(10 - 5\lambda)x_3\}\end{aligned}$$

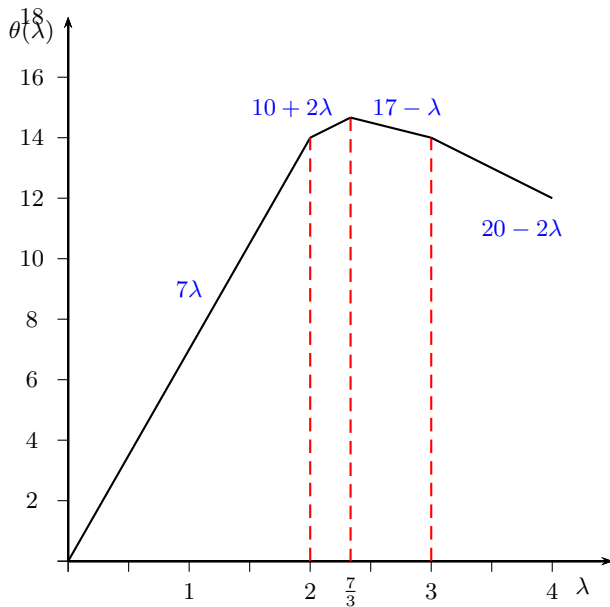
- $X(\lambda)$  is obtained by setting

$$x_j = \begin{cases} 1 \\ 0 \end{cases} \quad \text{when the objective coefficient is } \begin{cases} \leq 0 \\ \geq 0 \end{cases}$$

## Example 2

| $\lambda \in$      | $x_1(\lambda)$ | $x_2(\lambda)$ | $x_3(\lambda)$ | $g(\mathbf{x}(\lambda))$ | $\theta(\lambda)$ |
|--------------------|----------------|----------------|----------------|--------------------------|-------------------|
| $[-\infty, 2]$     | 0              | 0              | 0              | 7                        | $7\lambda$        |
| $[2, \frac{7}{3}]$ | 0              | 0              | 1              | 2                        | $10+2\lambda$     |
| $[\frac{7}{3}, 3]$ | 0              | 1              | 1              | -1                       | $17-\lambda$      |
| $[3, \infty]$      | 1              | 1              | 1              | -2                       | $20-2\lambda$     |

## Example 2





## Example 2

- $\theta$  is concave, but non-differentiable at break points  $\lambda \in \{2, \frac{7}{3}, 3\}$
- Check that the slope equals the value of the constraint function
- The slope of  $\theta$  is negative for objective pieces corresponding to feasible solutions to the original problem
- The one variable function  $\theta$  has a “derivative” which is non-increasing; this is a property of every concave function of one variable
- $\lambda^* = \frac{7}{3}, \theta(\lambda^*) = \frac{44}{3} = 14\frac{2}{3}$
- $\mathbf{x}^* = (0, 1, 1), f(\mathbf{x}^*) = 17$
- A **positive** duality gap!
- $X(\lambda^*) = \{(0, 0, 1), (0, 1, 1)\}$

## Example 3

### The problem

$$\begin{array}{ll}\text{minimize:} & \mathbf{c}^T \mathbf{x} \\ \text{subject to:} & \mathbf{Ax} \geq \mathbf{b} \\ & \mathbf{x} \text{ free}\end{array}$$

- Objective:

$$f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

- Identifying  $\mathbf{g}(\mathbf{x})$

$$\mathbf{g}(\mathbf{x}) = \mathbf{Ax} - \mathbf{b}$$

- Lagrangian dual function:

$$\theta(\boldsymbol{\lambda}) = \min_{\mathbf{x}} \{ \mathbf{c}^T \mathbf{x} + \boldsymbol{\lambda}^T (\mathbf{b} - \mathbf{Ax}) \} = \boldsymbol{\lambda}^T \mathbf{b}$$

## Example 3

- Provided that the following condition is satisfied

$$A^T \lambda - c = 0$$

- That is, we get the following problem

$$\begin{array}{ll}\text{maximize:} & b^T \lambda \\ \text{subject to:} & A^T \lambda = c \\ & \lambda \geq 0\end{array}$$

- Compare with Dual: min.  $b^T \lambda$  s.t.  $A^T \lambda = c, \lambda \geq 0$

### Class exercise 1

minimize:  $x$   
subject to:  $x^2 + y^2 = 1$

- Solve the problem
- Formulate and solve the dual
- Check whether the objective functions are equal

## Class Exercise 2

$$\begin{aligned} \text{minimize:} \quad & -2x_1 + x_2 \\ \text{subject to:} \quad & x_1 + x_2 = 3 \\ & (x_1, x_2) \in X \end{aligned}$$

- ① Suppose  $X = \{(0,0), (0,4), (4,4), (4,0), (1,2), (2,1)\}$
- ② Formulate the Lagrangian Dual Problem
- ③ Plot the Lagrangian Dual Problem
- ④ Find the optimal solution to the primal and dual problems
- ⑤ Check whether the objective functions are equal
- ⑥ Explain your observation in 5

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