Linear Programming & Duality

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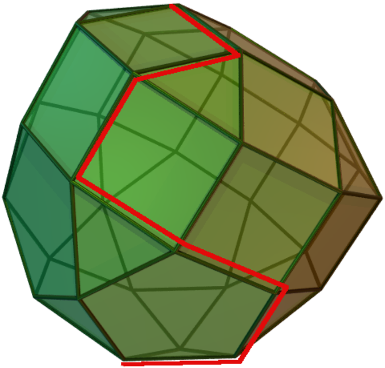
Today's Agenda

- Linear Programming
- Revised Simplex
- Duality

Solution Method



Simplex Method

- Convert $\leq$ inequalities by adding slack variables
- Put data into simplex tableau
- Perform simplex iterations by pivoting
- Entering Variable (*pivot column*)
  - Most negative coefficient in top row
- Leaving Variable (*pivot row*)
  - Minimum ratio: right hand sides and positive pivot column entries
- We disregard complications here
  - Phase 1, no feasible solution, unbounded solutions

First and Final Tableau

Z	x_1	x_2	s_1	s_2	s_3	
1	-3	-5				0
	1	0	1			4
	0	2		1		12
	3	2			1	18

Z	x_1	x_2	s_1	s_2	s_3	
1				$\frac{3}{2}$	1	36
			1	$\frac{1}{3}$	$-\frac{1}{3}$	2
		1		$\frac{1}{2}$	0	6
	1			$-\frac{1}{3}$	$\frac{1}{3}$	2

Matrix formulation

- General LP

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to:} \quad & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{aligned}$$

- becomes ..

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} + 0\boldsymbol{s} \\ \text{subject to:} \quad & A\boldsymbol{x} + I\boldsymbol{s} = \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \\ & \boldsymbol{s} \ge 0 \end{aligned}$$

Matrix formulation

- In each tableau each variable in x, s is designated as a basic variable or a nonbasic variable
- The tableau represents the equation system solved with respect to the basic variables.
- The basis matrix B is formed by the columns in the first tableau of the current basic variables
- The inverse basis matrix appears under the slack variables in each tableau

Matrix formulation

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{c}_B^T \boldsymbol{x_B} + \boldsymbol{c}_N^T \boldsymbol{x_N} & \\
\text{subject to:} \quad & B\boldsymbol{x_B} + N\boldsymbol{x_N} & = \boldsymbol{b} \\
& \boldsymbol{x_B} & \geq 0 \\
& \boldsymbol{x_N} & \geq 0
\end{aligned}$$

First and Later Tableau

First tableau ...

Z	$\boldsymbol{x}$	$\boldsymbol{s}$	
1	$-c$	0	0
	A	I	b

Later tableau ...

Z	$\boldsymbol{x}$	$\boldsymbol{s}$	
1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
	$B^{-1} A$	B^{-1}	$B^{-1} b$

- The current solution is $x_B = B^{-1}b,\ x_N = 0,\ Z = c_B B^{-1} b$
- At Optimality we have $c_B B^{-1} \geq 0, c_B B^{-1} A \geq c$
- The shadow prices are $c_B B^{-1}$

Duality

- The *primal* problem

$$
\begin{array}{rrrrl}
\text{maximize} & Z_P = & 3x_1 & +5x_2 & \\
\text{subject to:} & & x_1 & & \leq 4 \\
 & & & +2x_2 & \leq 12 \\
 & & 3x_1 & 2x_2 & \leq 18 \\
 & & x_1 & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}
$$

Continued

- The corresponding *dual* problem

$$
\begin{array}{rrrrrl}
\text{minimize} & Z_D = & 4y_1 & +12y_2 & +18y_3 & \\
\text{subject to:} & Z_D = & y_1 & & +3y_3 & \geq 3 \\
 & & & +2y_2 & +2y_3 & \geq 5 \\
 & & y_1 & & & \geq 0 \\
 & & & y_2 & & \geq 0 \\
 & & & & y_3 & \geq 0
\end{array}
$$

Class exercise 1

The problem

Each unit of product 1 requires 1 hour in department A and 1 hour in department B, and yields a profit of 1. The corresponding numbers for product 2 are 1 and 3, and 2. There are 3 and 7 hours available in departments A and B, respectively.

- Formulate an LP model and set up the first tableau
- Write the dual problem

Weak duality theorem

$$\begin{array}{rrl} \text{Primal: maximize} & \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to:} & A\boldsymbol{x} & \leq \boldsymbol{b} \\ & \boldsymbol{x} & \geq 0 \end{array}$$

$$\begin{array}{rrl} \text{Dual: minimize} & \boldsymbol{b}^T\boldsymbol{y} & \\ \text{subject to:} & A^T\boldsymbol{y} & \geq \boldsymbol{c} \\ & \boldsymbol{y} & \geq 0 \end{array}$$

Weak Duality Theorem

If $\boldsymbol{x}$ is primal feasible and $\boldsymbol{y}$ is dual feasible, then $c^T\boldsymbol{x} \leq \boldsymbol{y}^T A\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{y}$

Proof?

Strong duality theorem

Strong Duality Theorem

If one of the problems has an optimal solution the other one also has an optimal solution and the optimal objective function values are equal

- The optimal dual solution appears in the optimal primal tableau, under the slack variables (Proof?)
- The two other possibilities are
  - One problem is infeasible, the other is unbounded
  - Both problems are infeasible

Complementary Slackness

$$\begin{aligned} \text{Primal: maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} & \\ \text{subject to:} \quad A\boldsymbol{x} + \boldsymbol{s} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq 0 \\ \boldsymbol{s} &\geq 0 \end{aligned}$$

$$\begin{aligned} \text{Dual: minimize} \quad & \boldsymbol{b}^T\boldsymbol{y} & \\ \text{subject to:} \quad A^T\boldsymbol{y} - \boldsymbol{e} &= \boldsymbol{c} \\ \boldsymbol{y} &\geq 0 \\ \boldsymbol{e} &\geq 0 \end{aligned}$$

Definition

A primal solution and a dual solution exhibit complementary slackness if $\boldsymbol{e}^T\boldsymbol{x} = 0$ and $\boldsymbol{y}^T\boldsymbol{s} = 0$, i.e., corresponding $\boldsymbol{x}$- and $\boldsymbol{y}$-values are not both positive

Complementary Slackness

Complementary Slackness Theorem

Theorem: A primal solution and a dual solution are optimal iff they are feasible and complementary (proof?)

Example

Correspondences: x_1 and e_1, x_2 and e_2
y_1 and s_1, y_2 and s_2, y_3 and s_3

Class exercise 2

- The final tableau for the exercise 1 problem is

Z	x_1	x_2	s_1	s_2	
1			$\frac{1}{2}$	$\frac{1}{2}$	5
	1		$\frac{3}{2}$	$-\frac{1}{2}$	1
		1	$-\frac{1}{2}$	$\frac{1}{2}$	2

- Read off the optimal solution and the dual solution.
- Read off B^{-1} and verify that $B^{-1}B = I$.
- Given the primal solution, find the dual solution using complementary slackness.
- Use complementary slackness to show that $x_1 = 0$, $x_2 = \frac{7}{3}$ is not optimal.

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