42111: Static and Dynamic Optimzation

Introduction

Richard M. Lusby

DTU Management Engineering

People & Places

Primary Contacts

- Static: Associate Professor Richard Lusby
    - rmlu@dtu.dk, office: 424/208
- Static: Assistant Professor Evelien van der Hurk
    - evdh@dtu.dk, office: 424/226
- Dynamic: Associate Professor Niels Kjølstad Poulsen
    - nkpo@dtu.dk, office: 303b/016

Location

All lectures and exercises will take place in B421-A73

Course Objectives

Main Objective

To give a well-founded knowledge, both theoretical and practical, of static and dynamic optimization models for data-based decision making. You will be able to formulate and solve operations research and technical-economic models, and to appreciate the interplay between optimization models and the real-life problems described by these.

My part of the course

- Static Optimization
- Non-Linear Programs
- Theoretical properties, Formulations, Solution Methods

Course Objectives

Main Objective

To give a well-founded knowledge, both theoretical and practical, of static and dynamic optimization models for data-based decision making. You will be able to formulate and solve operations research and technical-economic models, and to appreciate the interplay between optimization models and the real-life problems described by these.

My part of the course

- Static Optimization
- Non-Linear Programs
- Theoretical properties, Formulations, Solution Methods

Learning Objectives (1)

- **Analyze** a given problem in order to **formulate** an optimization model
- **Formulate** and **analyze** models as these are met in static and dynamic optimization.
- **Describe** and **explain** the assumptions underlying models and computations
- **Analyze** an optimization problem in order to identify an appropriate solution method
- **Understand** and - using software - **solve** systems of equations for given optimization problems
- **Interpret** the solutions from a given optimization model

Learning Objectives (2)

- **Describe** and **explain** the mathematical background for the applied solution methods.
- **Perform** sensitivity analysis as part of the evaluating of what-if scenarios in decision making.
- **Make use** of the possibilities for sensitivity analysis in standard optimization software.

Course Evaluation

- Two written projects (one for static, one for dynamic)
- To be completed individually
- If you collaborate with another student, this must be documented
- Your final mark will be based on the evaluation of the two projects
- Reports must be submitted electronically via CampusNet
  - Static report due: Friday, October 13th
  - Dynamic report due: Friday, December 22nd
- Project one will be distributed next week

Calendar - Lecture Plan

Date	Topic
04 September	Introduction
11 September	Linear programming and duality
18 September	Convexity and optimality
25 September	Optimization with (in)equality constraints
02 October	More on Lagrange
09 October	(Approximate) Solution Algorithms for Non-linear problems

Recommended Reading Material:

- HL: Hillier and Lieberman: Introduction to Operations Research, McGraw- Hill 2010 (or earlier)
- BS: Bazaraa and Shetty: Non-linear Programming - Theory and Algorithms, Wiley 1979 (or later)
- BT: Brinkhuis and Tikhomirov Optimization: Insights and Applications, Princeton University Press 2005

Mathematical Programming

- Components:
  - Decision variables
  - Objective function
  - Constraints
- Model types:
  - Linear Programming
  - Integer Programming
  - Non-linear Programming
- Today's Agenda:
  - Linear programming review
  - Non-linear programming intro
  - Chapter 12 Hillier & Liberman 9^{th} Edition

Linear Programming (1)

Example

The WYNDOR GLASS CO. produces high quality glass products, including windows and glass doors. It has 3 plants. Aluminium frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products. Management is introducing the following two products (in batches of 20):

- Product 1: An 8-foot glass door with an aluminum frame (ppb = $3,000)
- Product 2: A 4×6 foot double hung wood-framed window (ppb = $5,000)

Data

	Production Time Required (hrs)		
Plant	Product 1	Product 2	Hours avail.
1	1	0	4
2	0	2	12
3	3	2	18

Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +5x_2 & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & & 2x_2 & \leq 12 \\ & & 3x_1 & +2x_2 & \leq 18 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +5x_2 & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & & 2x_2 & \leq 12 \\ & & 3x_1 & +2x_2 & \leq 18 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & {\color{red}Z =} & {\color{red}3x_1} & {\color{red}+\,5x_2} & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & & 2x_2 & \leq 12 \\ & & 3x_1 & +\,2x_2 & \leq 18 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +5x_2 & \\ \text{subject to:} & & \color{red}{x_1} & & \color{red}{\leq 4} \\ & & & 2x_2 & \leq 12 \\ & & 3x_1 & +2x_2 & \leq 18 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming (2)

$$
\left(
\begin{array}{rrrrl}
\text{maximize} & Z = & 3x_1 & +\,5x_2 & \\
\text{subject to:} & & x_1 & & \leq 4 \\
 & & & \color{red}{2x_2} & \color{red}{\leq 12} \\
 & & 3x_1 & +\,2x_2 & \leq 18 \\
 & & x_1 & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}
\right)
$$

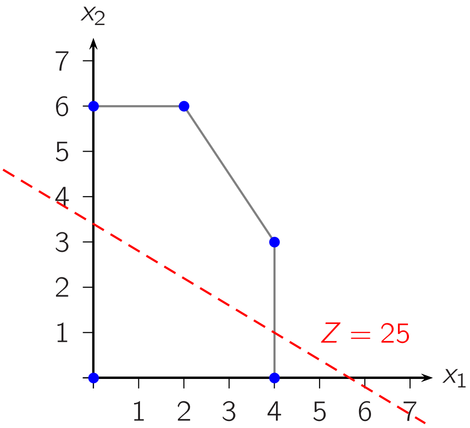
Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +\,5x_2 & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & & 2x_2 & \leq 12 \\ & & \color{red}{3x_1} & \color{red}{+\,2x_2} & \color{red}{\leq 18} \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming (2)

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +5x_2 & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & & 2x_2 & \leq 12 \\ & & 3x_1 & +2x_2 & \leq 18 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Linear Programming



Linear Programming

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to:} \quad & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Linear Programming

- Consider the polyhedron $\mathcal{X} = A\boldsymbol{x} \leq \boldsymbol{b}$
- A solution $\boldsymbol{x'}$ is feasible if $A\boldsymbol{x'} \leq \boldsymbol{b}$
- A solution $\boldsymbol{x^*}$ is optimal if $A\boldsymbol{x^*} \leq \boldsymbol{b}, \boldsymbol{c^T}\boldsymbol{x}^* \geq, \boldsymbol{c^T}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathcal{X}$
- The set of feasible solutions to an LP is termed the feasible region
  - forms a (possibly unbounded) convex set.
- An extreme point of $\mathcal{X}$ cannot be written as a linear combination of other points in $\mathcal{X}$
$$\nexists \lambda \in (0,1) : \boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}, \quad \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{X}, \boldsymbol{x} \neq \boldsymbol{y}, \boldsymbol{x} \neq \boldsymbol{z}$$
- A solution $\boldsymbol{x'}$ is a basic feasible solution if $A\boldsymbol{x}' \leq \boldsymbol{b}$, $\exists B : \boldsymbol{x}' = A_B^{-1}\boldsymbol{b}$
- or, $\boldsymbol{x}'$ is a basic feasible solution if and only if $\boldsymbol{x}'$ is an extreme point

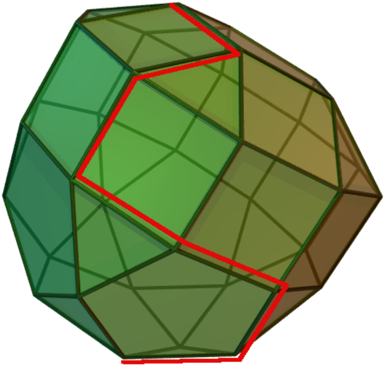
Linear Programming

- Every linear program can be one of : infeasible, unbounded, or have a unique optimal solution value
- Every linear program has an extreme point that is an optimal solution
- A solution approach need only consider extreme points
- A constraint of a linear program is binding at a point x' if the inequality is met with equality at x'. It is nonbinding/slack otherwise.
- What is a dual variable?
- If I have an unbounded polyhedron is my formulation unbounded?
- Can an LP have multiple optimal solutions? What does this mean?
- How do we solve linear programs?

Linear Programming

- Every linear program can be one of : infeasible, unbounded, or have a unique optimal solution value
- Every linear program has an extreme point that is an optimal solution
- A solution approach need only consider extreme points
- A constraint of a linear program is binding at a point x' if the inequality is met with equality at x'. It is nonbinding/slack otherwise.
- What is a dual variable?
- If I have an unbounded polyhedron is my formulation unbounded?
- Can an LP have multiple optimal solutions? What does this mean?
- How do we solve linear programs?

Solution Method



Linear Programming

$$
\begin{array}{rl}
\text{maximize} & \begin{bmatrix} 3 \\ 5 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\text{subject to:} & \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 18 \\ 12 \end{bmatrix} \\
 & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{array}
$$

Non Linear Programming

$$
\begin{aligned}
\text{maximize} \quad & f(\boldsymbol{x}) \\
\text{subject to:} \quad & g_i(\boldsymbol{x}) \leq b_i \quad \forall i \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \leq b_i & \forall i \\ & x_j & \geq 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_j \pi(x_j) & & \\ \text{subject to:} \quad & \textstyle\sum_j a_i(x_j) & \le b_i & \quad \forall i \\ & x_j & \ge 0 & \quad \forall j \end{aligned}$$

max. $f(x)$, s.t. $g(x) \le 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \leq b_i & \forall i \\ & x_j & \geq 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_j \pi(x_j) & & \\ \text{subject to:} \quad & \textstyle\sum_j a_i(x_j) & \leq b_i & \quad \forall i \\ & x_j & \geq 0 & \quad \forall j \end{aligned}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \le b_i & \forall i \\ & x_j & \ge 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \le 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \leq b_i & \forall i \\ & x_j & \geq 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \leq b_i & \forall i \\ & x_j & \geq 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Product-mix problem

- Production: y
- Price required to sell y units: $p(y)$
- Total cost of producing y units: $c(y)$
- Profit:

$$\pi(y) = yp(y) - c(y)$$

- hours needed in department i: $a_i(y)$

Formulation

$$\begin{array}{rlll} \text{maximize} & \sum_j \pi(x_j) & & \\ \text{subject to:} & \sum_j a_i(x_j) & \leq b_i & \forall i \\ & x_j & \geq 0 & \forall j \end{array}$$

max. $f(x)$, s.t. $g(x) \leq 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$\begin{array}{llll} \text{maximize} & \sum_i \sum_j C_{ij}(x_{ij}) & & \\ \text{subject to:} & \sum_j x_{ij} & = s_i & \forall i \\ & \sum_i x_{ij} & = d_j & \forall j \\ & x_j & \geq 0 & \forall i, j \end{array}$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$
\begin{array}{llll}
\text{maximize} & \sum_i \sum_j C_{ij}(x_{ij}) & & \\
\text{subject to:} & \sum_j x_{ij} & = s_i & \forall i \\
 & \sum_i x_{ij} & = d_j & \forall j \\
 & x_j & \geq 0 & \forall i, j
\end{array}
$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$\begin{array}{llll} \text{maximize} & \sum_i \sum_j C_{ij}(x_{ij}) & & \\ \text{subject to:} & \sum_j x_{ij} & = s_i & \forall i \\ & \sum_i x_{ij} & = d_j & \forall j \\ & x_j & \geq 0 & \forall i, j \end{array}$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$\begin{array}{rrll} \text{maximize} & \sum_i \sum_j C_{ij}(x_{ij}) & & \\ \text{subject to:} & \sum_j x_{ij} & = s_i & \forall i \\ & \sum_i x_{ij} & = d_j & \forall j \\ & x_j & \geq 0 & \forall i, j \end{array}$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$
\begin{array}{rrll}
\text{maximize} & \sum_i \sum_j C_{ij}(x_{ij}) & & \\
\text{subject to:} & \sum_j x_{ij} & = s_i & \forall i \\
 & \sum_i x_{ij} & = d_j & \forall j \\
 & x_j & \geq 0 & \forall i, j
\end{array}
$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Transportation problem

- Cost of shipping one extra unit: stair-case constant
- Cost of shipping y units: $c(y)$, piecewise linear
- x_{ij}: The volume shipped from source i to destination j

Formulation

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_i \sum_j C_{ij}(x_{ij}) & & \\ \text{subject to:} \quad & \textstyle\sum_j x_{ij} &= s_i \qquad & \forall i \\ & \textstyle\sum_i x_{ij} &= d_j \qquad & \forall j \\ & x_j &\geq 0 \qquad & \forall i, j \end{aligned}$$

max. $f(x)$, s.t. $g(x) \geq 0$ and $h(x) = 0$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- Share price of stock j: p_j
- Return on one share of stock j is a random variable with mean μ_j and variance σ_{jj}
- Covariance between stock returns of two stocks i and j: σ_{ij}
- Portfolio is to consist of x_j shares of stock $j \quad \forall j$
- Mean of portfolio return:

$$\sum_j \mu_j x_j$$

- Variance of portfolio return:

$$\sum_i \sum_j \sigma_{ij} x_i x_j$$

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$\begin{array}{lll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$\begin{array}{lll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$\begin{array}{lll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$\begin{array}{rll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$\begin{array}{rll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

$$
\begin{array}{rll}
\text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\
\text{subject to:} & \sum_j \mu_j x_j & \geq r \\
 & \sum_j p_j x_j & \leq b
\end{array}
$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Portfolio Selection

- We usually want a high return with a low risk
- Minimum acceptable expected return: r
- Maximum budget available: b

Formulation

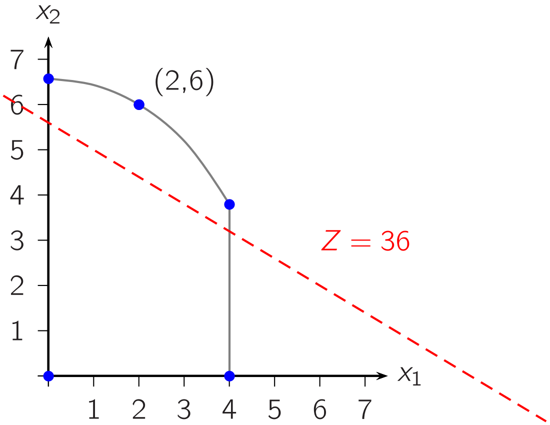
$$\begin{array}{rll} \text{minimize} & \sum_i \sum_j \sigma_{ij} x_i x_j & \\ \text{subject to:} & \sum_j \mu_j x_j & \geq r \\ & \sum_j p_j x_j & \leq b \end{array}$$

- Non-negativity constraints?
- Parametric programming on r yields the efficient frontier

Non-Linear Programming Example 1

$$\left(\begin{array}{rrrrl} \text{maximize} & Z = & 3x_1 & +5x_2 & \\ \text{subject to:} & & x_1 & & \leq 4 \\ & & 9x_1^2 & +5x_2^2 & \leq 216 \\ & & x_1 & & \geq 0 \\ & & & x_2 & \geq 0 \end{array}\right)$$

Graphical Illustration (1)



Graphical Illustration (2)

- The optimal solution remains unchanged
- It is NOT a corner point of the feasible region
- Depending on the objective function a corner point can be optimal
- We cannot limit the search to just corner points

Graphical Illustration (2)

- The optimal solution remains unchanged
- It is NOT a corner point of the feasible region
- Depending on the objective function a corner point can be optimal
- We cannot limit the search to just corner points

Graphical Illustration (2)

- The optimal solution remains unchanged
- It is NOT a corner point of the feasible region
- Depending on the objective function a corner point can be optimal
- We cannot limit the search to just corner points

Graphical Illustration (2)

- The optimal solution remains unchanged
- It is NOT a corner point of the feasible region
- Depending on the objective function a corner point can be optimal
- We cannot limit the search to just corner points

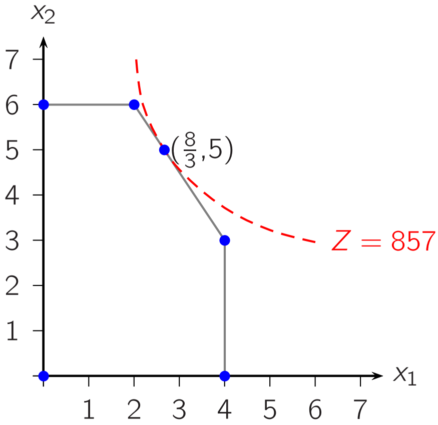
Graphical Illustration (2)

- The optimal solution remains unchanged
- It is NOT a corner point of the feasible region
- Depending on the objective function a corner point can be optimal
- We cannot limit the search to just corner points

Non-Linear Programming Example 2

$$
\left(
\begin{array}{rrrrl}
\text{maximize} & Z = & 126x_1 - 9x_1^2 & +182x_2 - 13x_2^2 & \\
\text{subject to:} & & x_1 & & \leq 4 \\
 & & & 2x_2 & \leq 12 \\
 & & 3x_1 & +2x_2 & \leq 18 \\
 & & x_1 & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}
\right)
$$

Graphical Illustration (1)



Graphical Illustration (2)

- The optimal solution is also not a corner point
- The locus of points with $Z = 857$ intersects the feasible region at this point only
- The locus of points with $Z > 857$ does not intersect the feasible region at all

Graphical Illustration (2)

- The optimal solution is also not a corner point
- The locus of points with $Z = 857$ intersects the feasible region at this point only
- The locus of points with $Z > 857$ does not intersect the feasible region at all

Graphical Illustration (2)

- The optimal solution is also not a corner point
- The locus of points with $Z = 857$ intersects the feasible region at this point only
- The locus of points with $Z > 857$ does not intersect the feasible region at all

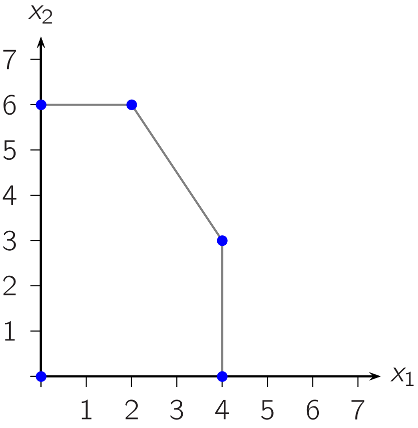
Graphical Illustration (2)

- The optimal solution is also not a corner point
- The locus of points with $Z = 857$ intersects the feasible region at this point only
- The locus of points with $Z > 857$ does not intersect the feasible region at all

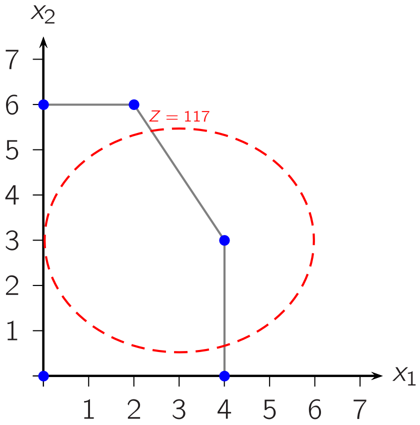
Non-Linear Programming Example 3

$$
\left(\begin{array}{rrrr}
\text{maximize} & Z = & 54x_1 - 9x_1^2 & +78x_2 - 13x_2^2 & \\
\text{subject to:} & & x_1 & & \leq 4 \\
 & & & 2x_2 & \leq 12 \\
 & & 3x_1 & +2x_2 & \leq 18 \\
 & & x_1 & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}\right)
$$

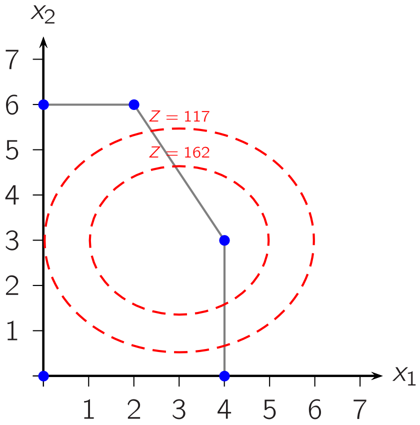
Graphical Illustration (1)



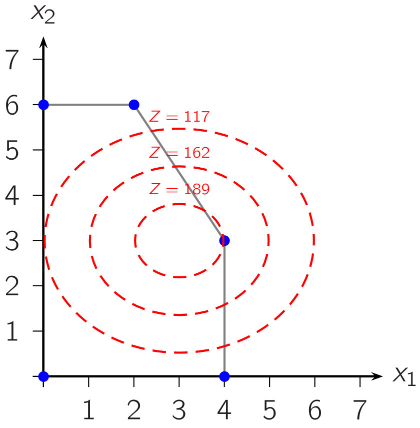
Graphical Illustration (1)



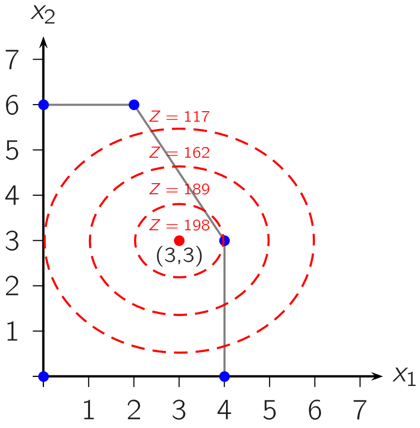
Graphical Illustration (1)



Graphical Illustration (1)



Graphical Illustration (1)



Graphical Illustration (2)

- The optimal solution is an interior point of the feasible region
- Solution to the unconstrained global maximum
- A general solution approach needs to consider all points

Graphical Illustration (2)

- The optimal solution is an interior point of the feasible region
- Solution to the unconstrained global maximum
- A general solution approach needs to consider all points

Graphical Illustration (2)

- The optimal solution is an interior point of the feasible region
- Solution to the unconstrained global maximum
- A general solution approach needs to consider all points

Graphical Illustration (2)

- The optimal solution is an interior point of the feasible region
- Solution to the unconstrained global maximum
- A general solution approach needs to consider all points

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

- Convex Function
- Concave Function
- Convex Set
- *Pseudoconvex Function*
- *Quasiconvex Function*
- Local Optimum
- Global Optimum

Convexity

Convex Programming Problem

$$\begin{aligned} \text{maximize} \quad & f(x) & & \\ \text{subject to:} \quad & g_i(x) & \leq b_i & \qquad \forall i \\ & x & \geq 0 & \end{aligned}$$

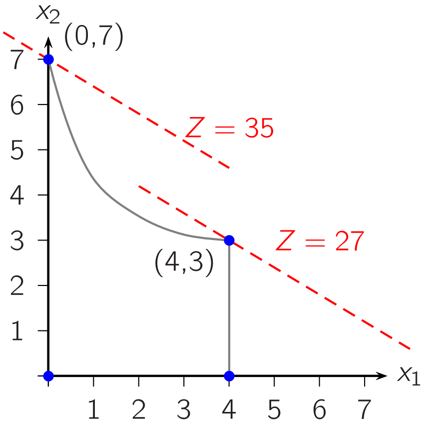
- f is concave, each g_i is convex

$\rightarrow$ a local maximum is also global

Non-Linear Programming Example 4

$$
\left(\begin{array}{rrrrl}
\text{maximize} & Z = & 3x_1 & +5x_2 & \\
\text{subject to:} & & x_1 & & \leq 4 \\
 & & 8x_1 - x_1^2 & +14x_2 - x_2^2 & \leq 49 \\
 & & x_1 & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}\right)
$$

Graphical Illustration (1)



Problem Types

- Unconstrained Optimization
  - No constraints
- Linearly Constrained Optimization
  - linear constraints
- Quadratic Programming
  - quadratic objective function, linear constraints
- Convex Programming
  - minimize a convex function on a convex set
- Separable Programming
  - objective function and constraints are separable i.e. sum of functions of individual decision variables

Class Exercises

Problem 1

$$
\begin{array}{llll}
\text{minimize} & (x_1 - 255)^2 & +10(x_1 - 220) & \\
& +(x_2 - x_1)^2 & +10(x_2 - 240) & \\
& +(x_3 - x_2)^2 & +10(x_3 - 200) & +(255 - x_3)^2 \\
\text{subject to:} & x_1 \geq 220 & x_2 \geq 240 & x_3 \geq 200
\end{array}
$$

Problem 2

$$
\begin{array}{ll}
\text{maximize} & 36x_1 + 9x_1^2 - 6x_1^3 + 36x_2 - 3x_2^3 \\
\text{subject to:} & x_1 + x_2 \leq 3, x_1 \geq 0, x_2 \geq 0
\end{array}
$$

Richard M. Lusby
DTU Management Engineering, Technical University of Denmark

Building 424, Room 208
2800 Kgs. Lyngby, Denmark
http://www.man.dtu.dk

rmlu@dtu.dk
phone +45 4525 3084