Linear Programming and Duality

Simplex method
Matrix formulation
Duality

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Simplex method

a. Convert $\leq$ inequalities by adding slack variables
b. Put data into a Simplex tableau
c. Do Simplex iterations by pivoting

Entering variable, pivot column:
Most negative coefficient in top row

Leaving variable, pivot row:
Smallest ratio between right hand sides and positive coefficients in pivot column

(We disregard complications here:
Phase 1, no feasible solution, unbounded solutions)

First and final tableau

Z	x_1	x_2	x_3	x_4	x_5	
1	-3	-5				0
	1	0	1			4
	0	2		1		12
	3	2			1	18
1				$3/2$	1	36
			1	$1/3$	$-1/3$	2
		1		$1/2$	0	6
	1			$-1/3$	$1/3$	2

Matrix formulation

$\max\ cx$ s.t. $Ax \leq b$, $x \geq 0$

$\max\ cx + 0x_s$ s.t. $Ax + Ix_s = b$, $x \geq 0,\ x_s \geq 0$

In each tableau each variable in x, x_s is designated as a basic variable or a nonbasic variable
The tableau represents the equation system solved with respect to the basic variables
The basis matrix B is formed by the columns in the first tableau of the current basic variables
The inverse basis matrix appears under the slack variables in each tableau

After re-arranging the variables:
$\max\ c_B x_B + c_N x_N$ s.t. $Bx_B + Nx_N = b$
$x_B \geq 0,\ x_N \geq 0$

First and later tableau

Z	x	x_s	
1	$-c$	0	0
	A	I	b
1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
	$B^{-1} A$	B^{-1}	$B^{-1} b$

The current solution is
$x_B = B^{-1} b,\ x_N = 0,\ Z = c_B B^{-1} b$

At optimality we have
$c_B B^{-1} \geq 0$, $c_B B^{-1} A \geq c$

$c_B B^{-1}$ are the shadow prices

Duality

The primal problem

$$\begin{array}{rrcrcl} \text{max.} & 3x_1 & + & 5x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 4 \\ & & & 2x_2 & \leq & 12 \\ & 3x_1 & + & 2x_2 & \leq & 18 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

The corresponding dual problem

$$\begin{array}{rrcrcrcl} \text{min.} & 4y_1 & + & 12y_2 & + & 18y_3 & & \\ \text{s.t.} & y_1 & & & + & 3y_3 & \geq & 3 \\ & & & 2y_2 & + & 2y_3 & \geq & 5 \\ & y_1, & & y_2, & & y_3 & \geq & 0 \end{array}$$

Weak duality theorem

Primal:
max. cx
s.t. $Ax \leq b,\ x \geq 0$

Dual:
min. yb
s.t. $yA \geq c,\ y \geq 0$
(note that y is a row vector)

If x is primal feasible and y is dual feasible, then
$cx \leq yAx \leq yb$

Proof?

Strong duality theorem

If one of the problems has an optimal solution,
the other one also has an optimal solution,
and the optimal objective function values are equal

The optimal dual solution appears in the optimal primal tableau, under the slack variables

Proof?

The two other possibilities are
One problem is infeasible, the other is unbounded
Both problems are infeasible

Complementary Slackness

Primal: max. cx s.t. $Ax + x_s = b$; $x, x_s \geq 0$

Dual: min. yb s.t. $yA - y_s = c$; $y, y_s \geq 0$

Correspondances: x_1 and y_4, x_2 and y_5
y_1 and x_3, y_2 and x_4, y_3 and x_5

Definition: A primal solution and a dual solution exhibit complementary slackness if $y_s x = 0$ and $y x_s = 0$, i.e., corresponding x- and y-values are not both positive

Theorem: A primal solution and a dual solution are optimal iff they are feasible and complementary

Proof?

Class exercises 1

Each unit of product 1 requires 1 hour in department A and 1 hour in department B, and yields a profit of 1. The corresponding numbers for product 2 are 1 and 3, and 2. There are 3 and 7 hours available in departments A and B, respectively.

Formulate an LP model and set up the first tableau. Write the dual problem.

Class exercises 2

The final tableau is

Z	x_1	x_2	x_3	x_4	
1			$1/2$	$1/2$	5
	1		$3/2$	$-1/2$	1
		1	$-1/2$	$1/2$	2

Read off the optimal solution and the dual solution.
Read off B^{-1} and verify that $B^{-1}B = I$.
Given the primal solution, find the dual solution using complementary slackness.
Use complementary slackness to show that $x_1 = 0$, $x_2 = 7/3$ is not optimal.