#### **Linear Programming and Duality**

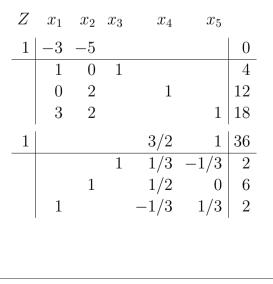
Simplex method Matrix formulation Duality

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# First and final tableau



## **Simplex method**

a. Convert  $\leq$  inequalities by adding slack variables

b. Put data into a Simplex tableau

c. Do Simplex iterations by pivoting

Entering variable, pivot column: Most negative coefficient in top row

Leaving variable, pivot row: Smallest ratio between right hand sides and positive coefficients in pivot column

(We disregard complications here: Phase 1, no feasible solution, unbounded solutions)

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### **Matrix formulation**

max cx s.t.  $Ax \leq b, x \geq 0$ 

max  $cx + 0x_s$  s.t.  $Ax + Ix_s = b, x \ge 0, x_s \ge 0$ 

In each tableau each variable in  $x, x_s$  is designated as a basic variable or a nonbasic variable The tableau represents the equation system solved with respect to the basic variables The basis matrix B is formed by the columns in the first tableau of the current basic variables The inverse basis matrix appears under the slack variables in each tableau

After re-arranging the variables: max  $c_B x_B + c_N x_N$  s.t.  $B x_B + N x_N = b$  $x_B \ge 0, x_N \ge 0$ 

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#### First and later tableau

Ζ	x	$x_s$	
1	-c	0	0
	A	Ι	b
1	$c_B B^{-1} A - c$		$c_B B^{-1} b$
	$B^{-1}A$	$B^{-1}$	$B^{-1}b$

The current solution is  $x_B = B^{-1}b, x_N = 0, Z = c_B B^{-1}b$ At optimality we have  $c_B B^{-1} > 0, c_B B^{-1}A > c$ 

 $c_B B^{-1}$  are the shadow prices

#### Weak duality theorem

Primal: max. cxs.t.  $Ax \le b, x \ge 0$ Dual: min. ybs.t.  $yA \ge c, y \ge 0$ 

(note that y is a row vector)

If x is primal feasible and y is dual feasible, then  $cx \le yAx \le yb$ Proof? Duality

The primal problem

max.  $3x_1 + 5x_2$ s.t.  $x_1 \leq 4$   $2x_2 \leq 12$   $3x_1 + 2x_2 \leq 18$  $x_1, \quad x_2 \geq 0$ 

The corresponding dual problem

min.  $4y_1 + 12y_2 + 18y_3$ s.t.  $y_1 + 3y_3 \ge 3$  $2y_2 + 2y_3 \ge 5$  $y_1, y_2, y_3 \ge 0$ 

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#### **Strong duality theorem**

If one of the problems has an optimal solution, the other one also has an optimal solution, and the optimal objective function values are equal

The optimal dual solution appears in the optimal primal tableau, under the slack variables

Proof?

The two other possibilities are One problem is infeasible, the other is unbounded Both problems are infeasible

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### **Complementary Slackness**

Primal: max. cx s.t.  $Ax + x_s = b$ ;  $x, x_s \ge 0$ 

Dual: min. yb s.t.  $yA - y_s = c; y, y_s \ge 0$ 

Correspondences:  $x_1$  and  $y_4$ ,  $x_2$  and  $y_5$  $y_1$  and  $x_3$ ,  $y_2$  and  $x_4$ ,  $y_3$  and  $x_5$ 

Definition: A primal solution and a dual solution exhibit complementary slackness if  $y_s x = 0$  and  $yx_s = 0$ , i.e., corresponding x- and y-values are not both positive

Theorem: A primal solution and a dual solution are optimal iff they are feasible and complementary

Proof?

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## **Class exercises 1**

Each unit of product 1 requires 1 hour in department A and 1 hour in department B, and yields a profit of 1. The corresponding numbers for product 2 are 1 and 3, and 2. There are 3 and 7 hours available in departments A and B, respectively.

Formulate an LP model and set up the first tableau. Write the dual problem.

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## **Class exercises 2**

The final tableau is

Z	$x_1$	$x_2$	$x_3$	$x_4$	
1				1/2	
	1		3/2	$-1/2 \\ 1/2$	1
		1	-1/2	1/2	2

Read off the optimal solution and the dual solution. Read off  $B^{-1}$  and verify that  $B^{-1}B = I$ . Given the primal solution, find the dual solution using complementary slackness. Use complementary slackness to show that  $x_1 = 0$ ,  $x_2 = 7/3$  is not optimal.

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