02465: Introduction to reinforcement learning and control

Bellmans equations and exact planning

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Lecture Schedule

Dynamical programming

1. The finite-horizon decision problem
   7 February
2. Dynamical Programming
   14 February
3. DP reformulations and introduction to Control
   21 February

Control

4. Discretization and PID control
   28 February
5. Direct methods and control by optimization
   7 March
6. Linear-quadratic problems in control
   14 March
7. Linearization and iterative LQR
   21 March

Reinforcement learning

8. Exploration and Bandits
   28 March
9. **Bellmans equations and exact planning**
   4 April
10. Monte-carlo methods and TD learning
    11 April
11. Model-Free Control with tabular and linear methods
    25 April
12. Eligibility traces
    2 May
13. Deep-Q learning
    9 May

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Reading material:

- [SB18, Chapter 3; 4]

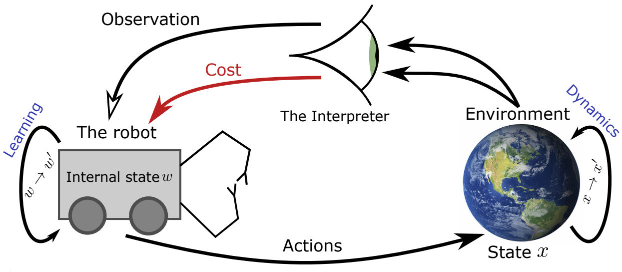
Learning Objectives

- Markov decision process
- Value/action value function and other tools
- Dynamical programming for policy evaluation and control

Housekeeping

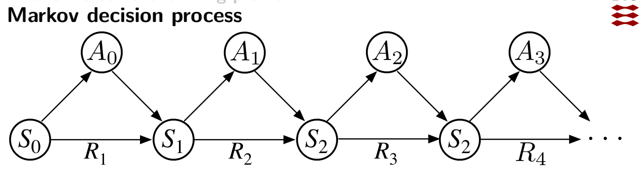
- Feedback on project 2 in about 2 weeks
- Project 3 is online
- Due to a combination of illness+baby I might have opened but not answered some emails. Please contact me again if I do not respond timely.

Today: Dynamical programming...again!



- Last time: Exploration and exploitation (**+No effects**)
- This time: Value functions and recursions (**+Known dynamics**)
- Next time: The full reinforcement-learning problem

Markov decision process



- Agent/system interacts at times $t = 0, 1, 2, \dots$
  - Agent observes state $S_t \in \mathcal{S}$
  - Agent takes action $A_t \in \mathcal{A}(S_t)$
  - Agent obtains a reward $R_{t+1} \in \mathbb{R}$; time increments to $t+1$
- Dynamics described using conditional probabilities

$$p(s', r|s, a) = \Pr\{S_{t+1} = s', R_{t+1} = r \mid S_t = s, A_t = a\}$$
$$= \Pr\{w \mid \text{s.t. } s' = f_t(s, a, w) \text{ and } r = -g_t(s, a, w)\}$$

- If the environments stops we call it **episodic**

`unf_gridworld.py`

Markov decision process (MDP)

Assumptions in a Markov Decision Process

- $\mathcal{S}, \mathcal{A}(s)$ are finite
- Markov property

$$\Pr\{S_{t+1}, R_{t+1} \mid S_t, A_t\} = \Pr\{S_{t+1}, R_{t+1} \mid S_0, A_0, \dots, S_t, A_t\}$$

- The **transition probabilities** are **stationary** (time-independent)

$$p(s_{t+1}, r_{t+1}|s_t, a_t) = p(s_{t'+1}, r_{t'+1}|s_{t'}, a_{t'})$$

Markov decision process (MDP)

Markov Decision Process - practically speaking

- A function that says which actions are available in a given state $\mathcal{A}(s)$
- The transition probability $p(s', r|s, a)$
- The initial state s_0
- A function which determines
  - if a state is **non-terminal,** $s_t \in \mathcal{S}$
  - or **terminal,** $s_T \notin \mathcal{S}$
- $\mathcal{S}, \mathcal{A}(s)$ are finite

An episode is $S_0, A_0, R_1, S_1, A_1, R_2, \dots, S_{T-1}, A_{T-1}, R_T, S_T$

Policy

Policy

A **policy** is a distribution over actions

$$\pi(a|s) = \Pr\{A_t = a \mid S_t = s\}$$

- Policy is time-independent
- Now a **Distribution** rather than **function** $a = \pi(s)$ because we want to **explore**

Return and discount

Return

For $0 \leq \gamma \leq 1$ and any t we define the accumulated γ-discounted return

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- Equivalent to:

$$\lim_{N \to \infty} \left[\gamma^N g_N(x_N) + \sum_{k=0}^{N} \gamma^k g_k(s_k, a_k, w_k) \right]$$

- **Fancy rationale for $\gamma < 1$:**
  - Don't worry about the far and uncertain future
- **Actual rationale for $\gamma < 1$:**
  - Avoids infinities when $\gamma = 1$; simpler convergence theory
- **tl;dr:** Use $\gamma > 0.9$ unless you have good reasons not to.

Value and action-value function

The **state-value function** $v_\pi(s)$ is the expected return starting in s and assuming actions are selected using π:

$$v_\pi(s) = \mathbb{E}_\pi[G_t|S_t = s], \quad A_t \sim \pi(\cdot|S_t)$$

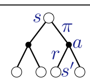
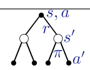
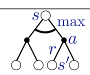
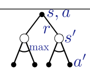
The **action-value function** $q_\pi(s, a)$ is the expected return starting in s, taking action a, and then follow π:

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t|S_t = s, A_t = a]$$

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots$$

Note that $J_\pi(s) = -v_\pi(s)$

Where we want to end up

Bellman equation	Learning algorithm	
Bellman expectation equation for v_π $v_\pi(s) = \mathbb{E}_\pi[R + \gamma v_\pi(S')\|s]$	**Iterative policy evaluation** to learn v_π $V(s) \leftarrow \mathbb{E}_\pi[R + \gamma V(S')\|s]$	
Bellman expectation equation for q_π $q_\pi(s,a) = \mathbb{E}_\pi[R + \gamma q_\pi(S', A')\|s, a]$	**Iterative policy evaluation** to learn q_π $Q(s,a) \leftarrow \mathbb{E}_\pi[R + \gamma Q(S', A')\|s, a]$	
Policy iteration: Use policy evaluation to estimate v_π or q_π Improve by acting greedily: $\pi'(s) \leftarrow \arg\max_a q_\pi(s,a)$		
Bellman optimality equation for v_* $v_*(s) = \max_a \mathbb{E}[R + \gamma v_*(S')\|s, a]$	**Value iteration** $V(s) \leftarrow \max_a \mathbb{E}[R + \gamma V(S')\|s, a]$	
Bellman optimality equation for q_* $q_*(s,a) = \mathbb{E}[R + \gamma \max_{a'} q_*(S', a')\|s, a]$	**Q-value iteration** $Q(s,a) \leftarrow \mathbb{E}[R + \gamma \max_{a'} Q(S', a')\|s, a]$	

Fundamental properties of value function

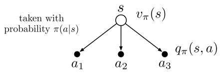
Fundamental properties of value/action-value functions

- Fundamental recursion

$$G_t = R_{t+1} + \gamma G_{t+1}$$

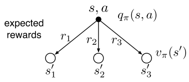
- Action-value to value function

$$v_\pi(s) = \mathbb{E}_{a\sim\pi(s)}\left[q_\pi(s,a)\right]$$



- value-function to action-value

$$q_\pi(s,a) = \mathbb{E}\left[R_{t+1} + \gamma v_\pi\left(S_{t+1}\right)|S_t = s, A_t = a\right] \quad (1)$$



 $v_\pi(s_t) = \mathbb{E}[R_{t+1} + \gamma G_{t+1}|s] = \mathbb{E}\left[R_{t+1} + \gamma \underbrace{\mathbb{E}\left[G_{t+1}|s_{t+1}\right]}_{=v_\pi(S_{t+1})}|s\right]$

Two first two Bellman equations

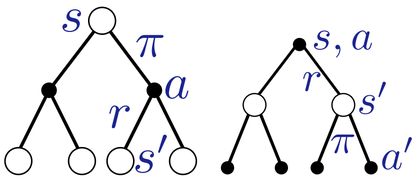
Bellman equations

- Recursive decomposition of value function. $V: \mathcal{S} \mapsto \mathbb{R}$ **initialized randomly**

$$v_\pi(s)V(s) = \leftarrow \mathbb{E}\left[R_{t+1} + \gamma v_\pi V\left(S_{t+1}\right)|S_t = s\right]$$

- Recursive decomposition of action-value function **(Q initialized randomly)**

$$q_\pi(s,a) = Q(s,a) \leftarrow \mathbb{E}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1})Q(S_{t+1}, A_{t+1})|S_t = s, A_t = a\right]$$



`unf_policy_evalution_stepwise_gridworld.py`

Task 1: Evaluate a policy

Iterative policy evaluation

- Given a policy π, initialize V randomly. For all s perform updates:

$$V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s,a)\left[r + \gamma V(s')\right]$$

until terminal condition is met. $V(s)$ will converge to $v_\pi(s)$

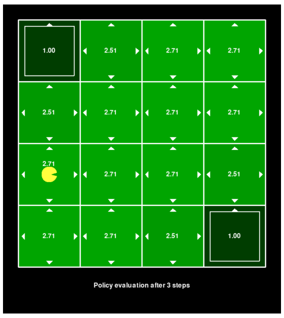
- Initialize Q randomly. For all s, a perform updates:

$$Q(s,a) \leftarrow \sum_{s',r} p(s', r|s,a)\left[r + \gamma \sum_{a'} \pi(a'|s')Q(s',a')\right]$$

until terminal condition is met. Q will converge to q_π

`unf_policy_improvement_gridworld.py`

Quiz: Policy evaluation



The environment has a living reward of $R = 1$ and if it moves into the wall it stays in the current state.

The value function v_π for the policy $\pi(a|s) = \frac{1}{4}$ is is estimated using Policy Evaluation with $\gamma = 0.9$. What is the value function in the state indicated by Pacman in the next step?

a. 3.41

b. 3.39

c. 3.31

d. 3.28

e. Don't know.

Optimal value function

The optimal state-value function v_* is the maximum value function over all policies

$$v_*(s) = \max_\pi v_\pi(s)$$

The optimal action-value function q_* is the maximum action-value function over all policies

$$q_*(s,a) = \max_\pi q_\pi(s,a)$$

We define a partial ordering over policies as

$$\pi \geq \pi' \text{ if for all } s\text{: } v_\pi(s) \geq v_{\pi'}(s)$$

Value/action value to policy

- Given any function $q: \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$ we can define the **greedy policy** π' **wrt.** q

$$\pi'(s) = \arg\max_a q(s,a)$$

- Given any function $v: \mathcal{S} \mapsto \mathbb{R}$ we can define **greedy policy** π' **wrt.** v

$$\pi'(s) = \arg\max_a \mathbb{E}_{s',r}\left[r + \gamma v(s')|s,a\right]$$

Policy improvement theorem

Policy improvement theorem

Let π and π' be any pair of deterministic policies such that for all $s \in \mathcal{S}$:

$$q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad (2)$$

Then $\pi' \geq \pi$ meaning for all $s \in \mathcal{S}$

$$v_{\pi'}(s) \geq v_\pi(s)$$

Inequality is strict if any inequality in eq. (2) is strict.

Idea

Given v_π, define new policy π' to be greedy with respect to v_π. Then:

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_{a\sim\pi(s)}\left[q_\pi(s,a)\right] \\ &\leq \max_a q_\pi(s,a), \quad \text{True by simple properties of expectations} \\ &= q_\pi(s, a^*), \quad a^* = \arg\max_a q_\pi(s,a) \\ &= q_\pi(s, \pi'(s)), \quad \pi' \text{ greedy policy wrt. } v_\pi \end{aligned}$$

Observations:

- Being greedy wrt. π means $\pi' \geq \pi$ by the policy-improvement theorem

Quiz: Optimal action-value function (Exam spring 2023)

Let v_*, q_* be the optimal value and action-value functions of an MDP, let π be any policy and finally let v_π and q_π be the value/action-value function associated with π. Which one of the following statements are true in general?

a. $\max_s q_*(s,a) = v_*(a)$

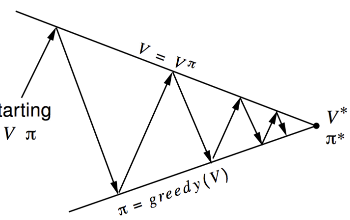
b. There is a policy π, a state s and an action a so that $q_*(s,a) < q_\pi(s,a)$

c. For all π and a it is true that $q_*(s,a) > q_\pi(s,a)$

d. There is a policy π and state s so that $\max_a q_*(s,a) = v_\pi(s)$

e. Don't know.

Policy iteration



- Given initial policy π
- Compute v_π using policy evaluation
- Let π' be greedy policy wrt. v_π
- Repeat until $v_\pi = v_{\pi'}$

`lecture_09_policy_improvement.py`

Policy iteration algorithm

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$
2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))\left[r + \gamma V(s')\right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$
3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma V(s')\right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

- In each step, the PI theorem guarantees that $\pi \leq \pi'$
- There is a limited number of policies so improvement cannot continue
- If $\pi = \pi'$, then the policy is in fact optimal
  - (it satisfy the Bellman optimality equation as we will see in a moment)

Bellmans optimality equations

Suppose π_* is the policy corresponding to the optimal value function $v_*(s)$

$$\begin{aligned} v_*(s) &= \max_a q_{\pi_*}(s,a) \\ &= \max_a \mathbb{E}\left[R + v_{\pi_*}(S')|s,a\right] \end{aligned}$$

Bellmans optimality equations

- Recursion of optimal value function v_*: **Given any V**

$$v_*(s) = V(s) \leftarrow \max_a \mathbb{E}\left[R_{t+1} + \gamma v_*(S_{t+1}) V(S_{t+1}) | S_t = s, A_t = a\right] \quad (3)$$

- Recursion of optimal action-value function q_*:

$$q_*(s,a) = \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') | S_t = s, A_t = a\right] \quad (4)$$

- **Theorem:** v_* (or q_*) satisfies the above recursions if (and only if) they corresponds to the optimal value function

Value Iteration

Bellmans optimality equations Value Iteration

- Recursion of optimal value function v_*: **Given any V**

$$v_*(s) = V(s) \leftarrow \max_a \mathbb{E}\left[R_{t+1} + \gamma v_*(S_{t+1}) V(S_{t+1}) | S_t = s, A_t = a\right] \quad (5)$$

- Recursion of optimal action-value function q_*: **Given any Q**

$$q_*(s,a) = Q(s,a) \leftarrow \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, A'_{t+1}) Q(S_{t+1}, A_{t+1}) | S_t = s, A_t = a\right] \quad (6)$$

- **Theorem:** VI converge to optimal v_* (or q_*)

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$
Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$
Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$

`lecture_09_vi_v.py`

Dimitri P Bertsekas and Huizhen Yu.
Distributed asynchronous policy iteration in dynamic programming.
In *2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1368–1375. IEEE, 2010.

Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning: An Introduction.
The MIT Press, second edition, 2018.
(Freely available online).

Note from lecture 3: Stationary problem = stationary policy

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}\left(f_k(x_k, u_k, w_k)\right) + g_k\left(x_k, u_k, w_k\right)\right]$$

Assume the problem is independent of k:

$$J_k(x) = \min_u \mathbb{E}\left[J_{k+1}\left(f(x,u,w)\right) + g\left(x,u,w\right)\right]$$

- It will be true that $J_0 \approx J_1 \approx J_2$ etc.
- Policies will be the same initially $\pi_0 \approx \pi_1$ etc.

In fact just iterate to convergence:

$$J(x) \leftarrow \min_u \mathbb{E}\left[J\left(f(x,u,w)\right) + g\left(x,u,w\right)\right]$$

This is in fact value iteration

Note from lecture 3: Action-value formulation

$$J_k(x_k) = \min_{u_k} \mathbb{E}[J_{k+1}(f_k(x_k,u_k,w_k)) + g_k(x_k,u_k,w_k)]$$

We want to remove the green part

$$J_k(x_k) = \min_{u_k} Q(x_k, u_k)$$
$$Q(x_k, u_k) = \mathbb{E}[\underbrace{J_{k+1}(f_k(x_k,u_k,w_k))}_{=\min_{u_{k+1}} Q(x_{k+1}, u_{k+1})} + g_k(x_k,u_k,w_k)]$$

Substituting, the entire equation becomes red:

$$Q(x_k,u_k) = \mathbb{E}\left[\min_{u_{k+1}} Q\left(f_k(x_k,u_k,w_k), u_{k+1}\right) + g_k\left(x_k,u_k,w_k\right)\right]$$

- Simply VI for Q-functions!

Asynchronous updates

- In **synchronous updates**, we do
  - For each $s \in \mathcal{S}$ compute:

$$v'_\pi(s) \leftarrow \mathbb{E}_\pi[R + \gamma v_\pi(S')|s]$$

  - When done, set $v_\pi \leftarrow v'_\pi$
- In **asynchronous updates**, we re-use the updated values within one sweep
  - For each $s \in \mathcal{S}$ compute:

$$v_\pi(s) \leftarrow \mathbb{E}_\pi[R + \gamma v_\pi(S')|s]$$

Both converge: You implement the **asynchronous version**, but most analysis is done in the **synchronous version**. It is also possible to structure sweeps for efficiency (see [BY10])

Convergence results

We will focus on the value function as the action-value results are very similar. First we define the operators $\mathcal{T}$ and $\mathcal{T}_\pi$:

$$(\mathcal{T}_\pi v)(s) = \mathbb{E}_\pi\left[R + \gamma v(S')|s\right] \quad (7)$$
$$(\mathcal{T} v)(s) = \max_a \mathbb{E}\left[R + \gamma v(S')|s,a\right] \quad (8)$$

If the state space is discrete $\mathcal{S} = \{s_1, \dots, s_N\}$ we can define the vector

$$v_i = v(s_i)$$

then the operators act on these vectors $\mathcal{T} : \mathbb{R}^N \to \mathbb{R}^N$

Fixed-point theorem

Let $T : A \mapsto A$ be a function and $A \subset \mathbb{R}^n$ a compact subset of $\mathbb{R}^n$. Then if for all $\boldsymbol{x}, \boldsymbol{z} \in A$

$$\|T(\boldsymbol{x}) - T(\boldsymbol{z})\| \leq \gamma \|\boldsymbol{x} - \boldsymbol{z}\|, \quad 0 \leq \gamma < 1$$

then repeatedly applying T to any $\boldsymbol{x}$ will converge to a single, unique fixed point $\boldsymbol{x}^* = T(\boldsymbol{x}^*)$

Asynchronous updates

- In synchronous updates, we iterate for all $s \in \mathcal{S}$:

$$v'_\pi(s) \leftarrow \mathbb{E}_\pi[R + \gamma v_\pi(S')|s]$$

  then $v_\pi \leftarrow v'_\pi$
- In synchronous updates, we re-use the updated values within one sweep

$$v_\pi(s) \leftarrow \mathbb{E}_\pi[R + \gamma v_\pi(S')|s]$$

Both converge. It is also possible to structure sweeps for efficiency (see [BY10])

Existence of solutions to Bellmans equations

- Both the operators $\mathcal{T}$ and $\mathcal{T}_\pi$ are contractions in the max-norm $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$. Example:

$$\|\mathcal{T}_\pi \boldsymbol{v} - \mathcal{T}_\pi \boldsymbol{w}\|_\infty = \max_i |\mathbb{E}_\pi [R + \gamma v(S')|s_i] - \mathbb{E}_\pi [R + \gamma w(S')|s_i]| \quad (9)$$

$$= \max_i \left| \sum_{s'} p(s'|s_i, a) (\gamma v(s') - \gamma w(s')) \right| \quad (10)$$

$$\leq \gamma \max_i \sum_{s'} p(s'|s_i, a) |v(s') - w(s')| \quad (11)$$

$$\leq \gamma \max_i \sum_{s'} p(s'|s_i, a) \|\boldsymbol{v} - \boldsymbol{w}\|_\infty = \gamma \|\boldsymbol{v} - \boldsymbol{w}\|_\infty \quad (12)$$

- Consequence: Repeatedly applying Bellmans operators will lead to a single, fixed point policy $\mathcal{T}\boldsymbol{v}_* = \boldsymbol{v}_*$ and $\mathcal{T}_\pi \boldsymbol{v}_\pi = \boldsymbol{v}_\pi$
- Therefore, PE/PI converge to v_π. VI also converges, but does it converge to the maximum?

VI and maximum

- We know: Value iteration converge to a unique fixed point

$$\boldsymbol{v}_* = (\mathcal{T}\mathcal{T} \cdots \mathcal{T})(\boldsymbol{v})$$

- Maximum value function is defined as

$$\tilde{v}(s) = \max_\pi v_\pi(s)$$

- It could be the case that $\tilde{v}(s) = v_\pi(s)$, $\tilde{v}(s') = v_{\pi'}(s')$, and neither was equal to $v_*(s), v_*(s')$

Value iteration solution corresponds to a policy

Show that $v_*(s) \leq \tilde{v}(s)$

- Value iteration gives us v_* as a fixed point
- From v_* we can construct the action-values

$$q_*(s, a) = \mathbb{E}[R + \gamma v_*(S')|s, a]$$

- From these we can define the greedy policy π_*

$$\pi_*(s) = \arg\max_a q_*(s, a)$$

- By definition now $v_*(s) = (T v_*)(s) = (\mathcal{T}_{\pi^*} v)(s)$
- Therefore v_* is the value function of the policy π_*, and so $v_*(s) \leq \tilde{v}(s)$ for all s

Value iteration is optimal

Show that $v_*(s) \geq \tilde{v}(s)$

- Assume $v_*(s) < \tilde{v}_\pi(s)$ for a specific s, π
- Let π_1 be the greedy policy according to $\tilde{v}_\pi$. We know that

$$\tilde{v}_\pi \leq v_{\pi_1}$$

  by the policy improvement theorem
- Therefore, $v_*(s) < \tilde{v}_\pi(s) \leq v_{\pi_1}(s)$
- Repeat again to obtain π_2 and notice we are doing policy iteration
- Since we are doing policy iteration eventually $\pi_k \to \pi_\infty$
- It must be the case v_{π_∞} is a fixed-point of $\mathcal{T}$, otherwise by the policy improvement theorem we could select a better (greedy) policy
- Since the fixed point is unique, $v_{\pi_\infty} = v_*$, which is a contradiction