02465: Introduction to reinforcement learning and control

Direct methods and control by optimization

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Lecture Schedule

Dynamical programming

1. The finite-horizon decision problem
   7 February
2. Dynamical Programming
   14 February
3. DP reformulations and introduction to Control
   21 February

Control

4. Discretization and PID control
   28 February
5. **Direct methods and control by optimization**
   7 March
6. Linear-quadratic problems in control
   14 March
7. Linearization and iterative LQR
   21 March

Reinforcement learning

8. Exploration and Bandits
   28 March
9. Bellmans equations and exact planning
   4 April
10. Monte-carlo methods and TD learning
    11 April
11. Model-Free Control with tabular and linear methods
    25 April
12. Eligibility traces
    2 May
13. Deep-Q learning
    9 May

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Reading material:

- [Her25, Chapter 15]

Learning Objectives

- Direct methods for optimal control
- Trajectory planning for linear-quadratic problems using optimization
- Trajectory planning using trapezoidal collocation

Project part 1

- Great job! Part 2 is online
- Survey on course experience on DTU Learn
- A TA caught a minor issue with $N \to N-1$ in the beginning of todays chapter; new version online. Exercise+slides+algorithm not affected.

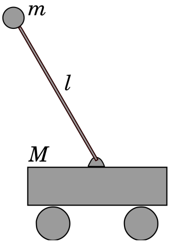
Dynamics

Dynamics of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

- $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a complete description of the system at t
- $\boldsymbol{u}(t) \in \mathbb{R}^d$ are the controls applied to the system at t
- The time t belongs to an interval $[t_0, t_F]$ of interest

Example: Cartpole



- Coordinates are $\boldsymbol{x} = \begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}$ (angle, angular velocity, cart position, cart velocity)
- Action u is one-dimensional; the force applied to cart
- Dynamics are

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

where $\boldsymbol{f}$ is a fairly complicated function

Constraints

$$\text{Equality constraint:} \quad x = c \tag{1}$$

$$\text{Inequality constraint:} \quad a \leq x \leq b \tag{2}$$

Any realistic physical system has constraints

- Simple boundary constraints

$$\boldsymbol{x}_{\text{low}} \leq \boldsymbol{x}(t) \leq \boldsymbol{x}_{\text{upp}}$$
$$\boldsymbol{u}_{\text{low}} \leq \boldsymbol{u}(t) \leq \boldsymbol{u}_{\text{upp}}$$

- End-point constraints:

$$\begin{aligned} \boldsymbol{x}_{0,\,\text{low}} &\leq \boldsymbol{x}(t_0) \leq \boldsymbol{x}_{0,\,\text{upp}} \\ \boldsymbol{x}_{F,\,\text{low}} &\leq \boldsymbol{x}(t_F) \leq \boldsymbol{x}_{F,\text{upp}}. \end{aligned} \tag{3}$$

- Time constraints

$$\begin{aligned} t_{0,\,\text{low}} &\leq t_0 \leq t_{0,\,\text{upp}} \\ t_{F,\,\text{low}} &\leq t_F \leq t_{F,\text{upp}}. \end{aligned} \tag{4}$$

Cost and policy

- The cost function is of the form

$$J_{\boldsymbol{u}}(\boldsymbol{x}, t_0, t_F) = \underbrace{c_F\left(t_0, t_F, \boldsymbol{x}\left(t_0\right), \boldsymbol{x}\left(t_F\right)\right)}_{\text{Mayer Term}} + \underbrace{\int_{t_0}^{t_F} c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) d\tau}_{\text{Lagrange Term}}$$

Cartpole

- Necessary constraint $-u_{\max} < u(t) < u_{\max}$ and $\boldsymbol{x}_0 = \begin{bmatrix} 0 & 0 & \pi & 0 \end{bmatrix}$
- Goal is to bring $\boldsymbol{x}$ to $\boldsymbol{x}^g = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$
- Up-right cartpole, version 1:
  - $$J_u(t_0, t_F, \boldsymbol{x}) = \|\boldsymbol{x}(t_F) - \boldsymbol{x}^g\|^2 + \lambda \int_{t_0}^{t_F} \boldsymbol{u}(t)^\top \boldsymbol{u}(t)$$
  - Constraints $t_0 = 0, t_F = 3$ (complete in 3 seconds)
- Up-right cartpole, version 2:
  - $$J_u(t_0, t_F, \boldsymbol{x}) = t_F - t_0$$
  - Constraints $\boldsymbol{x}_F = \boldsymbol{x}^g$

Endless combinations; depends on goal + method you are using

The continuous-time control problem

Given system dynamics for a system

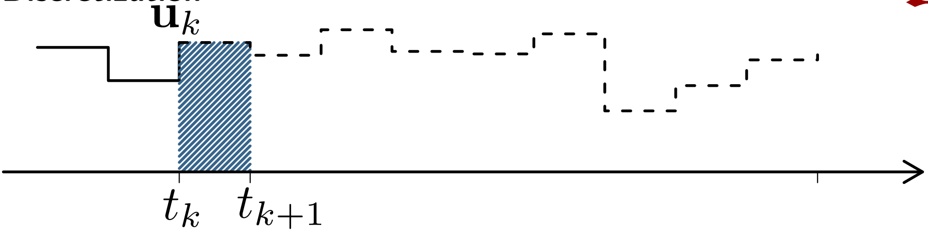
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$$

Obtain $\boldsymbol{u} : [t_0; t_F] \to \mathbb{R}^m$ as solution to

$$\boldsymbol{u}^*, \boldsymbol{x}^*, t_0^*, t_F^* = \underset{\boldsymbol{x}, \boldsymbol{u}, t_0, t_F}{\arg\min}\, J_{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, t_0, t_F).$$

(Minimization subject to all constraints)

Discretization



- Simplest choice: Eulers method
- Choose grid size N: $t_0, t_1, \dots,\quad t_{k+1} - t_k = \Delta$
- $\boldsymbol{x}_k = \boldsymbol{x}(t_k), \boldsymbol{u}_k = \boldsymbol{u}(t_k)$

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k) \\ &= \boldsymbol{x}_k + \Delta \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k) \end{aligned}$$

$$J_{\boldsymbol{u}=(\boldsymbol{u}_0, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})}(\boldsymbol{x}_0) = c_f(t_0, \boldsymbol{x}_0, t_F, \boldsymbol{x}_F) + \sum_{k=0}^{N-1} c_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

$$c_k(\boldsymbol{x}_k, \boldsymbol{u}_k) = \Delta c(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k)$$

- Simple but not very exact

Approaches to control

- Last week: Rule-based methods (build $\boldsymbol{u}(t) = \pi(\boldsymbol{x}, t)$ directly)
- **Today: Optimization-based methods:**

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} J_{\boldsymbol{u}}(\boldsymbol{x}_0)$$

  - Direct optimization of a discretized version of the problem
- Next week: DP-inspired planning methods

Infrastructure: Nonlinear program

A non-linear program is an optimization task of the form

$$\min_{\boldsymbol{z}\in\mathbb{R}^n} E(\boldsymbol{z}) \quad \text{subject to}$$

$$\boldsymbol{h}(\boldsymbol{z}) = 0$$

$$\boldsymbol{g}(\boldsymbol{z}) \leq 0$$

$$\boldsymbol{z}_{\text{low}} \leq \boldsymbol{z} \leq \boldsymbol{z}_{\text{upp}}$$

i.e. the objective is to find the $\mathbf{z}$ that minimizes E under the constraints.

- If problem is not too complex, can use methods such as **sequential convex programming** to find $\mathbf{z}^*$.
- Requires luck and engineering
  - Needs a good initial guess
  - Improves when given gradient of J and Jacobian of $\boldsymbol{f}$ and $\boldsymbol{h}$.

Infrastructure: Linear Quadratic program

A special case of the optimization task:

$$\min \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \quad \text{subject to}$$

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

$$F\boldsymbol{x} = \boldsymbol{g}$$

- When Q is positive definite and the problem is not very large the solution can always be found

Optimizing the Discrete Problem: Shooting

Consider the simplest form of a discrete control problem

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{d}_k$$

quadratic cost function

$$\boldsymbol{J}_{\boldsymbol{u}_0,\dots,\boldsymbol{u}_{N-1}}(\boldsymbol{x}_0) = \boldsymbol{x}_N^T Q_N \boldsymbol{x}_N + \sum_{k=0}^{N-1} (\boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k)$$

- Given $\boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}$, all the $\boldsymbol{x}_k$'s can be found form the system dynamics:

$$\boldsymbol{x}_2 = A_1 \boldsymbol{x}_1 + B_1 \boldsymbol{u}_1 + d_1 = A_1(A_0 \boldsymbol{x}_0 + B_0 \boldsymbol{u}_0 + \boldsymbol{d}_0) + B_1 \boldsymbol{u}_1 + d_1$$

- Problem equivalent to optimizing $J_{\boldsymbol{u}_0,\dots,\boldsymbol{u}_{N-1}}(\boldsymbol{x}_0)$ (which is quadratic) wrt. $\boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}$
- This method is called **shooting**
- **+ A single linear-quadratic optimization problem**
- **+ Easy to understand**

Optimizing the Discrete Problem: Shooting

- General case

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

$$J_{\boldsymbol{u}=(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1})}(\boldsymbol{x}_0) = c_f(t_0, \boldsymbol{x}_0, t_F, \boldsymbol{x}_F) + \sum_{k=0}^{N-1} c_k(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

- Get rid of all the $\boldsymbol{x}_k$'s except $\boldsymbol{x}_0$:

$$\boldsymbol{x}_2 = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{u}_1) = \boldsymbol{f}(\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0), \boldsymbol{u}_1)$$

So just optimize $J_{\boldsymbol{u}=(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1})}(\boldsymbol{x}_0)$ wrt. $\boldsymbol{u}$

  - **+ Easy to understand**
  - A big, non-linear program (we cannot avoid that for general dynamics)
  - **- Unstable: small changes in u_0 can mean big changes in x_N**
  - **- Eulers method is imprecise**
  - **- No bueno**. To overcome these issues, we have to take a step back

The continuous-time control problem

Given system dynamics for a system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{5}$$

Step 1: Must evaluate this ODE somehow

Subject to a number of dynamical and constant path and end-point constraints, obtain $\boldsymbol{u} : [t_0; t_F] \to \mathbb{R}^m$ as solution to

Step 2: Computing this integral

$$\min_{t_0, t_F, \boldsymbol{x}(t), \boldsymbol{u}(t)} \underbrace{c_F\left(t_0, t_F, \boldsymbol{x}\left(t_0\right), \boldsymbol{x}\left(t_F\right)\right)}_{\text{Mayer Term}} + \underbrace{\int_{t_0}^{t_F} c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau) d\tau}_{\text{Lagrange Term}}$$

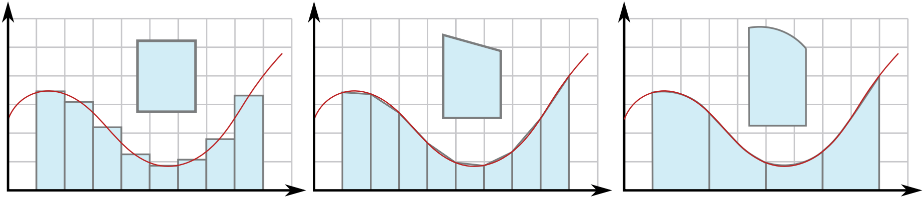
Step 3:
Minimize over all functions?
What about constraints?

subject to eq. (5) and whatever constraints are imposed on the system.

This is a nasty constrained minimization problem

Numerical integration

Suppose we wish to approximate a function $f(x)$. Divide interval into a partition $a = x_0 < x_1 < \cdots < x_n = b$

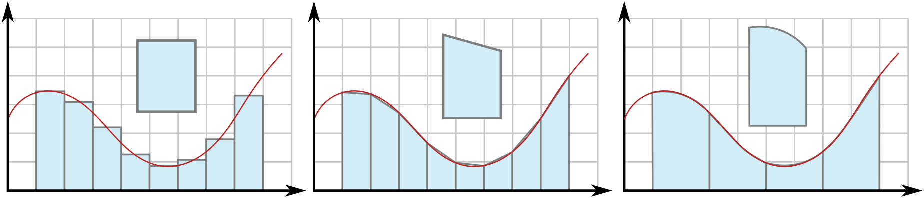


Choices corresponds to

- Piecewise constant
- Piecewise linear
- Piecewise 2nd order polynomial (use midpoint to fit the three parameters)

Approximation and integration

Each provide an approximation for the integral: $\int_a^b f(x)dx$



- Midpoint rule: $\approx \sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right)\Delta_i$
- Trapezoid rule: $\approx \frac{\Delta x}{2}\left(f\left(x_0\right)+2f\left(x_1\right)+2f\left(x_2\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right)$
- Simpson's rule: $\approx \frac{\Delta x}{3}\left(f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+2f\left(x_4\right)+\cdots+4f\left(x_{n-1}\right)+f\left(x_n\right)\right)$

General Collocation: Time discretization

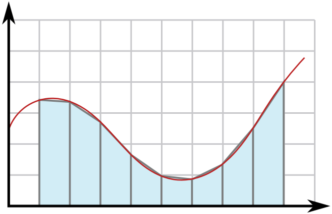
- Given t_0 and t_F and N
- We discretize the time into N intervals:

$$t_0 < t_1 < t_2 < \cdots < t_{N-1} = t_F$$

- Specifically $t_k = t_0 + \frac{k}{N-1}(t_F - t_0)$
- For later use we define:

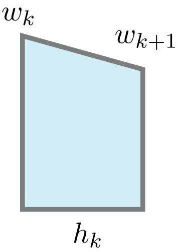
$$\begin{aligned}
h_k &= t_{k+1} - t_k, \quad k = 0, \ldots, N-2 \\
\boldsymbol{x}_k &= \boldsymbol{x}\left(t_k\right), \quad k = 0, \ldots, N-1 \\
\boldsymbol{u}_k &= \boldsymbol{u}\left(t_k\right) \\
c_k &= c\left(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k\right) \\
\boldsymbol{f}_k &= \boldsymbol{f}\left(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k\right)
\end{aligned}$$

Trapezoid collocation



Trapezoid collocation assumes

$$\int_{t_0}^{t_F} c(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau) d\tau \quad \approx \sum_{k=0}^{N-2} \frac{1}{2} h_k \left(c_k + c_{k+1}\right)$$



We can at this point evaluate the cost if we know $\boldsymbol{x}$ and $\boldsymbol{u}$!

$$c_F\left(t_0, t_F, \boldsymbol{x}_0, \boldsymbol{x}_N\right) + \frac{1}{2} \sum_{k=0}^{N-2} h_k \left(c_k + c_{k+1}\right)$$

Collocating system dynamics

Recall

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$$

Integrating both sides

$$\int_{t_k}^{t_{k+1}} \dot{\boldsymbol{x}}(t)dt = \int_{t_k}^{t_{k+1}} \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt$$

Using **trapezoid collocation** we on the right-hand side and integrating the left

$$\boldsymbol{x}_{k+1} - \boldsymbol{x}_k \approx \frac{1}{2} h_k \left(\boldsymbol{f}_{k+1} + \boldsymbol{f}_k \right)$$

Trapezoid collocation: System dynamics

- Constraints are translated to simply apply to their knot points:

$$
\begin{aligned}
x < 0 \quad &\rightarrow \quad x_k < 0 \\
u < 0 \quad &\rightarrow \quad u_k < 0 \\
\boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u}) < \mathbf{0} \quad &\rightarrow \quad \boldsymbol{h}\left(t_k, \boldsymbol{x}_k, \boldsymbol{u}_k\right) < \mathbf{0}
\end{aligned}
$$

- Boundary constraints still just apply at boundary:

$$
\boldsymbol{g}\left(t_0, \boldsymbol{x}\left(t_0\right), \boldsymbol{u}\left(t_0\right)\right) < \mathbf{0} \quad \rightarrow \quad \boldsymbol{g}\left(t_0, \boldsymbol{x}_0, \boldsymbol{u}_0\right) < \mathbf{0}
$$

Trapezoid collocation: First attempt

Optimize over $\boldsymbol{z} = (\boldsymbol{x}_0, \boldsymbol{u}_0, \dots, \boldsymbol{u}_{N-1}, t_0, t_f)$

$$\min_{\boldsymbol{z}} \left[c_F\left(t_0, t_F, \boldsymbol{x}_0, \boldsymbol{x}_N\right) + \frac{1}{2}\sum_{k=0}^{N-2} h_k\left(c_k + c_{k+1}\right) \right]$$

Such that

$$\boldsymbol{h}\left(t_k, \boldsymbol{x}_k, \boldsymbol{u}_k\right) < \mathbf{0}$$
$$\boldsymbol{g}\left(t_0, t_F, \boldsymbol{x}_0, \boldsymbol{x}_F\right) \leq \mathbf{0}$$

with convention we iteratively compute $\boldsymbol{x}_{k+1}$ from $\boldsymbol{x}_k$ starting at $k = 0$

$$k = 0, \dots, N-2: \quad \boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \frac{1}{2} h_k\left(\boldsymbol{f}_{k+1} + \boldsymbol{f}_k\right)$$

Wait, did we just solve it?

Almost! The final idea:

- Suppose we let $\boldsymbol{x}_k, \boldsymbol{u}_k$ vary freely (ensure everything can be evaluated)
- But we add the $N-1$ constraints:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \frac{1}{2} h_k \left(\boldsymbol{f}_{k+1} + \boldsymbol{f}_k \right)$$

- The key observation is local changes in $\boldsymbol{x}_k$ and $\boldsymbol{u}_k$ have local effects

Trapezoid collocation method

Optimize over $\boldsymbol{z} = (\boldsymbol{x}_0, \boldsymbol{u}_0, \boldsymbol{x}_1, \boldsymbol{u}_1, \dots, \boldsymbol{x}_{N-1}, \boldsymbol{u}_{N-1}, t_0, t_F)$

$$\min_{\boldsymbol{z}} \left[c_F\left(t_0, t_F, \boldsymbol{x}_0, \boldsymbol{x}_N\right) + \frac{1}{2} \sum_{k=0}^{N-2} h_k\left(c_k + c_{k+1}\right) \right] \tag{6}$$

$$\text{Such that } \boldsymbol{z}_{\text{lb}} \leq \boldsymbol{z} \leq \boldsymbol{z}_{\text{ub}} \tag{7}$$

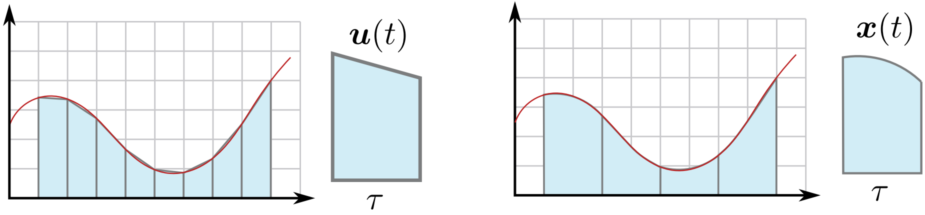
$$\boldsymbol{h}\left(t_k, \boldsymbol{x}_k, \boldsymbol{u}_k\right) \leq \boldsymbol{0} \tag{8}$$

$$\boldsymbol{x}_k - \boldsymbol{x}_{k+1} + \frac{1}{2} h_k\left(\boldsymbol{f}_{k+1} + \boldsymbol{f}_k\right) = 0 \tag{9}$$

- Optimizer also need initial point $\boldsymbol{z}_0$
- Recall $\boldsymbol{f}_k = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k)$ so last constraint is non-linear

Reconstruction

Given $\boldsymbol{z}$, how do we reconstruct the (predicted) path $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$?



- $\boldsymbol{u}(t)$ was assumed to be linear, using $\tau = t - t_k$:

$$\boldsymbol{u}(t) \approx \boldsymbol{u}_k + \frac{\tau}{h_k}\left(\boldsymbol{u}_{k+1} - \boldsymbol{u}_k\right)$$

- For $\boldsymbol{x}(t)$ we assumed

$$\dot{\boldsymbol{x}}(t) \approx \boldsymbol{f}_k + \frac{\tau}{h_k}\left(\boldsymbol{f}_{k+1} - \boldsymbol{f}_k\right)$$

- Integrating both sides and using $\boldsymbol{x}(\boldsymbol{t}_k) = \boldsymbol{x}_k$

$$\boldsymbol{x}(t) = \boldsymbol{x}_k + \boldsymbol{f}_k\tau + \frac{\tau^2}{2h_k}\left(\boldsymbol{f}_{k+1} - \boldsymbol{f}_k\right)$$

Implementation

Algorithm 1 Direct solver

function $\textsc{Direct-Solve}(N, \textsc{guess}{=}(t_0^g, t_F^g, \boldsymbol{x}^g, \boldsymbol{u}^g)\ )$
  Define $z \leftarrow (\boldsymbol{x}_0, \boldsymbol{u}_0, \dots, \boldsymbol{x}_{N-1}, \boldsymbol{u}_{N-1}, t_0, t_F)$ as all optimization variables
  Define grid time points $t_k = \frac{k}{N-1}(t_F - t_0) + t_0, \quad k = 0, \dots, N-1$ ▷ eq. (15.11)
  Define h_k, $\boldsymbol{f}_k = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k)$ and $c_k = c(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k)$.
  Define I_{eq} and I_{ineq} as empty lists of inequality/equality constraints
  for $k = 0, \dots, N-2$ **do**
    Append constraint $\boldsymbol{x}_{k+1} - \boldsymbol{x}_k = \frac{h_k}{2}(\boldsymbol{f}_{k+1} + \boldsymbol{f}_k)$ to I_{eq} ▷ eq. (15.20)
    Add all other path-constraints eq. (15.21) to I_{ineq} and I_{eq}
  end for
  Add possible end-point constraints on $\boldsymbol{x}_0, \boldsymbol{x}_F$ and t_0, t_F to I_{eq} and I_{ineq}
  Build optimization target $E(\boldsymbol{z}) = c_f(t_0, t_F, \boldsymbol{x}_0, \boldsymbol{x}_{N-1}) + \sum_{k=0}^{N-2} \frac{h_k}{2}(c_{k+1} + c_k)$
  Construct guess time-grid: $t_k^g \leftarrow \frac{k}{N-1}(t_F^g - t_0^g) + t_0^g$
  Construct guess states $\boldsymbol{z}^g \leftarrow (\boldsymbol{x}^g(t_0^g), \boldsymbol{u}^g(t_0^g), \cdots, \boldsymbol{x}^g(t_{N-1}^g), \boldsymbol{u}^g(t_{N-1}^g), t_0^g, t_F^g)$
  Let $\boldsymbol{z}^*$ be minimum of E optimized over $\boldsymbol{z}$ subject to I_i and I_{eq} using guess $\boldsymbol{z}^g$
  Re-construct $\boldsymbol{u}^*(t), \boldsymbol{x}^*(t)$ from $\boldsymbol{z}^*$ using eq. (15.22) and eq. (15.26)
  Return $\boldsymbol{u}^*, \boldsymbol{x}^*$ and t_0^*, t_F^*
end function

Making it work well

- For small N, method is imprecise, but less sensitive to $\boldsymbol{z}_0$
- For moderate N, method is **very** sensitive to $\boldsymbol{z}_0$
- Initially we do linear interpolation to get $\boldsymbol{z}_0$
- An idea is to use an optimizer for low value of N, obtain solution $\boldsymbol{z}'$
- From this $\boldsymbol{z}'$, we can construct $\boldsymbol{x}'(t)$ and $\boldsymbol{u}'(t)$
- We run optimizer with higher N and an initial guess as $\boldsymbol{x}_k = \boldsymbol{x}'(t_k)$

Implementation

Algorithm 2 Iterative direct solver

Require: An initial guess $\boldsymbol{z}_0^g = (\boldsymbol{x}^g, \boldsymbol{u}^g, t_0^g, t_F^g)$ found using simple linear interpolation

Require: A sequence of grid sizes $10 \approx N_0 < N_1 < \cdots < N_T$

1: **for** $t = 0, T$ **do**

2: $\quad \boldsymbol{x}^*, \boldsymbol{u}^*, t_0^*, t_F^* \leftarrow \textsc{Direct-Solve}(N_t, \boldsymbol{z}_t^g)$

3: $\quad \boldsymbol{z}_{t+1} \leftarrow \boldsymbol{x}^*, \boldsymbol{u}^*, t_0^*, t_F^*$

4: **end for**

5: Return $\boldsymbol{u}^*, \boldsymbol{x}^*$ and t_0^*, t_F^*

Implementation:

```
# sample.py
ineq_cons = {'type': 'ineq',
             'fun': lambda x: np.array([1 - x[0] - 2 * x[1],
                                         1 - x[0] ** 2 - x[1],
                                         1 - x[0] ** 2 + x[1]]),
             'jac': lambda x: np.array([[-1.0, -2.0],
                                         [-2 * x[0], -1.0],
                                         [-2 * x[0], 1.0]])}
eq_cons = {'type': 'eq',
           'fun': lambda x: np.array([2 * x[0] + x[1] - 1]),
           'jac': lambda x: np.array([2.0, 1.0])}
from scipy.optimize import Bounds
z_lb, z_ub = [0, -0.5], [1.0, 2.0]
bounds = Bounds(z_lb, z_ub)  # Bounds(z_low, z_up)
z0 = np.array([0.5, 0])
res = minimize(J_fun, z0, method='SLSQP', jac=J_jac,
               constraints=[eq_cons, ineq_cons], bounds=bounds)
```

We use sympy because of the gradient/Jacobians

Example: Pendulum

Example: Cartpole, the Kelly task

Task is taken from the excellent [Kel17]

- Constraints: $t_0 = 0, t_F = 2$, end-point constraints $\boldsymbol{x}_0$ and $\boldsymbol{x}_F = \boldsymbol{x}^g$ and $-20 < u(t) < 20$
- $c(\boldsymbol{x}, \boldsymbol{u}, t) = u(t)^2$
- Grid refinement: $N = 10$ then $N = 60$

`lecture_05_cartpole_kelly`

Example: Cartpole, the minimum-time task

From the (also great!) `https://github.com/MatthewPeterKelly/OptimTraj/blob/master/demo/cartPole/MAIN_minTime.m`

- Constraints: $t_0 = 0, t_F > 0$, end-point constraints $\boldsymbol{x}_0$ and $\boldsymbol{x}_F = \boldsymbol{x}^g$ and $-50 < u(t) < 50$
- $c(\boldsymbol{x}, \boldsymbol{u}, t) = t_F - t_0$
- $N = 8, 16, 32, 70$

`lecture_05_cartpole_time`

Optimizing the Discrete Problem - Collocation

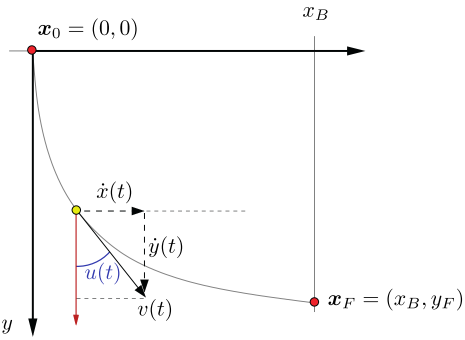
- We can also optimize over both action/state values

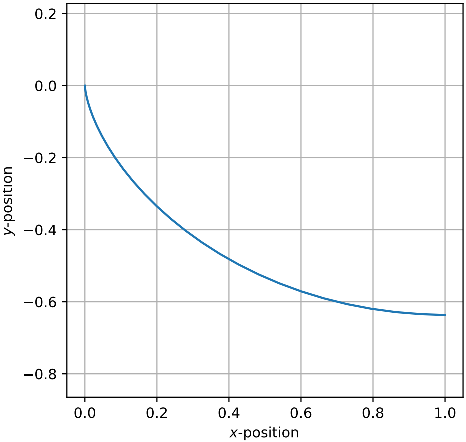
The optimisation problem is then defined as

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{x}_N^T Q_N \boldsymbol{x}_N + \sum_{k=0}^{N-1} (\boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k) \\ \text{subject to} \quad & F' \boldsymbol{x} \leq \boldsymbol{h}' \\ & F'' \boldsymbol{x} \leq \boldsymbol{h}'' \\ & A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{d}_k - \boldsymbol{x}_{k+1} = 0 \end{aligned}$$

Example: Brachistochrone

What is the fastest path for a bead to travel x_B distance in the x-direction?



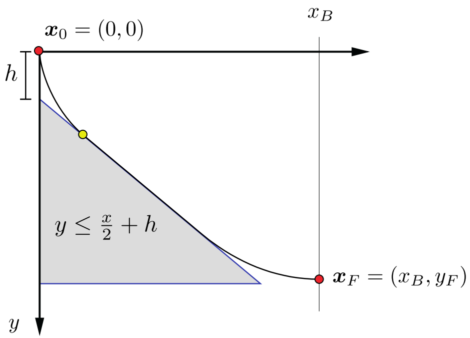


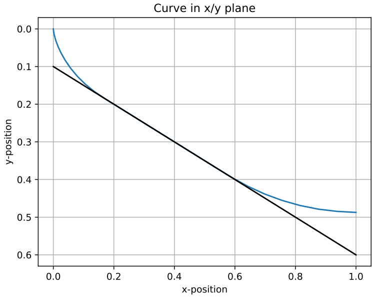
- Cost: $\min t_F$
- Actions is the angle $u(t)$. Dynamics:

$$\dot{x} = v \sin u, \quad \dot{y} = v \cos u, \quad \dot{v} = g \cos u \tag{10}$$

Example: Brachistochrone with dynamical constraints

Same as before but bead cannot pass through solid object





- Dynamical constraint

$$h(\boldsymbol{x}) = y - \frac{x}{2} - h \leq 0 \tag{11}$$

Extra: Hermite-Simpson

Hermite-Simpson collocation refers to replacing the Trapezoid rule

$$\int_{t_0}^{t_F} c(\tau) d\tau \approx \sum_{k=0}^{N-1} \frac{h_k}{6} \left(c_k + 4c_{k+\frac{1}{2}} + c_{k+1} \right)$$

For dynamics

$$\boldsymbol{x}_{k+1} - \boldsymbol{x}_k = \frac{1}{6} h_k \left(\boldsymbol{f}_k + 4\boldsymbol{f}_{k+\frac{1}{2}} + \boldsymbol{f}_{k+1} \right)$$

- **Generally better for small N**
- **Scales worse in N**

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An introduction to trajectory optimization: How to do your own direct collocation.
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