02465: Introduction to reinforcement learning and control

DP reformulations and introduction to Control

Tue Herlau

DTU Compute, Technical University of Denmark (DTU)

Lecture Schedule

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Reading material:

- [Her25, Section 6.3; Chapter 10-11] Alternative formulations of DP

Learning Objectives

- Reformulations of DP
- The control problem
- Simulating a control problem

Recap: Discrete stochastic decision problem

- The states are $x_0, \dots, x_N$, and the controls are $u_0, \dots, u_{N-1}$
- $w_k \sim P_k(W_k = w_k | x_k, u_k)$, $k = 0, \dots, N-1$ are random disturbances
- The system evolves as

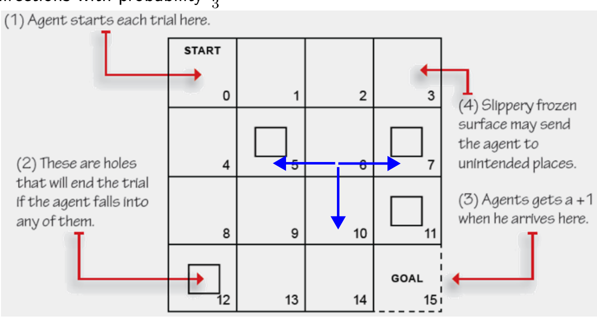
$$x_{k+1} = f_k(x_k, \mu_k(x_k), w_k), \quad k = 0, \dots, N-1$$

- At time k, the possible states/actions are $x_k \in S_k$ and $u_k \in \mathcal{A}_k(x_k)$

`lecture_03_frozen_lake.py`

DP Recap: Frozen lake

If agent takes action **down** in square 6, it will slide in either of the blue directions with probability $\frac{1}{3}$



- Implementation: w_k is **'slide forward'**, **'slide left'**, **'slide right'**
- $p(w_k|x_k, u_k) = \frac{1}{3}$ and $f_k(x_k, u_k, w_k)$ computes effect of action + slide

The Dynamical Programming algorithm

The Dynamical Programming algorithm

For every initial state x_0, the optimal cost $J^*(x_0)$ is equal to $J_0(x_0)$, and optimal policy π^* is $\pi^* = \{\mu_0, \dots, \mu_{N-1}\}$, computed by the following algorithm, which proceeds backward in time from $k = N$ to $k = 0$ and for each $x_k \in S_k$ computes

$$J_N(x_N) = g_N(x_N) \tag{1}$$

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \mathbb{E}_{w_k} \{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\} \tag{2}$$

$$\mu_k(x_k) = u_k^* \quad (u_k^* \text{ is the } u_k \text{ which minimizes the above expression}). \tag{3}$$

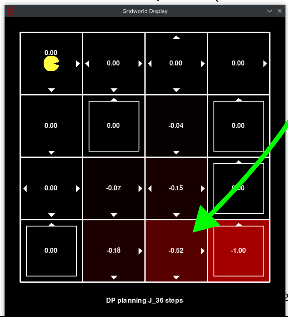
The optimal value function is expected future cost from a given state x_k at a given time k.

Quiz: Frozen lake

Consider the DP update equation:

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \mathbb{E}_{w_k} \{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \}$$

What will be the expected cost $J_{35}(x_k)$ of the indicated square? (hint: What action is best at this stage?)



a. $J_{35}(x_k) = -0.607$

b. $J_{35}(x_k) = -0.587$

c. $J_{35}(x_k) = -0.567$

d. $J_{35}(x_k) = -0.543$

e. Don't know.

Evaluate a policy

- Suppose the policy π is fixed
- We want to now how well it does

$$J_\pi(x_0) = \mathbb{E}_\pi \left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k) \mid x_0 \right].$$

- Just move expectation:

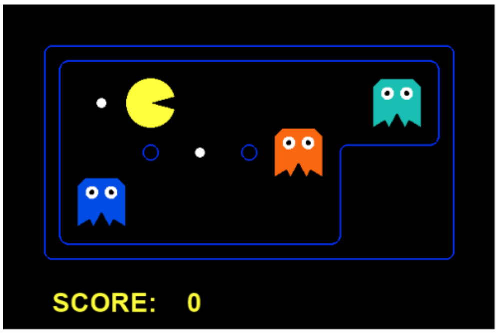
$$J_\pi(x_0) = \mathbb{E}\left[g_0(x_0, u_0, w_0) + \mathbb{E}\left[g_N(x_N) + \sum_{k=1}^{N-1} g_k(x_k, u_k, w_k) \mid x_1 \right]\right]$$
$$= \mathbb{E}\left[g_0(x_0, u_0, w_0) + J_{1,\pi}(x_1) \right]$$

- Initialize at $J_{N,\pi}(x_N) = g_N(x_N)$ and iterate:

$$J_{\pi,k}(x_k) = \mathbb{E}\left[g_k(x_k, u_k, w_k) + J_{k+1,\pi}(x_{k+1}) \right]$$

- Applications: Many RL algorithms

The DP algorithm is often not practical



- Too many states! $\{\text{tiles}\}^{\{\text{players}\}} \times 2^{\{\text{pellets}\}}$
- We often don't know dynamics/distribution over opponents moves

N	J_0	Win pct	Length	$\lvert\mathcal{S}\rvert$
1	0.00	0.00	1.00	12.0
2	0.00	0.00	2.00	41.0
3	0.00	0.00	2.50	155.0
4	0.75	0.72	3.72	278.0
6	0.81	0.81	4.30	1098.0
8	0.82	0.82	4.33	3565.0
12	0.85	0.86	4.54	18956.0
16	0.85	0.84	4.51	37516.0
20	0.85	0.84	4.56	47811.0

Table: Results of the DP algorithm to the pacman level with three ghosts

Stationary problem = stationary policy

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}(f_k(x_k, u_k, w_k)) + g_k(x_k, u_k, w_k) \right]$$

Assume the problem is independent of k:

$$J_k(x) = \min_{u} \mathbb{E}\left[J_{k+1}(f(x, u, w)) + g(x, u, w) \right]$$

- Will be true that $J_0 \approx J_1 \approx J_2$ etc.
- Policies will be the same initially $\pi_0 \approx \pi_1$ etc.
- The horizon N is irrelevant assuming it is *long enough*

In fact just iterate to convergence:

$$J(x) \leftarrow \min_{u} \mathbb{E}\left[J(f(x, u, w)) + g(x, u, w) \right]$$

Applications: This is nearly always the case.

Action-value formulation

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}(f_k(x_k, u_k, w_k)) + g_k(x_k, u_k, w_k) \right]$$

Rewrite using $Q(x_k, u_k)$ as the expected cost

- Foundation of Q-learning
- If we know the Q-functions, they give us the policy for free

Robust control

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}\left(f_k(x_k, u_k, w_k)\right) + g_k\left(x_k, u_k, w_k\right)\right]$$

- **Problem:** What if we don't know $p(w_k|x_k, u_k)$?
- **Assumes the worst possible thing always happen**

$$J_k(x_k) = \min_{u_k} \left[\arg\max_{w_k} \left[J_{k+1}\left(f_k(x_k, u_k, w_k)\right) + g_k\left(x_k, u_k, w_k\right)\right]\right]$$

RL Most game-playing methods (Alphago-zero, TD-gammon, etc.)

Control Robust control

Games (imperfect information, Nash-equilibrium) are generally a fairly open problem in RL [BBLG20]

Sample-based formulation

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}\left(f_k(x_k, u_k, w_k)\right) + g_k\left(x_k, u_k, w_k\right)\right]$$

- **Problem:** What if we **really** don't know $P(w_k|x_k, u_k)$?
- **Idea:** We can sample from it

$$J_k(x_k) \approx \min_{u_k} \frac{1}{S} \sum_{s=1}^{S} \left[J_{k+1}\left(f_k(x_k, u_k, w^{(s)})\right) + g_k\left(x_k, u_k, w^{(s)}\right)\right]$$

Foundation of RL: **Samples can be obtained by just observing what nature does in a state** (x_k, u_k)

Approximate dynamical programming

We solve the following problem at each step k

$$J_k(x_k) = \min_{u_k} \mathbb{E}\left[J_{k+1}\left(x_{k+1}\right) + g_k\left(x_k, u_k, w_k\right)\right]$$

To many damn states! (...although calculation for a single x_k is ok..)

- **Idea:** Use an approximating function $J_k(x_k) \approx \tilde{J}(x_k, \boldsymbol{w})$
- **How?:** The right-hand side gives us a *prediction* y_k for x_k which we use to **train** $\boldsymbol{w}_k$

$$\boldsymbol{w}^* = \min_{\boldsymbol{w}} \sum_{s=1}^{S} \left(y^{(s)} - \tilde{J}(x^{(s)}, \boldsymbol{w})\right)^2$$

This is the idea behind deep RL, and has applications to control and DP-based planning

d-step methods

DP applied in the starting state:

$$J^*(x_0) = \arg\min_{u_0} \mathbb{E}\left[J_1^*\left(x_1\right) + g_0\left(x_0, u_0, w_0\right)\right]$$

d-step rollout of DP:

$$J^*(x_0) = \arg\min_{\mu_0, \dots, \mu_{d-1}} \mathbb{E}\left[J_d^*\left(x_{k+d}\right) + \sum_{k=0}^{d-1} g_k\left(x_k, \mu_k\left(x_k\right), w_k\right)\right]$$

Instead of using J_d^*, perhaps use a **really** rough approximation

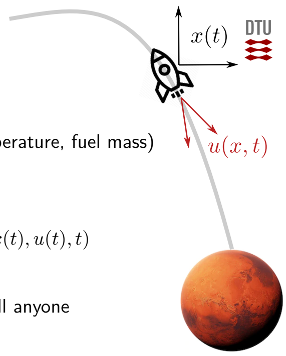
RL n-step methods (Impala, Alphastar, etc.)

Control Model-predictive control

- Often just ignore the terminal cost
- Often just assume model is deterministic
- Both assumptions are justifiable because the model wrong anyway

Control theory

Example: Mars landing



Time Continuous

State/Actions $x(t)$: (Position, velocity, temperature, fuel mass)
$u(t)$: thruster outputs

Dynamics Smooth and time-dependent

$$\dot{x}(t) = f(x(t), u(t), t)$$

Cost Land the right place,
and use little fuel **and** don't kill anyone

Constraints Thrusters deliver limited force,
ship cannot go into mars, etc.

Objective Determine $u(t)$ to minimize final cost

Really important constraints; no learning

`lecture_01_car_random.py`

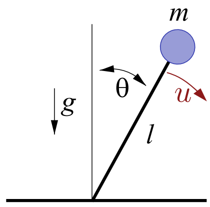
Control theory in general

- Why care?
  - More mature and practically important than RL
  - Ideas in control relevant for RL and beyond
- This course will teach **naive** but **real** control theory:
  - **Don't care about error analysis/analytical properties**
  - **Will emphasize real methods**
  - **Will distinguish between approximate model of environment/actual environment**

Differences and similarities to dynamical programming

- Similarities
  - A time-dependent problem
  - States and actions
  - Goal is still to minimize a cost function
  - Ideas from DP will carry over
- **Complications**
  - Time is continuous $t \in [t_0, t_F]$
  - Dynamics is an ODE
- **Simplifications**
  - No noise
  - Open-loop techniques play a more prominent role

Example: The pendulum environment



If u is a torque applied to the axis of rotation θ then:

$$\ddot{\theta}(t) = \frac{g}{l}\sin(\theta(t)) + \frac{u(t)}{ml^2}$$

If $\boldsymbol{x} = \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}^T$ this can be written as

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta} \\ \frac{g}{l}\sin(\theta) + \frac{u}{ml^2}\end{bmatrix} = f(\boldsymbol{x}, u) \quad (4)$$

All high-order ODEs are equivalent to systems of 1st order ODEs, see **theorem 12.4.1**: https://mat1a.compute.dtu.dk/_assets/12_system_diff_eqs.pdf

`lecture_04_pendulum_random.py`

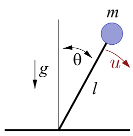
Dynamics

We assume the system we wish to control has dynamics of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t)$$

- $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a complete description of the system at t
- $\boldsymbol{u}(t) \in \mathbb{R}^d$ are the controls applied to the system at t
- The time t belongs to an interval $[t_0, t_F]$ of interest
- The evolution of the system $\boldsymbol{x}(t), \boldsymbol{u}(t)$ is called a **path** or **trajectory**

Quiz: Stopping the pendulum



$$\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta} \\ \frac{g}{l}\sin(\theta) + \frac{u}{ml^2}\end{bmatrix} = f(\boldsymbol{x}, u)$$

If the pendulum is at an angle of $\frac{\pi}{4}$ to vertical, how much torque u should we apply to keep it still?

a. $u(t) = -\frac{mgl}{\sqrt{2}}$ (**Correct: An earlier version was missing** l)

b. $u(t) = -\frac{m}{gl\sqrt{2}}$

c. $u(t) = -\frac{mg\sqrt{2}}{l^2}$

d. $u(t) = -\frac{g\sqrt{2}}{ml}$

e. Don't know.

Constraints I

Any realistic physical system has constraints. Examples:

- Simple boundary constraints

$$\boldsymbol{x}_{\text{low}} \leq \boldsymbol{x}(t) \leq \boldsymbol{x}_{\text{upp}}$$
$$\boldsymbol{u}_{\text{low}} \leq \boldsymbol{u}(t) \leq \boldsymbol{u}_{\text{upp}}$$

  Maximal acceleration of a car; that the acceleration of an airplane cannot exceed a certain safety limit

- Problem must terminate within a given time

$$t_{\text{low}} \leq t_0 < t_F \leq t_{\text{upp}}$$

  (or we could know t_0 and t_f; note this is different from DP case with x_0 and N!)
  Don't take forever

Constraints II

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- Boundary constraints

$$\boldsymbol{x}_{0,\text{ low}} \leq \boldsymbol{x}(t_0) \leq \boldsymbol{x}_{0,\text{ upp}}$$
$$\boldsymbol{x}_{F,\text{ low}} \leq \boldsymbol{x}(t_F) \leq \boldsymbol{x}_{F,\text{upp}}$$

I want you to be somewhere when you start or end

- Notice that for some coordinate the two boundaries can be equal to give equality constraints; they can also be ∞ for unconstrained problems

Cost and policy

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- State/action trajectories $\boldsymbol{x}, \boldsymbol{u}$ which satisfy the constraints are said to be **admissible**
- The cost function will be of this form:

$$J_{\boldsymbol{u}}(\boldsymbol{x}, t_0, t_F) = \underbrace{c_F(t_0, t_F, \boldsymbol{x}(t_0), \boldsymbol{x}(t_F))}_{\text{Mayer Term}} + \underbrace{\int_{t_0}^{t_F} c(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) d\tau}_{\text{Lagrange Term}}$$

- Note we sometimes write this as $J_{\boldsymbol{u}}(\boldsymbol{x}_0, t_0, t_F)$
- Very often $t_0 = 0$

Special cases

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- Minimum time $c_F = 0$, $c = 1$ and

$$\text{cost } = \int_{t_0}^{t_f} 1 d\tau = (t_f - t_0)$$

- Coordinate 3 takes a particular value $c_F(\cdots) = (x_3(t_f) - x_0)^2$, $c = 0$ and

$$\text{cost } = (x_3(t_f) - x_0)^2$$

- Minimize energy used $c(\cdots) =$ force $\times$ distance

$$\text{cost } = \int_{t_0}^{t_f} (\text{force} \times \text{velocity}) d\tau = \text{energy}$$

The continuous-time control problem

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Given system dynamics for a system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$$

Obtain $\boldsymbol{u} : [t_0; t_F] \rightarrow \mathbb{R}^m$ as solution to

$$\boldsymbol{u}^*, \boldsymbol{x}^*, t_0^*, t_F^* = \underset{\boldsymbol{x}, \boldsymbol{u}, t_0, t_F}{\arg\min} J_{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, t_0, t_F).$$

(Minimization subject to all constraints)

Today:

- Simulate the system

Example: The harmonic oscillator

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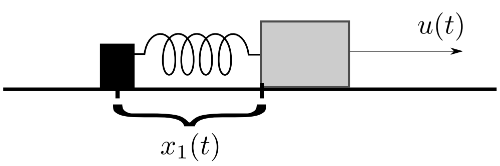


A mass attached to a spring which can move back-and-forth

$$\ddot{x}(t) = -\frac{k}{m} x(t) + \frac{1}{m} u(t) \quad (5)$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \quad (6)$$

$$J(\boldsymbol{x}_0) = \int_0^{t_F} \left(\boldsymbol{x}(t)^\top \boldsymbol{x}(t) + u(t)^2 \right) dt. \quad (7)$$

`lecture_04_harmonic.py`

Simulation: Euler integration

DTU

Apply a Taylor expansion:

$$\boldsymbol{x}(t + \delta) = \boldsymbol{x}(t) + \dot{\boldsymbol{x}}(t)\delta + \frac{1}{2}\ddot{\boldsymbol{x}}(t)\delta^2 + \mathcal{O}(\delta^3)$$

Define $\Delta = \frac{t_F - t_0}{N}$ and introduce

$$t_1 = t_0 + \Delta$$
$$t_2 = t_0 + 2\Delta$$
$$t_k = t_0 + k\Delta$$
$$t_N = t_0 + N\Delta = t_F$$

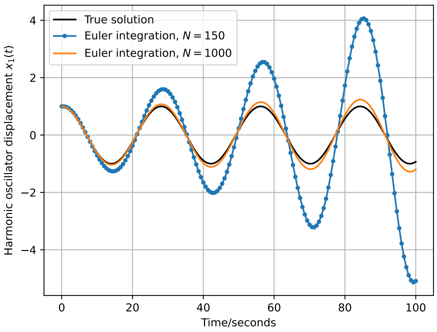
Then we can iteratively update:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k, t_k)$$

Practical issues

A harmonic oscillator with no force $\ddot{x} = -\frac{k}{m}x$

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \Delta \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \boldsymbol{x}_k, \quad \Delta = \frac{t_F}{N}. \tag{8}$$

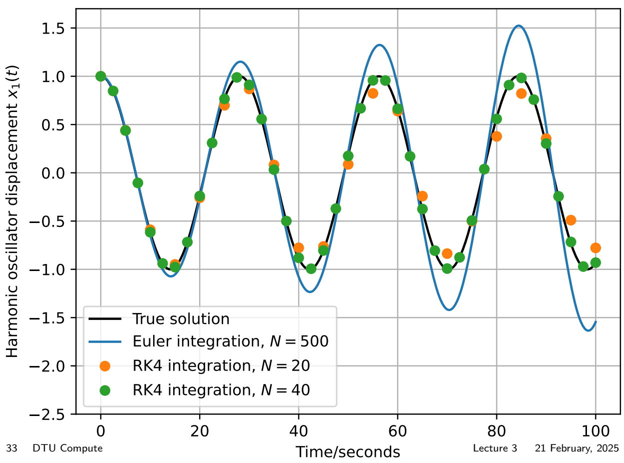


Simulation: Runge-Kutta 4 (RK4)

- Discretize time similar to Euler $t_k = t_0 + k\Delta$
- Compute

$$\begin{aligned} k_1 &= \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k) \\ k_2 &= \boldsymbol{f}\left(\boldsymbol{x}_k + \Delta\frac{k_1}{2}, \boldsymbol{u}\left(t_k + \frac{\Delta}{2}\right), t_k + \frac{\Delta}{2}\right) \\ k_3 &= \boldsymbol{f}\left(\boldsymbol{x}_k + \Delta\frac{k_2}{2}, \boldsymbol{u}\left(t_k + \frac{\Delta}{2}\right), t_k + \frac{\Delta}{2}\right) \\ k_4 &= \boldsymbol{f}\left(\boldsymbol{x}_k + \Delta k_3, \boldsymbol{u}(t_{k+1}), t_{k+1}\right) \end{aligned}$$

- Set $\boldsymbol{x}_{k+1} \leftarrow \boldsymbol{x}_k + \frac{1}{6}\Delta(k_1 + 2k_2 + 2k_3 + k_4)$
- Repeat for all k



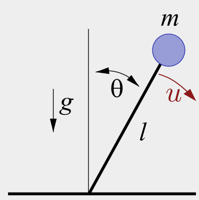
Implementation: Continuous symbolic model

```python
# basic_pendulum.py
class BasicPendulumModel(ControlModel):
    def sym_f(self, x, u, t=None):
        g = 9.82
        l = 1
        m = 2
        theta_dot = x[1]  # Parameterization: x = [theta, theta']
        theta_dot_dot = g / l * sym.sin(x[0]) + 1 / (m * l ** 2) * u[0]
        return [theta_dot, theta_dot_dot]

    def get_cost(self) -> SymbolicQRCost:
        return SymbolicQRCost(Q=np.eye(2), R=np.eye(1))

    def u_bound(self) -> Box:
        return Box(np.asarray([-10]), np.asarray([10]), dtype=float)

    def x0_bound(self) -> Box:
        return Box(np.asarray( [np.pi, 0] ), np.asarray( [np.pi, 0]), dtype=float)
```



Implements:

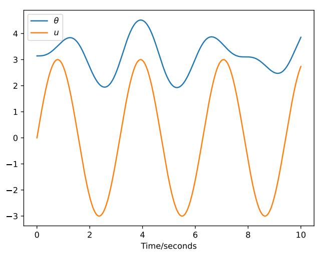
- $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \boldsymbol{f}\left(\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, u\right) = \begin{bmatrix} \dot{\theta} \\ \frac{g}{l}\sin(\theta) + \frac{1}{ml^2}u \end{bmatrix}$
- $J(\boldsymbol{x}_0) = \int_{t_0}^{t_F} \left(\frac{1}{2}\boldsymbol{x}(t)^\top Q \boldsymbol{x}(t) + \frac{1}{2}u(t)^\top R u(t)\right) dt = \frac{1}{2}\int_{t_0}^{t_F} \left(\|\boldsymbol{x}(t)\|^2 + u(t)^2\right) dt$
- $-10 \leq u(t) \leq 10$, and $\boldsymbol{x}_0 = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

Simulation

```python
# chapter7continuous/model_example_plot.py
cmodel = PendulumModel()
x0 = cmodel.x0_bound().low

def policy(x, t):
    return [3 * np.sin(2 * t)]

xx, uu, tt, cost = cmodel.simulate(x0, policy, t0=0, tF=10)
plt.plot(tt, xx[:, 0], label="$\\theta$")
plt.plot(tt, uu[:, 0], label="$u$")
```



Resources and references

https://en.wikipedia.org Overview of alternative discretization approaches of a ODE to discrete system (https://en.wikipedia.org/wiki/Discretization)

Noam Brown, Anton Bakhtin, Adam Lerer, and Qucheng Gong. Combining deep reinforcement learning and search for imperfect-information games, 2020.

Tue Herlau. Sequential decision making. (Freely available online), 2025.