02465: Introduction to reinforcement learning and control

Dynamical Programming

Tue Herlau

DTU Compute, Technical University of Denmark (DTU)

Lecture Schedule

Dynamical programming

1. The finite-horizon decision problem
   7 February
2. **Dynamical Programming**
   14 February
3. DP reformulations and introduction to Control
   21 February

Control

4. Discretization and PID control
   28 February
5. Direct methods and control by optimization
   7 March
6. Linear-quadratic problems in control
   14 March
7. Linearization and iterative LQR
   21 March

Reinforcement learning

8. Exploration and Bandits
   28 March
9. Bellmans equations and exact planning
   4 April
10. Monte-carlo methods and TD learning
    11 April
11. Model-Free Control with tabular and linear methods
    25 April
12. Eligibility traces
    2 May
13. Deep-Q learning
    9 May

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

Reading material:

- [Her25, Chapter 5-6.2] Formalization of the decision problem and the DP algorithm

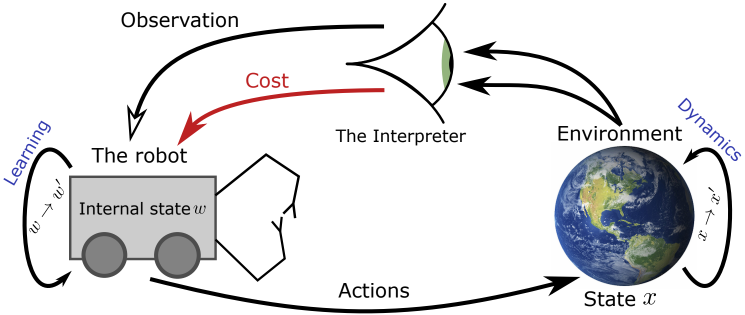
Learning Objectives

- Dynamical Programming
- Principle of optimality
- Optimal policy/value function using DP

Practicals

- Issue with recording 1
- Numpy-core issue fixed (old numpy; see guide)
- Fixed page numbering (thanks!)

The decision problem

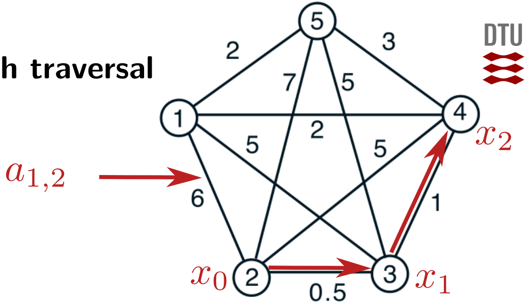


State The configuration of the environment x

Action What we do u

Cost/reward A number which depends on the state and action

Example: Shortest path graph traversal



Find shortest path from starting node $x_0 = 2$ to final node $t = 5$

State Current node $x_k = 4$

Actions next possible node: $u_k \in \{1, 2, \ldots, 5\}$

Dynamics Deterministic, known

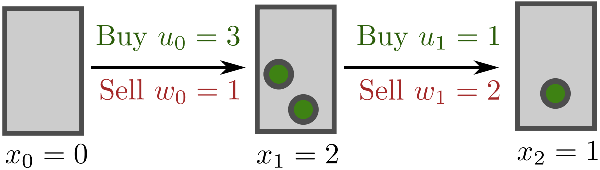
$$x_{k+1} = f(x_k = 4, u_k = 5) = 5$$

Cost Sum of edge weights

$$\sum_{k=0}^{N-1} a_{x_k, u_k} + \begin{cases} 0 & \text{if } x_N = t \\ \infty & \text{otherwise} \end{cases}$$

We want optimal path $\{2, 3, 4, 5\}$

Inventory control



- We order a quantity of an item at period $k = 0, \ldots, N$ so as to meet a stochastic demand

x_k stock available at the beginning of the kth period,
$u_k \geq 0$ stock ordered (and immediately delivered) at the beginning of the kth period.
$w_k \geq 0$ Demand during the k'th period

- Dynamics: $x_{k+1} = x_k + u_k - w_k$
- Cost function (in each step)

$$u_k + (x_k + u_k - w_k)^2$$

- Select actions $u_0, \ldots, u_{N-1}$ to minimize cost

We want proven optimal rule for ordering

Basic control setup: Environment dynamics

Finite time Problem starts at time 0 and terminates at *fixed* time N. Indexed as $k = 0, 1, \dots, N$.

State space The states x_k belong to the **state space** $\mathcal{S}_k$

Control The available controls u_k belong to the **action space** $\mathcal{A}_k(x_k)$, which may depend on x_k

Dynamics

$$x_{k+1} = f_k\left(x_k, u_k, w_k\right), \quad k = 0, 1, \dots, N-1$$

Disturbance/noise A random quantity w_k with distribution

$$w_k \sim P_k(W_k | x_k, u_k)$$

Cost and control

Agent observe x_k, agent choose u_k, environment generates w_k

Cost At each stage k we obtain cost

$$g_k(x_k, u_k, w_k), \quad k = 0, \dots, N-1 \quad \text{and} \quad g_N(x_k) \text{ for } k = N.$$

Action choice Chosen as $u_k = \mu_k(x_k)$ using a function $\mu_k : \mathcal{S}_k \to \mathcal{A}_k(x_k)$

$$\mu_k(x_k) = \{\text{Action to take in state } x_k \text{ in period } k\}$$

Policy The collection $\pi = \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$

Rollout of policy Given x_0, select $u_k = \mu_k(x_k)$ to obtain a **trajectory** $x_0, u_0, x_1, \dots, x_N$ and **accumulated cost**

$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)$$

Expected cost/value function

Expected cost Given π, x_0 it is the average cost of all trajectories:

$$J_\pi(x_0) = \mathbb{E}\left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right]$$

Optimal policy Given x_0, an optimal policy π^* is one that minimizes the cost

$$\pi^*(x_0) = \underset{\pi=\{\mu_0,\dots,\mu_{N-1}\}}{\arg\min} J_\pi(x_0)$$

Optimal cost function The optimal cost, given x_0, is denoted $J^*(x_0)$ and is defined as

$$J^*(x_0) = \min_{\pi=\{\mu_0,\dots,\mu_{N-1}\}} J_\pi(x_0)$$

J_π is the key quantity in control/reinforcement learning

Open versus closed loop

Our goal is to find the policy π which minimize:

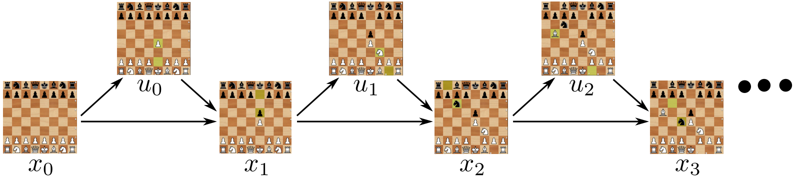
$$J_\pi(x_0) = \mathbb{E}\left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right]$$

Closed-loop minimization Select u_k last-minute as $u_k = \mu_k(x_k)$ when information x_k is available

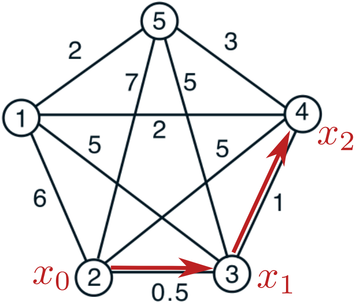
Open-loop minimization Select actions $u_0, \dots, u_{N-1}$ at $k = 0$

- Open-loop minimization is simpler

Open or closed loop



- If environment is stochastic, we need a closed-loop controller
- If environment is deterministic, we know the position x_k with certainty given $u_0, \dots, u_{k-1}$. Therefore, there is no advantage in delaying choice



Quiz: Chess and DP

Suppose the game of chess was formulated as dynamical programming (N, $\mathcal{S}_k$, $\mathcal{A}_k$, etc.) with the intention of obtaining a good policy μ_k using dynamical programming.

This will lead to several practical problems, however, focusing just on the potential problems listed below, which one will be a main obstacle?

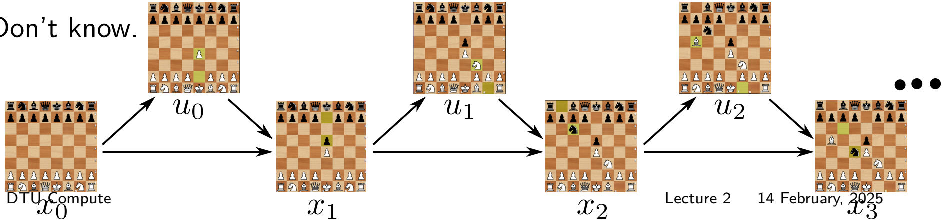
a. The policy function μ_k will require too much memory to store

b. Given a state $\boldsymbol{x}_k$, it is not practical to define the action spaces $\mathcal{A}_k(x_k)$

c. It will require too much space to store the state space $\mathcal{S}_2$.

d. We cannot define a meaningful cost function g_k.

e. Don't know.



Summary: Discrete stochastic decision problem

- The states are $x_0, \dots, x_N$, and the controls are $u_0, \dots, u_{N-1}$
- $w_k \sim P_k(W_k = w_k | x_k, u_k)$, $k = 0, \dots, N-1$ are random disturbances
- The system evolves as

$$x_{k+1} = f_k(x_k, \mu_k(x_k), w_k), \quad k = 0, \dots, N-1$$

- At time k, the possible states/actions are $x_k \in S_k$ and $u_k \in \mathcal{A}_k(x_k)$
- Policy is a sequence of functions $\pi = \{\mu_0, \dots, \mu_{N-1}\}$, $\mu_k : S_k \mapsto \mathcal{A}_k(x_k)$
- The cost starting in x_0 is:

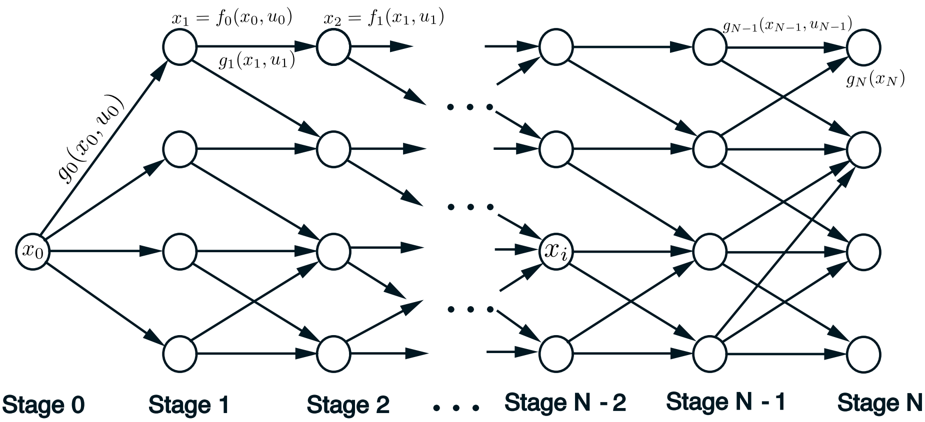
$$J_\pi(x_0) = \mathbb{E}\left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right]$$

- **The control problem**: Given x_0, determine optimal policy by minimizing

$$\pi^*(x_0) = \underset{\pi=\{\mu_0,\dots,\mu_{N-1}\}}{\arg\min} J_\pi(x_0)$$

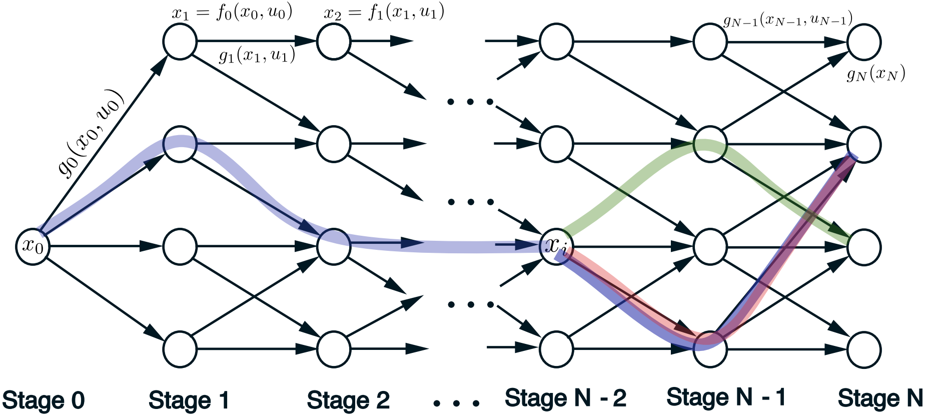
Graph representation

Starting in x_0, decision problem can be seen as traversing a graph



- Nodes are states, edges are possible transitions, cost is sum of edges
- In deterministic case, actions are edges and a policy is just a path

Principle of optimality (PO), deterministic case



The **blue line** is a path corresponding to an **optimal** policy

$$J^*(x_0) = J_{\pi^*}(x_0) = \min_{\pi} J_\pi(x_0)$$

Suppose at stage i optimal path $\pi^* = \left\{\mu_0^*, \mu_1^*, \dots, \mu_{N-1}^*\right\}$ pass through x_i

- **PO:** The **tail policy** $\left\{\mu_i^*, \mu_{i+1}^*, \dots, \mu_{N-1}^*\right\}$ is optimal from x_i to x_N
- **Why?** Suppose **alternative tail policy** ... is better than

Definitions

For any policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$

- For any $k = 0, \ldots, N-1$, $\pi^k = \{\mu_k, \mu_{k+1}, \ldots, \mu_{N-1}\}$ is a **tail policy**
- For any x_k the **cost of the tail policy** is

$$J_{k,\pi}(x_k) = \mathbb{E}\left\{g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_i(x_i), w_i)\right\}$$

- And the **optimal cost of a tail policy** starting in x_k

$$J_k^*(x_k) = \min_{\pi^k} J_{k,\pi_k}(x_k)$$

- Note that $J_0^*(x_0) = J^*(x_0)$

Proof of PO in deterministic case

$$J_{\pi^*}(x_0) = g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k^*(x_k)) =$$

$$\left(\sum_{k=0}^{i-1} g_k(x_k, \mu_k^*(x_k))\right) + \left(g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, \mu_k^*(x_k))\right)$$

$$\geq \left(\sum_{k=0}^{i-1} g_k(x_k, \mu_k^*(x_k))\right) + g_N(x'_N) + \sum_{k=i}^{N-1} g_k(x'_k, \mu'_k(x'_k))$$

$$= J_{\pi=(\mu_0,\dots,\mu_{i-1},\pi'_k)}$$

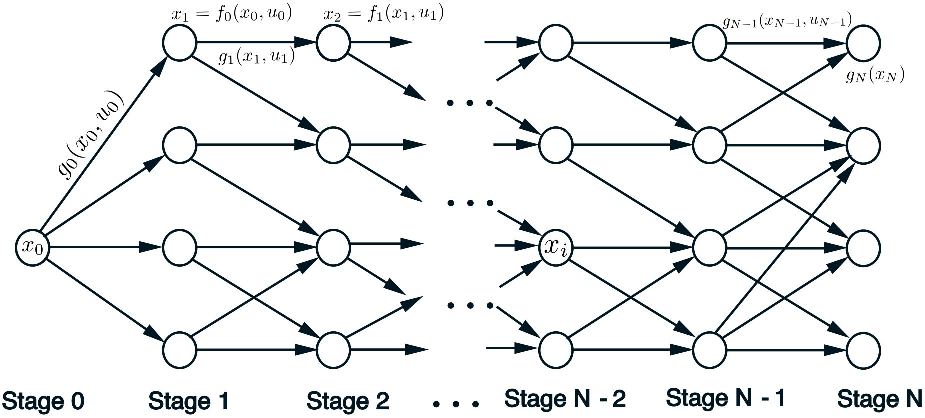
If the optimal tail policy π'_i had a lower tail cost than the tail of optimal policy this means:

$$g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, \mu_k^*(x_k)) > g_N(x'_N) + \sum_{k=i}^{N-1} g_k(x'_k, \mu'_k(x'_k))$$

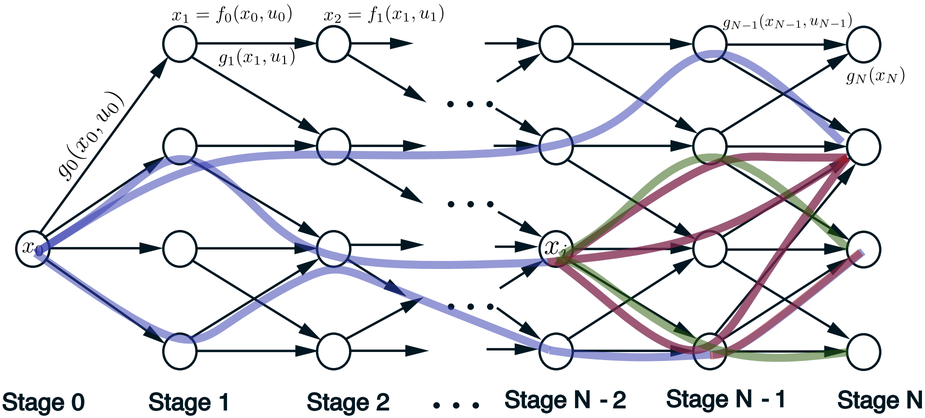
and so the combined policy $(\mu_0, \dots, \mu_{i-1}, \pi'_i)$ would have lower cost than optimal policy π^*

The stochastic case

Consider the stochastic case. Trajectories are now random



The stochastic case



- Consider **tail policy** of π^*: $J_{i,\pi^*}(x_0)$
- Suppose **optimal tail policy** $J_i^*(x_i)$ is an improvement
- It seems true the combined policy is an improvement over π^* [Her25, appendix A]

Principle of optimality

Consider a general, stochastic/discrete finite-horizon decision problem

The principle of optimality

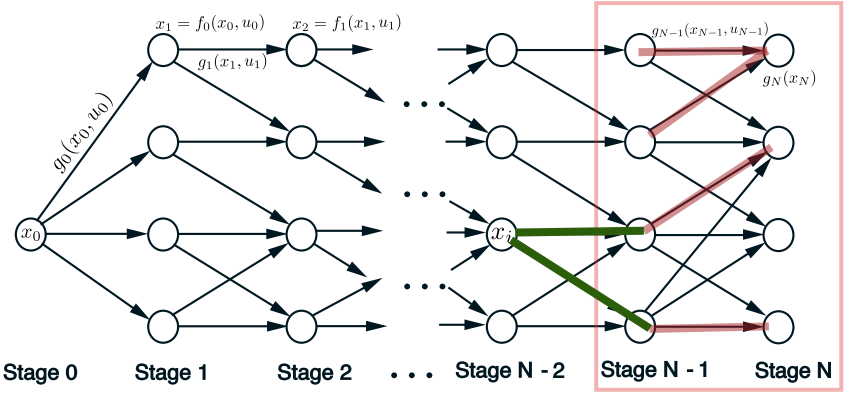
Let $\pi^* = \{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ be an optimal policy for the problem, and assume that when using π^*, a given state x_i occurs at stage i with positive probability. Suppose $\tilde{\pi}_k^*$ is the optimal tail policy obtained by minimizing the tail cost starting from x_i

$$J_{k,\pi}(x_i) = \mathbb{E}\left\{ g_N(x_N) + \sum_{i=k}^{N-1} g_i(x_i, \mu_i(x_i), w_i) \right\}.$$

Then the truncated policy $\{\mu_i^*, \mu_{i+1}^*, \ldots, \mu_{N-1}^*\}$ of π^* is optimal for the tail problem

$$J_{k,\tilde{\pi}_{*,k}}(x_k) = J_{k,\pi^*}(x_k).$$

The dynamical programming algorithm: Informal



- Suppose we know the optimal tail policy at stage $k+1$ for all x_{k+1}
- Cost of optimal path π_k^* from k to N is the cost of optimal path $x_k \to x_{k+1}$ and then $x_{k+1} \to x_N$
- The later part is the same as $J_{k+1}^*(x_{k+1})$ by the **PO**
- We find optimal cost by minimizing

$$J_k^*(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \left[g_k(x_k, u_k) + J_{k+1}^*(x_{k+1}) \right], \quad \mu_k(x_k) = u_k^*$$

The Dynamical Programming algorithm

The Dynamical Programming algorithm

For every initial state x_0, the optimal cost $J^*(x_0)$ is equal to $J_0(x_0)$, and optimal policy π^* is $\pi^* = \{\mu_0, \dots, \mu_{N-1}\}$, computed by the following algorithm, which proceeds backward in time from $k = N$ to $k = 0$ and for each $x_k \in S_k$ computes

$$J_N(x_N) = g_N(x_N) \quad (1)$$

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \mathbb{E}_{w_k} \{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\} \quad (2)$$

$$\mu_k(x_k) = u_k^* \quad (u_k^* \text{ is the } u_k \text{ which minimizes the above expression}). \quad (3)$$

- There are N μ's and $N+1$ J's. This will also be the case in the code
- In the deterministic case:

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \{g_k(x_k, u_k) + J_{k+1}(f_k(x_k, u_k))\}$$

Example: Inventory control

- Consider the inventory control problem where we plan over $N = 3$ stages
- Customers can buy $w_k = 0$ to $w_k = 2$ units and we can order $u_k = 0$ to $u_k = 2$ units
- We assume the stock can hold from 0 to 2 units (no excess stock; no backlog)

$$x_{k+1} = f_k(x_k, u_k, w_k) = x_k + u_k - w_k \text{ (threshold s.t. } 0 \leq x_{k+1} \leq 2)$$

- The cost to buy an item is 1 plus quadratic penalty for excess stock and unmet demand:

$$u_k + (x_k + u_k - w_k)^2$$

- There is no terminal cost $g_N(x_N) = 0$
- The demand has distribution

$$p(w_k = 0) = 0.1, \quad p(w_k = 1) = 0.7, \quad p(w_k = 2) = 0.2$$

Implementation

```python
# inventory.py
class InventoryDPModel(DPModel):
    def __init__(self, N=3):
        super().__init__(N=N)

    def A(self, x, k): # Action space A_k(x)
        return {0, 1, 2}

    def S(self, k): # State space S_k
        return {0, 1, 2}

    def g(self, x, u, w, k): # Cost function g_k(x,u,w)
        return u + (x + u - w) ** 2

    def f(self, x, u, w, k): # Dynamics f_k(x,u,w)
        return max(0, min(2, x + u - w ))

    def Pw(self, x, u, k): # Distribution over random disturbances
        return {0:.1, 1:.7, 2:0.2}

    def gN(self, x):
        return 0
```

Option 1: Pen-and-paper

First step: $J_3(x_3) = 0$ (for all x_3)

Step $k = 2$ For $x_2 = 0$

$$J_2(0) = \min_{u_2=0,1,2} \mathbb{E}_{w_2} \left\{ u_2 + (u_2 - w_2)^2 \right\}$$
$$= \min_{u_2=0,1,2} \left[u_2 + 0.1\,(u_2)^2 + 0.7\,(u_2 - 1)^2 + 0.2\,(u_2 - 2)^2 \right]$$
$$= \min_{u_2=0,1,2} \{0.7\cdot 1 + 0.2\cdot 4, 1 + 0.1\cdot 1 + 0.2\cdot 1, 2 + 0.1\cdot 4 + 0.7\cdot 1\}$$
$$= \min_{u_2=0,1,2} \{1.5, 1.3, 3.1\}$$

Therefore $\mu_2^*(0) = 1$ and $J_2^*(0) = 1.3$

Until nails bleed Keep at it for $x_2 = 1, 2$ and then for $k = 1$ and finally $k = 0$...

Quiz: Manual DP

Suppose that for a given k:

- $\mathcal{A}_k(x_k) = \{0, 1\}$, $f_k(x_k, u_k, w_k) = x_k + u_k w_k$
- $g_k(x_k, u_k, w_k) = -x_k u_k$, $J_{k+1}(x_{k+1}) = x_{k+1}$
- $\mathbb{E}[w_k] = 1$

What is the value of $J_k(x_k = 1)$?. **Tip:**

$$J_k(x_k) = \min_{u_k \in \mathcal{A}_k(x_k)} \mathbb{E}_{w_k} \{ g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k)) \}$$

a. $J_k(1) = -2$

b. $J_k(1) = -1$

c. $J_k(1) = 0$

d. $J_k(1) = 1$

e. $J_k(1) = 2$

f. Don't know.

Option 2: Computer

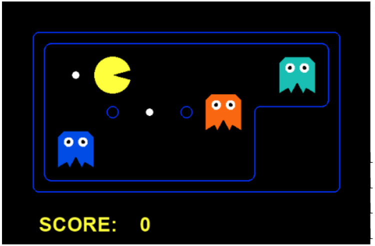
```python
# inventory.py
inv = InventoryDPModel()
J,pi = DP_stochastic(inv)
print(f"Inventory control optimal policy/value functions")
for k in range(inv.N):
    print(", ".join([f" J_{k}(x_{k}={i}) = {J[k][i]:.2f}" for i in inv.S(k)] ) )
for k in range(inv.N):
    print(", ".join([f"pi_{k}(x_{k}={i}) = {pi[k][i]}" for i in inv.S(k)] ) )
```

```
Inventory control optimal policy/value functions
 J_0(x_0=0) = 3.70,  J_0(x_0=1) = 2.70,  J_0(x_0=2) = 2.82
 J_1(x_1=0) = 2.50,  J_1(x_1=1) = 1.50,  J_1(x_1=2) = 1.68
 J_2(x_2=0) = 1.30,  J_2(x_2=1) = 0.30,  J_2(x_2=2) = 1.10
pi_0(x_0=0) = 1, pi_0(x_0=1) = 0, pi_0(x_0=2) = 0
pi_1(x_1=0) = 1, pi_1(x_1=1) = 0, pi_1(x_1=2) = 0
pi_2(x_2=0) = 1, pi_2(x_2=1) = 0, pi_2(x_2=2) = 0
```

`lecture_02_optimal_dp_g1.py`

`lecture_02_optimal_inventory.py`

Project 1: Pacman



How?

- Define a DP model
- Apply the DP algorithm

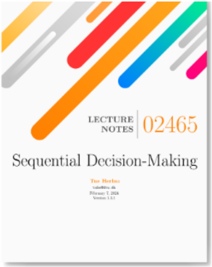
```python
# chapter3dp/pacman_dp_excerpt.py
from irlc.pacman.pacman_environment import PacmanEnvi
from irlc.pacman.gamestate import GameState
from irlc.ex02.dp_model import DPModel
from irlc.ex02.dp import DP_stochastic

class MyPacmanDPModel(DPModel):
    def __init__(self, env, N : int):
        self.env = env
        super().__init__(N)

    def A(self, x : GameState, k):
        # See online documentation!
        return x.A()

    # remember f, g, gN, S, ...

if __name__ == "__main__":
    model = MyPacmanDPModel(PacmanEnvironment())
    J, pi = DP_stochastic(model)
```

https://www2.compute.dtu.dk/courses/02465/models/models_dp.html

https://www2.compute.dtu.dk/courses/02465/models/week2_pacman_g1.html

Part 1 of the project

- you *should* be all set!



- Information
- Models and Environments
- Exercises
- Projects
  - Project 1: Dynamical Programming
  - Project 2: Control theory

Project 1: Dynamical Programming

Note

When?	**Thursday** [redacted]**. Before 23:59**
What?	To get started, download the project description here: 02465project1.pdf
Where?	Under assignments on DTU Learn 02465
What to hand in?	(see project description) • `irlc/project1/Project1_handin_k_of_n.token` • irlc/project1/Latex/02465project1_handin.tex • `irlc/project1/Latex/02465project1_handin.pdf`

Consult the project description (above) for details about the problems. To get the newest version of the course material, please see Making sure your files are up to date.

Creating your hand-in

Tue Herlau.
Sequential decision making.
(Freely available online), 2025.