# 02465: Introduction to reinforcement learning and control

Eligibility traces

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# Lecture Schedule

Dynamical programming

1. The finite-horizon decision problem
   7 February
2. Dynamical Programming
   14 February
3. DP reformulations and introduction to Control
   21 February

Control

4. Discretization and PID control
   28 February
5. Direct methods and control by optimization
   7 March
6. Linear-quadratic problems in control
   14 March
7. Linearization and iterative LQR
   21 March

Reinforcement learning

8. Exploration and Bandits
   28 March
9. Bellmans equations and exact planning
   4 April
10. Monte-carlo methods and TD learning
    11 April
11. Model-Free Control with tabular and linear methods
    25 April
12. **Eligibility traces**
    2 May
13. Deep-Q learning
    9 May

Syllabus: `https://02465material.pages.compute.dtu.dk/02465public`
Help improve lecture by giving feedback on DTU learn

## Reading material:

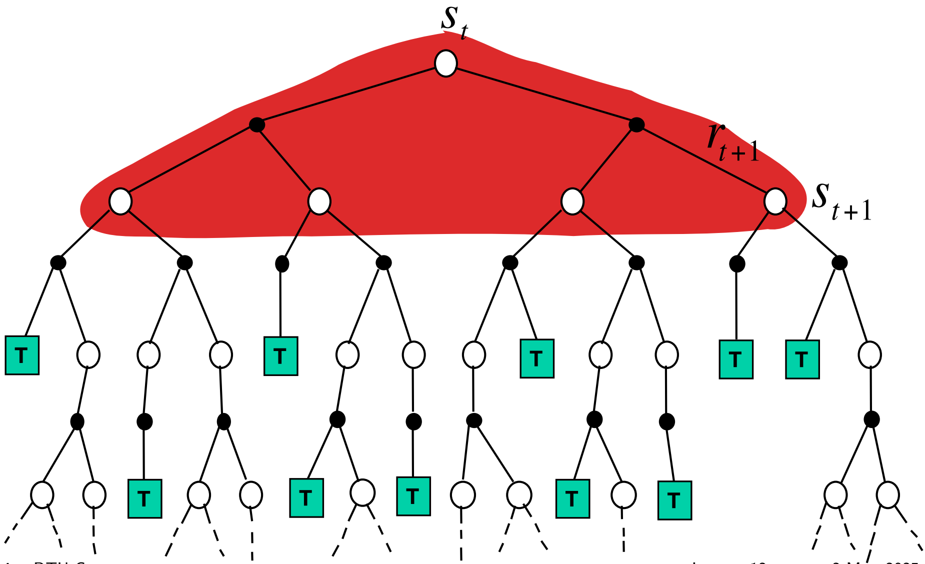
- [SB18, Chapter 10.2; 12-12.7]

### Learning Objectives

- Using the TD-lambda return to interpolate between MC and TD(0)
- Eligibility traces as an efficient implementation of TD(lambda) and Sarsa(lambda)
- Function approximators and Sarsa(lambda)
- The online lambda-return, with emphasis on linear function approximators

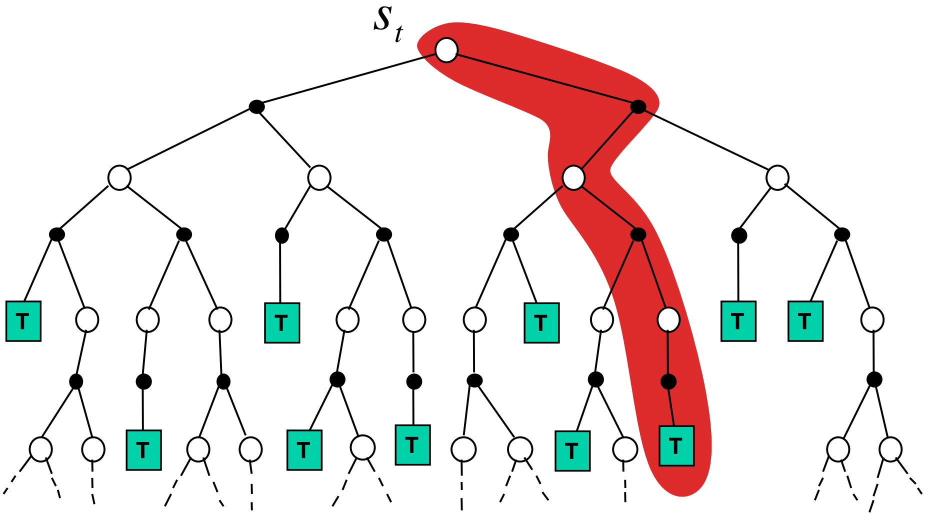
# DP backups

$$V(S_t) \leftarrow \mathbb{E}_\pi\left[R_{t+1} + \gamma V(S_{t+1})\right]$$



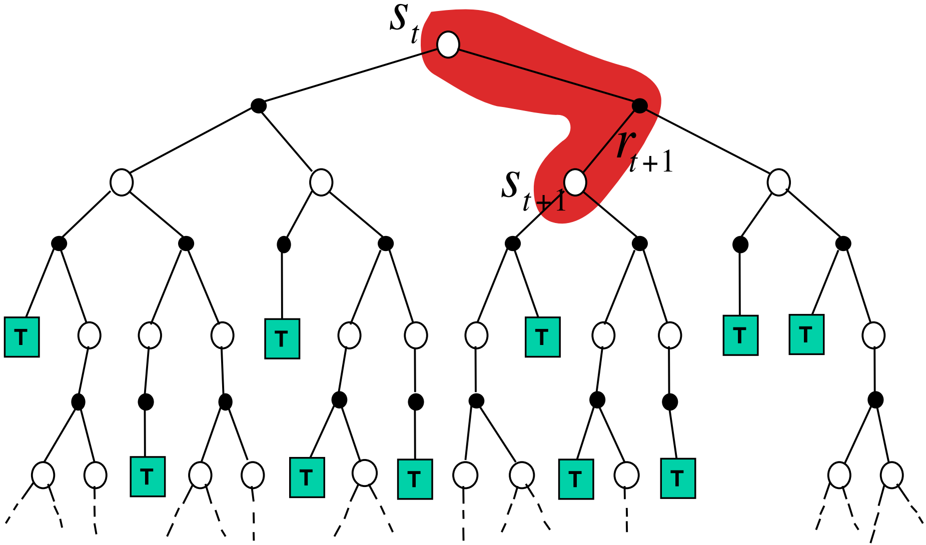
# Last week: MC backups

$$V(S_t) \leftarrow V(S_t) + \alpha \left( G_t - V(S_t) \right)$$

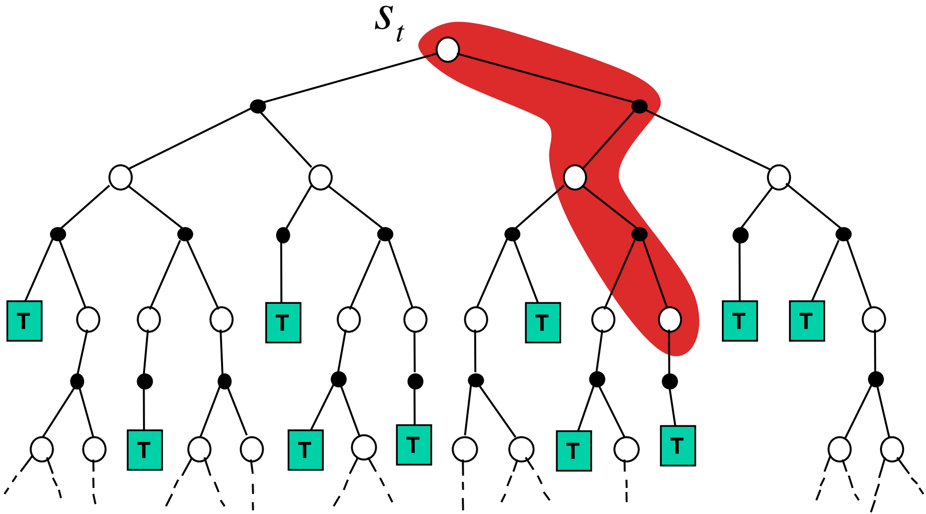


# Last week: TD backups

$$V(S_t) \leftarrow V(S_t) + \alpha \left( R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$



# Last week: $n$-step backup



# General plan

- The $\lambda$-return provides a method to interpolate between TD(0) and Monte-Carlo
- There are **forward** and **backward** variant of $\lambda$-return methods
  - **Forward:** Quite easy to understand; annoying to implement
  - **Backward:** Harder to understand; it has the same updates of value-function but applied immediately. Much easier to implement.
- Additionally, [SB18] distinguishes between (i) regular $\mathrm{TD}(\lambda)$ and a more advanced variant (ii) online TD($\lambda$)
  - ...and the online-version also has a forward and backward view...
  - ...and [SB18] presents the methods in context of function approximators...

**We will focus on the tabular version.**

## From last week: The $n$-step return

- Recall return is $G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots$

$n = 1$: **(TD)** $\quad G_t^{(1)} = R_{t+1} + \gamma G_{t+1}$

$n = 2$: $\quad G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 G_{t+2}$

$n$: $\quad G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n G_{t+n}$

$n = \infty$ **(MC)**: $\quad G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$

- Using the rules of expectations:

$$
\begin{aligned}
v_\pi(s) &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n G_{t+n} | s] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \mathbb{E}\left[\gamma^n G_{t+n} | S_{t+n}\right] | S_t = s\right] \\
&= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n v_\pi(S_{t+n}) | S_t = s]
\end{aligned}
$$

Therefore, the $n$-step return is an estimate of $V(S_t)$

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- This gives $n$-step temporal difference update:

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_{t:t+n} - V(S_t)\right)$$

# Averaging $n$-step returns

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V\left(S_{t+n}\right)$$

- We can average $n$-step returns for different $n$. The estimator

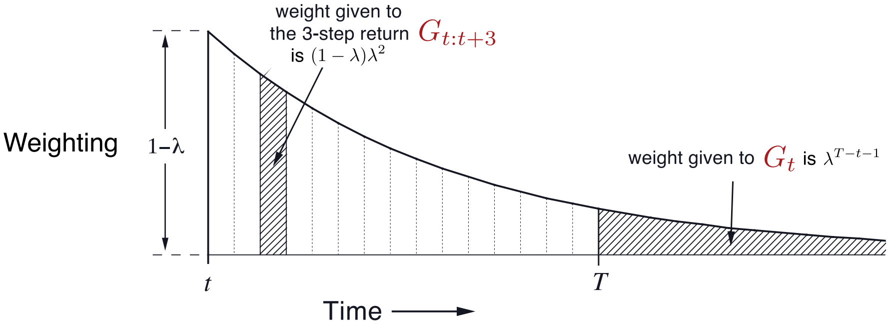
$$\bar{G}_t = \frac{1}{3} G_{t:t+2} + \frac{2}{3} G_{t:t+4}$$

  is still an estimator of the return
- More generally assuming that $\sum_{i=1}^{\infty} w_i = 1$ then

$$\bar{G}_t = \sum_{i=1}^{\infty} w_i G_{t:t+i}$$

  is an estimator of the return

# The $\lambda$-return

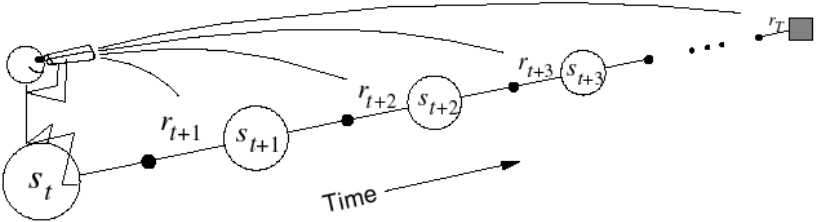



- Combine returns $G_{t:t+n}$ using weights $(1-\lambda)\lambda^{n-1}$ (note $\sum_{n=1}^{\infty}(1-\lambda)\lambda^{n-1} = 1$)

$$G_t^\lambda \doteq (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_{t:t+n}$$

- For $t+n > T$ it is the case that $G_{t:t+n} = G_t$:

**$\lambda$-return:** $$G_t^\lambda = (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_{t:t+n} + \lambda^{T-t-1}G_t$$

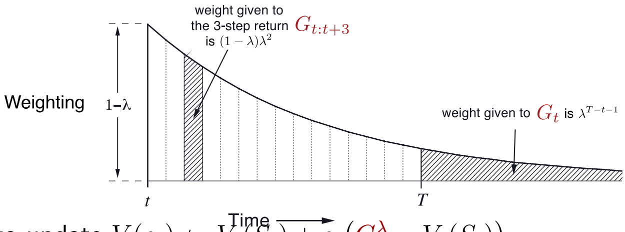



- Forward-view $\mathrm{TD}(\lambda)$ update rule is

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^\lambda - V(S_t)\right)$$

- Forward-view $\mathrm{TD}(\lambda)$ looks into the future to compute $G_t^\lambda$
- Like MC, it can only be computed from complete episodes
- Theoretically simple, but computationally impractical

# Backwards $\mathrm{TD}(\lambda)$




- We want to update $V(s_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_{t:t+n} + \lambda^{T-t-1}G_t$$
$$= (1-\lambda)G_{t:t+1} + (1-\lambda)\lambda G_{t:t+2} + (1-\lambda)\lambda^2 G_{t:t+3} + \cdots + \lambda^{T-t-1}G_t$$

- The return $G_t^\lambda$ includes the term $(1-\lambda)\lambda^2 G_{t:t+3}$
- This means $V(s_t)$ is updated towards

$$G_t^\lambda = \cdots + (1-\lambda)\lambda^2(R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3})) + \cdots$$

- **Idea:** Wait until time $t+3$, compute above terms and update $V(s_t)$ **in the past**
- The further **in the future** a term $R_{t+n}$ is, the less it influences **past** term $V(s_t)$

# Eligibility trace

- The eligibility trace $E_t$ is just af function of states: $E_t : \mathcal{S} \to \mathbb{R}$
- Measures both how frequent and how recent a state was visited
- Initialized to $E_{t=0}(s) = 0$
- Updated at each time step as

$$E_t(s) = \begin{cases} \gamma\lambda E_{t-1}(s) & \text{if } s \neq s_t \\ \gamma\lambda E_{t-1}(s) + 1 & \text{if } s = s_t \end{cases}$$

  - States decay at a rate of $\gamma\lambda$
  - Each time they are visited they get a bonus of $+1$,

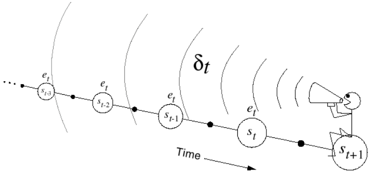
# Backward view TD($\lambda$)

- Initialize value function for each state.
- At start of each episode, initialize eligibility trace for each state to $E(s) = 0$
- For each transition $S_t = s \to S_{t+1} = s'$, giving reward $R_{t+1} = r$, compute ordinary TD error

$$\delta_t = r + \gamma V(s') - V(s)$$

- Update eligibility trace

$$E_t(s) = E_t(s) + 1$$




- For every state $s$ where $E_t(s) > 0$ update

$$V(s) \leftarrow V(s) + \alpha\delta E(s)$$
$$E(s) \leftarrow \gamma\lambda E(s)$$

- See `http://incompleteideas.net/book/ebook/node75.html`

# $\lambda = 0$ is equivalent TD(0)

- When $\lambda = 0$ only the current state is updated:

$$E_t(s) = 1 \text{ if and only if } s = S_t$$
$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$

- This means $\mathrm{TD}(\lambda)$ is equal to $\mathrm{TD}(0)$ when $\lambda = 0$

# Equivalence of forward/Backward $\mathrm{TD}(\lambda)$

Suppose a state $S_t = s$ is visited just once at time step $t$

Forward-view The change in value-function $V(s)$ in the forward-view update is $\alpha(G_t^\lambda - V(S_t))$

Eligibility traces Implied update is:

- At $t$ we change $E(S_t = s) = 1$
- In subsequent steps we iterate

$$V(s) \leftarrow V(s) + \alpha\delta E(s)$$
$$E(s) \leftarrow \gamma\lambda E(s)$$

- The last update means that at step $t+n$ we have $E(s) = (\gamma\lambda)^n$
- Total change to value function $V(s)$ is therefore

$$\alpha\left(\delta_t + \gamma\lambda\delta_{t+1} + (\gamma\lambda)^2\delta_{t+2} + \ldots\right)$$

Are these two updates the same (is the red stuff equal)?

## Proof:

Recall $G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$

$$\begin{aligned}
G_t^\lambda - V(S_t) &= -V(S_t) + (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n} \\
&= -V(S_t) + \left( \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n} \right) + \left( \sum_{n=1}^{\infty} -\lambda^n G_{t:t+n} \right) \\
&= -V(S_t) + \left( G_{t:t+1} + \sum_{n=2}^{\infty} \lambda^{n-1} G_{t:t+n} \right) + \left( \sum_{n=2}^{\infty} -\lambda^{n-1} G_{t:t+n-1} \right) \\
&= G_{t:t+1} - V(S_t) + \sum_{n=2}^{\infty} \lambda^{n-1} \left( G_{t:t+n} - G_{t:t+n-1} \right)
\end{aligned}$$

Recall that $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$ then

$$\begin{aligned}
G_{t:t+n} - G_{t:t+n-1} &= \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n}) - \gamma^{n-1} V(S_{t+n-1}) \\
&= \gamma^{n-1} \delta_{t+n-1}
\end{aligned}$$

# Proof II

$$
\begin{aligned}
G_t^\lambda - V(S_t) &= G_{t:t+1} - V(S_t) + \sum_{n=2}^{\infty} \lambda^{n-1} \left(G_{t:t+n} - G_{t:t+n-1}\right) \\
&= \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right) + \sum_{n=2}^{\infty} \lambda^{n-1} \left(\gamma^{n-1} \delta_{t+n-1}\right) \\
&= (\gamma\lambda)^0 \delta_t + \sum_{n=2}^{\infty} (\gamma\lambda)^{n-1} \delta_{t+n-1} \\
&= (\gamma\lambda)^0 \delta_t + (\gamma\lambda)^1 \delta_{t+1} + (\gamma\lambda)^2 \delta_{t+2} + \cdots
\end{aligned}
$$

# Forward/Backward TD

Suppose a state $S_t = s$ is visited just once at time step $t$

Forward-view The change in value-function $V(s)$ in the forward-view update is $\alpha(G_t^\lambda - V(S_t))$

Eligibility traces Implied update is:

- At $t$ we change $E(S_t = s) = 1$
- In subsequent steps we iterate

$$V(s) \leftarrow V(s) + \alpha \delta E(s)$$
$$E(s) \leftarrow \gamma \lambda E(s)$$

- The last update means that at step $t + n$ we have $E(s) = (\gamma\lambda)^n$
- Total change to value function $V(s)$ is therefore

$$\alpha \left( \delta_t + \gamma\lambda\delta_{t+1} + (\gamma\lambda)^2 \delta_{t+2} + \dots \right)$$

Same updates!

# Forward/Backward TD (Summary)

- Forward view is just using $G_t^\lambda$ is an estimate of return
- Forward/Backwards TD are equivalent
  - Both change the value function the same way
  - Forward-view just changes value-function during an episode
- $\mathrm{TD}(\lambda = 0)$ is equivalent to $\mathrm{TD}(0)$
- $\mathrm{TD}(1)$ corresponds to MC

# From last week: $n$-step Sarsa

Recall the decomposition:

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n G_{t+n}$$

- As before:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n G_{t+n} | S_t = s, A_t = a] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n q_\pi(S_{t+n}, A_{t+n}) | S_t = s, A_t = a] \end{aligned}$$

- Therefore, the following $n$-step action-value return is an unbiased estimate of $q_\pi$

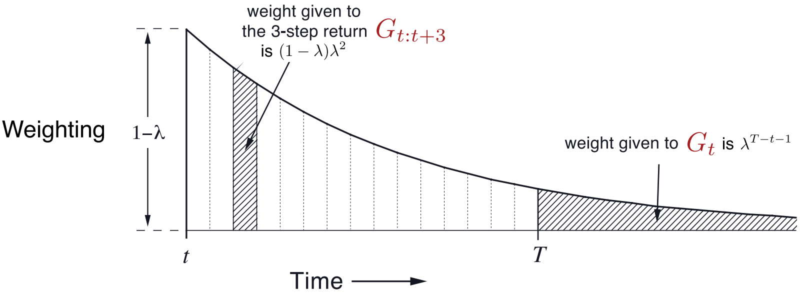
$$q_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n q_\pi \left(S_{t+n}, A_{t+n}\right)$$

- Suggest the following bootstrap update of the action-value function

$$Q\left(S_t, A_t\right) \leftarrow Q\left(S_t, A_t\right) + \alpha \left(q_t^{(n)} - Q\left(S_t, A_t\right)\right)$$

`lecture_12_sarsa_nstep_open.py`

# Forward-view Sarsa




- Use weights to combine returns $q_{t:t+n}$

$$q_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n Q\left(S_{t+n}, A_{t+n}\right)$$

- For $t + n \geq T$ it is the case $q_{t:t+n} = G_t$:

$$q_t^\lambda = (1-\lambda) \sum_{n=1}^{T-t-1} \lambda^{n-1} q_{t:t+n} + \lambda^{T-t-1} G_t$$

- We therefore obtain the following generalized update rule

$$Q\left(S_t, A_t\right) \leftarrow Q\left(S_t, A_t\right) + \alpha \left(q_t^\lambda - Q\left(S_t, A_t\right)\right)$$

# Backward view Sarsa($\lambda$)

- We once more introduce an eligibility trace $E_t$, updated as before:

$$E_t(s,a) = \begin{cases} \gamma\lambda E_{t-1}(s,a) + 1 & \text{if } s = s_t \text{ and } a = a_t; \\ \gamma\lambda E_{t-1}(s,a) & \text{otherwise.} \end{cases} \quad \text{for all } s, a$$

- Each each step, given $(s, a, r, s')$, update

$$\begin{aligned} \delta_t &= R_{t+1} + \gamma Q\left(S_{t+1}, A_{t+1}\right) - Q\left(S_t, A_t\right) \\ Q(s,a) &\leftarrow Q(s,a) + \alpha\delta_t E_t(s,a) \end{aligned}$$

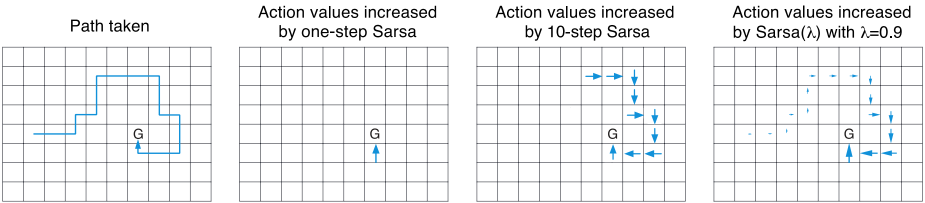
# Sarsa($\lambda$) control algorithm (tabular version)

See `http://incompleteideas.net/book/first/ebook/node77.html`

Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
Repeat (for each episode):
  $E(s,a) = 0$, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$
  Initialize $S$, $A$
  Repeat (for each step of episode):
    Take action $A$, observe $R$, $S'$
    Choose $A'$ from $S'$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
    $\delta \leftarrow R + \gamma Q(S', A') - Q(S, A)$
    $E(S,A) \leftarrow E(S,A) + 1$
    For all $s \in \mathcal{S}, a \in \mathcal{A}(s)$:
      $Q(s,a) \leftarrow Q(s,a) + \alpha \delta E(s,a)$
      $E(s,a) \leftarrow \gamma \lambda E(s,a)$
    $S \leftarrow S'$; $A \leftarrow A'$
  until $S$ is terminal

# Implied updates in the Open gridworld example

Recall only terminal state has a reward of $+1$




`lecture_12_sarsa_open.py`, `lecture_12_mc_open.py`, `lecture_12_sarsa_lambda_open.py`

# From last time: Feature vectors and linear representations

- Represent value function by a linear combination of features

$$\hat{v}(s, \mathbf{w}) = \mathbf{x}(s)^\top \mathbf{w}, \quad \boldsymbol{w} \in \mathbb{R}^d$$

  Where **feature vector** is defined as:

$$\mathbf{x}(s) = \begin{bmatrix} \mathbf{x}_1(s) \\ \vdots \\ \mathbf{x}_d(s) \end{bmatrix}$$

- The gradient is simply:

$$\nabla \hat{v}(s, \mathbf{w}) = \mathbf{x}(s)$$

- For $Q$-values we only need to change the feature vector:

$$\hat{q}(s, a, \boldsymbol{w}) = \boldsymbol{x}(s, a)^\top \boldsymbol{w}$$

# From last time: implementation details

- TD learning

$$\begin{aligned} V(s) &\leftarrow V(s) + \alpha(r + \gamma V(s') - V(s)) \\ \boldsymbol{w} &\leftarrow \boldsymbol{w} + \alpha\left(r + \gamma \hat{v}(s', \boldsymbol{w}) - \hat{v}(s, \boldsymbol{w})\right) \nabla \hat{v}(s, \boldsymbol{w}) \end{aligned}$$

- Sarsa learning

$$\begin{aligned} q(s,a) &\leftarrow q(s,a) + \alpha\left(r + \gamma q(s', a') - q(s,a)\right) \\ \boldsymbol{w} &\leftarrow \boldsymbol{w} + \alpha\left(r + \gamma \hat{q}(s', a', \boldsymbol{w}) - \hat{q}(s, a, \boldsymbol{w})\right) \nabla \hat{q}(s, a, \boldsymbol{w}) \end{aligned}$$

- Using a general estimator:

$$\begin{aligned} q(s,a) &\leftarrow q(s,a) + \alpha\left(G - q(s,a)\right) \\ \boldsymbol{w} &\leftarrow \boldsymbol{w} + \alpha\left(G - \hat{q}(s, a, \boldsymbol{w})\right) \nabla \hat{q}(s, a, \boldsymbol{w}) \end{aligned}$$

# Forward and backward view

Assuming linear function approximators: $\nabla \hat{q}(s, a, \boldsymbol{w}) = \boldsymbol{x}(s, a)$

- Forward view Sarsa($\lambda$) is exactly as before

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \alpha \left( G_t^\lambda - \hat{q}(s, a, \boldsymbol{w}) \right) \nabla \hat{q}(s, a, \boldsymbol{w})$$

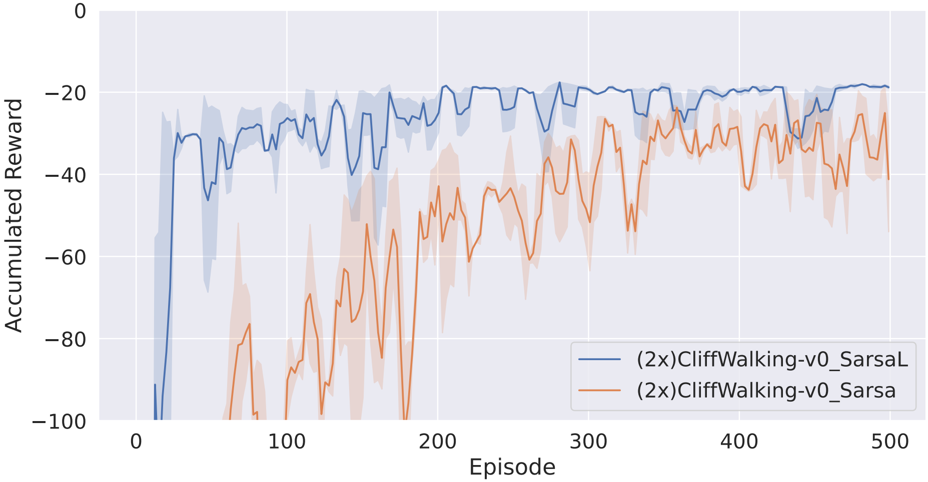
- Keep track of terms that include which gradient to get **backward view** of Sarsa($\lambda$):

$$\begin{aligned}
\delta_t &= R_{t+1} + \gamma \hat{q}\left(S_{t+1}, A_{t+1}, \mathbf{w}_t\right) - \hat{q}\left(S_t, A_t, \mathbf{w}_t\right) \\
\boldsymbol{z}_t &= \gamma \lambda \boldsymbol{z}_{t-1} + \nabla \hat{q}(S_t, A_t, \boldsymbol{w}_t) \\
\boldsymbol{w}_{t+1} &\leftarrow \boldsymbol{w}_t + \alpha \delta_t \boldsymbol{z}_t
\end{aligned}$$

- The gradient plays the role of state-action pairs visited. It is propagated into the future but attenuated by $\gamma\lambda$
- A change in the past (gradient) which lead to a **poor** (or good) result $\delta_t$ will be **penalized** (promoted)
- Forward/backward view equivalent in the linear case

# Cliffwalk example

Comparison of $\mathrm{Sarsa}(\lambda)$ and Sarsa on the cliffwalk example




(Note that results are somewhat sensitive to the to learning rate)

# Quiz: Exam problem spring 2023

Which one of the following questions are correct?

**a.** $\mathrm{TD}(\lambda)$ cannot be used with function approximators

**b.** The role of the eligibility trace is to let reward obtained earlier in an episode affect the change in the value function later in the episode

**c.** The eligibility trace cannot be negative

**d.** The eligibility trace is a measure of the amount of reward obtained in a given state weighted by an exponential factor

**e.** Don't know.

# Using binary features

**Sarsa($\lambda$) with binary features and linear function approximation for estimating $\mathbf{w}^\top \mathbf{x} \approx q_\pi$ or $q_*$**

Input: a function $\mathcal{F}(s, a)$ returning the set of (indices of) active features for $s, a$
Input: a policy $\pi$ (if estimating $q_\pi$)
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0, 1]$
Initialize: $\mathbf{w} = (w_1, \dots, w_d)^\top \in \mathbb{R}^d$ (e.g., $\mathbf{w} = \mathbf{0}$), $\mathbf{z} = (z_1, \dots, z_d)^\top \in \mathbb{R}^d$

Loop for each episode:
- Initialize $S$
- Choose $A \sim \pi(\cdot|S)$ or $\varepsilon$-greedy according to $\hat{q}(S, \cdot, \mathbf{w})$
- $\mathbf{z} \leftarrow \mathbf{0}$
- Loop for each step of episode:
  - Take action $A$, observe $R, S'$
  - ~~$\delta \leftarrow R$~~ $\delta \leftarrow R - \boldsymbol{w}^\top \boldsymbol{x}$
  - ~~Loop for $i$ in $\mathcal{F}(S, A)$:~~
    - ~~$\delta \leftarrow \delta - w_i$~~
    - ~~$z_i \leftarrow z_i + 1$~~ $z \leftarrow z + x$ (accumulating traces)
    - ~~or $z_i \leftarrow 1$~~ ~~(replacing traces)~~
  - If $S'$ is terminal then:
    - $\mathbf{w} \leftarrow \mathbf{w} + \alpha \delta \mathbf{z}$
    - Go to next episode
  - Choose $A' \sim \pi(\cdot|S')$ or near greedily $\sim \hat{q}(S', \cdot, \mathbf{w})$
  - ~~Loop for $i$ in $\mathcal{F}(S', A')$: $\delta \leftarrow \delta + \gamma w_i$~~ $\delta \leftarrow \delta + \gamma \boldsymbol{w}^\top \boldsymbol{x}'$
  - $\mathbf{w} \leftarrow \mathbf{w} + \alpha \delta \mathbf{z}$
  - $\mathbf{z} \leftarrow \gamma \lambda \mathbf{z}$
  - $S \leftarrow S'; A \leftarrow A'$

# Truncated, online and true online $\lambda$-return algorithms (advanced)

- Recall the $\lambda$-return is defined as:

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{T-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t-1} G_t$$

- Each $G_t$ is an estimate of the return and the sum of the weights is 1
- More generally the **truncated $\lambda$-return** estimator is

$$G_{t:h}^\lambda = (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{h-t-1} G_{t:h}, \quad 0 \leq t < h \leq T$$

# Using the estimator

- Recall the forward-view $\mathrm{TD}(\lambda)$ algorithm:

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t^\lambda - V(S_t))$$

- The **truncated $\lambda$** return fixes $h = n$ and do:

$$V(S_t) \leftarrow V(S_t) + \alpha(G_{t:t+n}^\lambda - V(S_t))$$

- Or as weight updates

$$\boldsymbol{w}_{t+n} = \boldsymbol{w}_{t+n-1} + \alpha \left( G_{t:t+n}^\lambda - \hat{v}(S_t, \boldsymbol{w}_{t+n-1}) \right) \nabla \hat{v}(S_t, \boldsymbol{w}_{t+n-1})$$

- This requires a fixed $n$ and that we store previous results. Can we do better?

# Online $\lambda$-return

$$G^\lambda_{t:h} = (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{h-t-1} G_{t:h}, \quad 0 \le t < h \le T$$

- Once we have observed $h$ steps of an episode, we can evaluate

$$G^\lambda_{0:h}, G^\lambda_{1:h}, \dots, G^\lambda_{h-1:h}$$

- **Online $\lambda$-return**: After $h$ steps, perform $h$ updates corresponding to all $h$ returns
- Repeat each time $h$ is increased

$$\begin{aligned}
h=1: \quad & \mathbf{w}_1^1 \doteq \mathbf{w}_0^1 + \alpha\left[G^\lambda_{0:1} - \hat{v}(S_0,\mathbf{w}_0^1)\right]\nabla\hat{v}(S_0,\mathbf{w}_0^1), \\
\\
h=2: \quad & \mathbf{w}_1^2 \doteq \mathbf{w}_0^2 + \alpha\left[G^\lambda_{0:2} - \hat{v}(S_0,\mathbf{w}_0^2)\right]\nabla\hat{v}(S_0,\mathbf{w}_0^2), \\
& \mathbf{w}_2^2 \doteq \mathbf{w}_1^2 + \alpha\left[G^\lambda_{1:2} - \hat{v}(S_1,\mathbf{w}_1^2)\right]\nabla\hat{v}(S_1,\mathbf{w}_1^2), \\
\\
h=3: \quad & \mathbf{w}_1^3 \doteq \mathbf{w}_0^3 + \alpha\left[G^\lambda_{0:3} - \hat{v}(S_0,\mathbf{w}_0^3)\right]\nabla\hat{v}(S_0,\mathbf{w}_0^3), \\
& \mathbf{w}_2^3 \doteq \mathbf{w}_1^3 + \alpha\left[G^\lambda_{1:3} - \hat{v}(S_1,\mathbf{w}_1^3)\right]\nabla\hat{v}(S_1,\mathbf{w}_1^3), \\
& \mathbf{w}_3^3 \doteq \mathbf{w}_2^3 + \alpha\left[G^\lambda_{2:3} - \hat{v}(S_2,\mathbf{w}_2^3)\right]\nabla\hat{v}(S_2,\mathbf{w}_2^3).
\end{aligned}$$

- I.e. for each new step $h-1 \rightarrow h$ repeat $t = 0, \dots, h-1$:

$$\boldsymbol{w}^h_{t+1} = \boldsymbol{w}^h_t + \alpha\left[G^\lambda_{t:h} - \hat{v}(S_t, \boldsymbol{w}^h_t)\right]\nabla\hat{v}(S_t, \boldsymbol{w}^h_t)$$

# True online $\mathrm{TD}(\lambda)$

- Online $\mathrm{TD}(\lambda)$ is computationally very wasteful
- For linear function approximators online $\mathrm{TD}(\lambda)$ allows a backwards view known as **True online** $\mathrm{TD}(\lambda)$

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t + \alpha\delta_t \boldsymbol{z}_t + \alpha(\boldsymbol{w}_t^\top \boldsymbol{x}_t - \boldsymbol{w}_{t-1}^\top \boldsymbol{x}_t)(\boldsymbol{z}_t - \boldsymbol{x}_t)$$

$$\boldsymbol{z}_t = \gamma\lambda \boldsymbol{z}_{t-1} + (1 - \alpha\gamma\lambda \boldsymbol{z}_{t-1}^\top \boldsymbol{x}_t)\boldsymbol{x}_t$$

- The control algorithm is **true online Sarsa**$(\lambda)$

# True online Sarsa($\lambda$)

**True online Sarsa($\lambda$) for estimating $\mathbf{w}^\top \mathbf{x} \approx q_\pi$ or $q_*$**

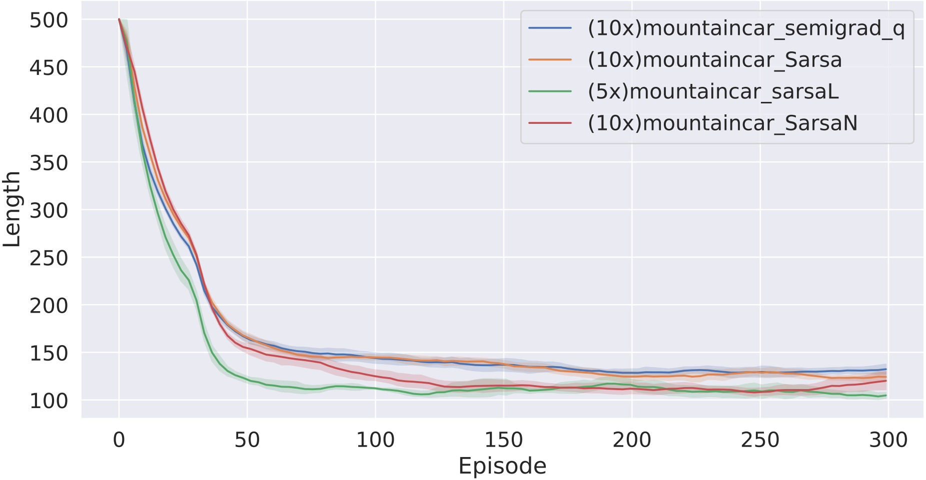
Input: a feature function $\mathbf{x} : \mathcal{S}^+ \times \mathcal{A} \rightarrow \mathbb{R}^d$ such that $\mathbf{x}(terminal, \cdot) = \mathbf{0}$
Input: a policy $\pi$ (if estimating $q_\pi$)
Algorithm parameters: step size $\alpha > 0$, trace decay rate $\lambda \in [0, 1]$
Initialize: $\mathbf{w} \in \mathbb{R}^d$ (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
  Initialize $S$
  Choose $A \sim \pi(\cdot|S)$ or near greedily from $S$ using $\mathbf{w}$
  $\mathbf{x} \leftarrow \mathbf{x}(S, A)$
  $\mathbf{z} \leftarrow \mathbf{0}$
  $Q_{old} \leftarrow 0$
  Loop for each step of episode:
  |  Take action $A$, observe $R$, $S'$
  |  Choose $A' \sim \pi(\cdot|S')$ or near greedily from $S'$ using $\mathbf{w}$
  |  $\mathbf{x}' \leftarrow \mathbf{x}(S', A')$
  |  $Q \leftarrow \mathbf{w}^\top \mathbf{x}$
  |  $Q' \leftarrow \mathbf{w}^\top \mathbf{x}'$
  |  $\delta \leftarrow R + \gamma Q' - Q$
  |  $\mathbf{z} \leftarrow \gamma\lambda\mathbf{z} + \left(1 - \alpha\gamma\lambda\mathbf{z}^\top\mathbf{x}\right)\mathbf{x}$
  |  $\mathbf{w} \leftarrow \mathbf{w} + \alpha(\delta + Q - Q_{old})\mathbf{z} - \alpha(Q - Q_{old})\mathbf{x}$
  |  $Q_{old} \leftarrow Q'$
  |  $\mathbf{x} \leftarrow \mathbf{x}'$
  |  $A \leftarrow A'$
  until $S'$ is terminal

(we will implement this during the exercises)

# Mountaincar example

Comparison of $\mathrm{Sarsa}(\lambda)$ and Sarsa on the Mountaincar example

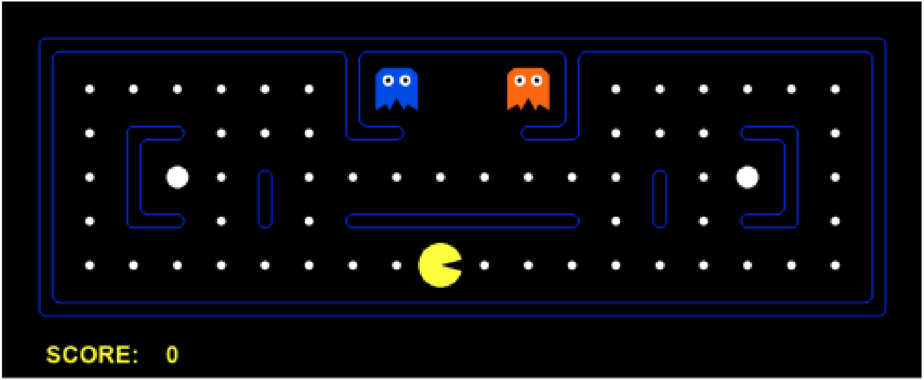


Richard S. Sutton and Andrew G. Barto.
*Reinforcement Learning: An Introduction*.
The MIT Press, second edition, 2018.
(Freely available online).

# Appendix

# A more challenging pacman environment

- Use successor representation: $\hat{q}(s, a, \boldsymbol{w}) = \boldsymbol{x}(s')^\top \boldsymbol{w}$, $s' = f(s, a)$



# A more challenging pacman environment