

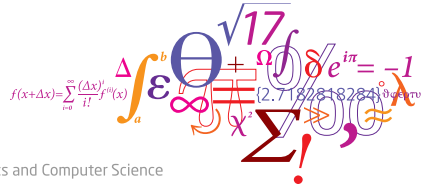
02465: Introduction to reinforcement learning and control

Monte-carlo methods and TD learning

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Lecture Schedule

Dynamical programming

- 1 The finite-horizon decision problem
7 February
- 2 Dynamical Programming
14 February
- 3 DP reformulations and introduction to Control
21 February

Control

- 4 Discretization and PID control
28 February
- 5 Direct methods and control by optimization
7 March
- 6 Linear-quadratic problems in control
14 March
- 7 Linearization and iterative LQR
21 March

Reinforcement learning

- 8 Exploration and Bandits
28 March
- 9 Bellmans equations and exact planning
4 April
- 10 Monte-carlo methods and TD learning
11 April
- 11 Model-Free Control with tabular and linear methods
25 April
- 12 Eligibility traces
2 May
- 13 Deep-Q learning
9 May

Syllabus: <https://02465material.pages.compute.dtu.dk/02465public>
Help improve lecture by giving feedback on DTU learn

Reading material:

- [SB18, Chapter 5-5.4+5.10; 6-6.3]

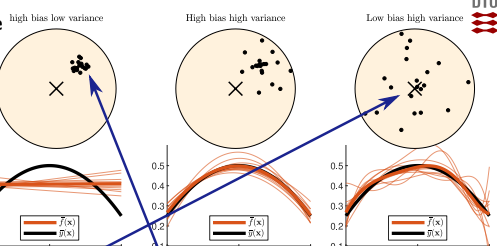
Learning Objectives

- Monte-Carlo rollouts to estimate the value function
- Monte-carlo rollouts for control
- Temporal difference learning

Housekeeping

- DTU Course survey is online; remember to give your TAs feedback
 - Remember that concrete feedback is easier to act on
- This week the theoretical exercise is a bit longer because MC methods are less nice to implement (but try the TD(0) problem)

Bias/Variance



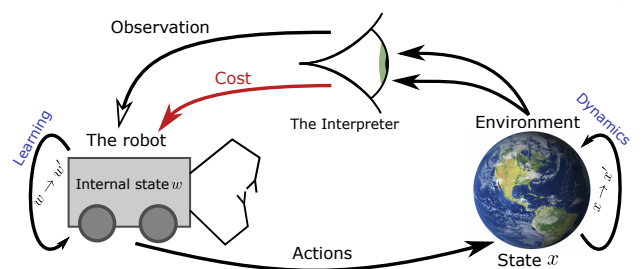
- An estimator can be **unbiased** and **biased**

$$\mathbb{E}[X] \approx \frac{1}{n} \sum_{i=1}^n x_i = \hat{\mu}_1, \quad \mathbb{E}[X] \approx \frac{1}{n + \sqrt{n}} \sum_{i=1}^n x_i = \hat{\mu}_2$$

- A biased estimator is **asymptotically consistent** if it is unbiased as $n \rightarrow \infty$:

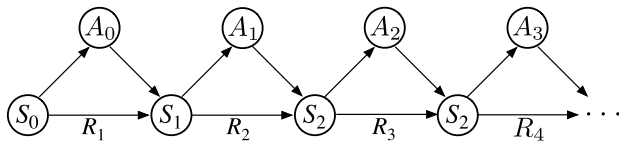
$$\hat{\mu}_2 = \frac{1}{n + \sqrt{n}} \sum_{i=1}^n x_i = \frac{n}{n + \sqrt{n}} \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{1 + \frac{1}{\sqrt{n}}} \hat{\mu}_1$$

Monte-Carlo estimation and control



- **Model free**; requires no knowledge of MDP
- Uses simplest possible idea: State value = mean return
- **Limitation**: Can only be used on episodic MDPs

From last time



Value and action-value function

The **state-value function** $v_\pi(s)$ is the expected return starting in s and assuming actions are selected using π :

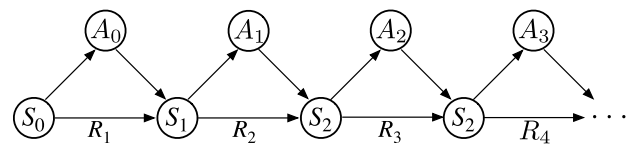
$$v_\pi(s) = \mathbb{E}_\pi [G_t | S_t = s], \quad A_t \sim \pi(\cdot | S_t)$$

The **action-value function** $q_\pi(s, a)$ is the expected return starting in s , taking action a , and then follow π :

$$q_\pi(s, a) = \mathbb{E}_\pi [G_t | S_t = s, A_t = a]$$

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots$$

Monte Carlo evaluation: Idea



- Recall return defined by

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots$$

- Each rollout by a policy π , starting in s , is an estimate of

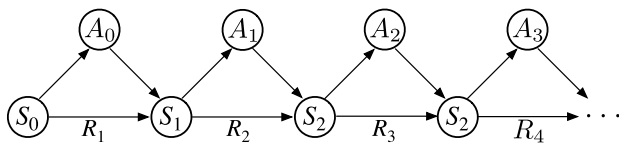
$$v_\pi(s) = \mathbb{E}_\pi [G_t | S_t = s], \quad A_t \sim \pi(\cdot | S_t)$$

- Each rollout of π , starting in s and taking action a , is an estimate

$$q_\pi(s, a) = \mathbb{E}_\pi [G_t | S_t = s, A_t = a]$$

`unf_policy_evaluation_gridworld.py` `mc_value_first_one_state.py`,
`mc_value_first_one_state_b.py`

Every-Visit Monte-Carlo value estimation



Simulate an episode of experience $s_0, a_0, r_1, s_1, a_1, r_2, \dots, r_T$ using π

- First** step t we visit a state s **Every** step t we visit a state s
- Increment number of times s visited $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value estimate is $V(s) = \frac{S(s)}{N(s)}$

Value estimate converge to $v_\pi(s)$

- Every-visit is biased but consistent (non-trivial)**

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π : $S_0, A_0, R_1, S_1, A_1, R_2, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in S_0, S_1, \dots, S_{t-1} :

Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

`lecture_09_mc_value_first.py`, `mc_value_every_one_state.py`,
`lecture_09_mc_value_every.py`

Quiz: A two-state gridworld

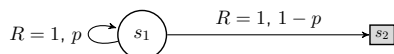


Figure: A simple MRP with one non-terminal state s_1 and one terminal state s_2 . With probability p the process stay in s_1 and with probability $1-p$ it jumps to s_2 , and in each jump it gets a reward of $R_t = 1$.

Assume that $\gamma = 1$ and we evaluate the agent for the episode:

- s_1, s_1, s_1, s_2 (accumulated reward 3)

What is the estimated return using (1) first visit and (2) every-visit Monte-Carlo?

- First-visit: $V^{\text{first}}(s_1) = 3$, every-visit: $V^{\text{every}}(s_1) = 2$
- First-visit: $V^{\text{first}}(s_1) = 3$, every-visit: $V^{\text{every}}(s_1) = 3$
- First-visit: $V^{\text{first}}(s_1) = 1$, every-visit: $V^{\text{every}}(s_1) = 2$
- First-visit: $V^{\text{first}}(s_1) = 1$, every-visit: $V^{\text{every}}(s_1) = 3$

Incremental mean

Recall from the bandit-lecture that:

$$\begin{aligned} \mu_n &= \frac{1}{n} \sum_{i=1}^n x_i \\ &= \frac{1}{n} \left(x_n + \frac{n-1}{n-1} \sum_{i=1}^{n-1} x_i \right) \\ &= \frac{1}{n} x_n + \mu_{n-1} - \frac{1}{n} \mu_{n-1} \\ &= \mu_{n-1} + \frac{1}{n} (x_n - \mu_{n-1}) \end{aligned}$$

Incremental updates



First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$
 $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π : $S_0, A_0, R_1, S_1, A_1, R_2, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in S_0, S_1, \dots, S_{t-1} :

Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

- **No α :** Update $N(s) \leftarrow N(s) + 1$, $S(s) \leftarrow S(s) + G$ and estimate $V(s) = \frac{S(s)}{N(s)}$
- **With α :** $V(s) \leftarrow V(s) + \alpha(G - V(s))$

TD(0) value-function estimation



Bellman equation

- Recursive decomposition of value function

$$v_\pi(s) = \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$$

- **Observation:** By the MC principle $R_{t+1} + \gamma v_\pi(S_{t+1})$ is an estimate of $v_\pi(s)$
- The estimate of v involves v . This is known as **bootstrapping**.
 - TD(0) uses **bootstrapping**
 - Monte-Carlo does not use **bootstrapping**

TD(0)

- MC learning: G_t estimate of $v_\pi(s)$; update:

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t))$$

- TD learning: $R_{t+1} + \gamma v_\pi(S_{t+1})$ estimate of $v_\pi(s)$; update:

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

Tabular TD(0) for estimating v_π

Input: the policy π to be evaluated

Algorithm parameter: step size $\alpha \in (0, 1]$

Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\text{terminal}) = 0$

Loop for each episode:

Initialize S

Loop for each step of episode:

$A \leftarrow$ action given by π for S

Take action A , observe R, S'

$V(S) \leftarrow V(S) + \alpha[R + \gamma V(S') - V(S)]$

$S \leftarrow S'$

until S is terminal



Comparisons



- TD can learn **online**
 - TD can learn after each step
 - MC must wait until the end of episode to learn
- TD can learn **without** knowing the final outcome
 - TD can learn from incomplete sequences
 - MC requires complete sequences
- TD works in **non-episodic** environments
 - TD work in non-terminating environments
 - MC only works in episodic environments

Bias variance tradeoff



- Return $G_t = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-t-1} R_T$ is an **unbiased** estimate of $v_\pi(S_t)$
- True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ is an **unbiased** estimate of $v_\pi(S_t)$
- Actual TD target $R_{t+1} + \gamma V(S_{t+1})$ is a **biased** estimate of $v_\pi(S_t)$
- TD target has lower variance than the return-target G_t :
 - Return is a sum over rewards involving many steps
 - TD target only depend on one action, transition, reward triplet

Bias variance tradeoff continued



- (first-visit) MC has high variance, no bias
 - Good convergence properties
 - (..even with function approximators)
 - Not very sensitive to initial value of V
 - Simple to use/understand (a bit annoying to implement)
- TD has low variance, some bias
 - Usually more efficient to learn than MC
 - Asymptotically consistent
 - (but not always with function approximators)
 - More sensitive to initial value (bootstrapping)

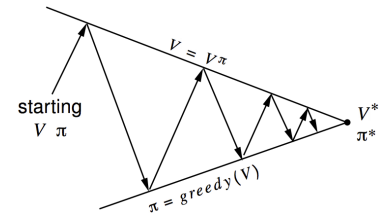
MC vs. TD



- TD exploits Markov property
 - Usually more efficient in Markov environments
- MC does not exploit Markov property:
 - Usually more efficient in non-Markovian environments

Control

How to turn value-function iteration to controller

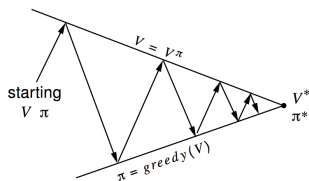


- Given initial policy π
- Compute v_π using policy evaluation
- Let π' be greedy policy wrt. v_π
- Repeat until $v_\pi = v_{\pi'}$

`unf_policy_improvement_gridworld.py`

Control

Two problems



- **Problem:** We need a model to do policy improvement

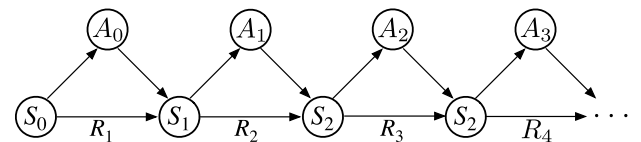
$$\pi'(s) = \arg \max_a \mathbb{E}[R + \gamma V(S') | s, a]$$
- **Solution:** Estimate/save $q_\pi(s, a)$ instead of $v_\pi(s)$:

$$\pi'(s) = \arg \max_a Q(s, a)$$
- **Problem:** Acting greedily means all $q(s, a)$ -values are not estimated by MC
- **Solution:** Be ϵ -greedy in π

$$\pi(a|s) = \begin{cases} \epsilon/m + 1 - \epsilon & \text{if } a^* = \arg \max_{a \in \mathcal{A}} Q(s, a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

Control

First-Visit Monte-Carlo value estimation



Simulate an episode of experience $s_0, a_0, r_1, s_1, a_1, r_2, \dots, r_T$ using π

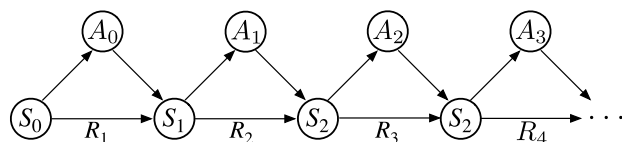
- **First** step t we visit a state s
- Increment number of times s visited $N(s) \leftarrow N(s) + 1$
- Increment total return $S(s) \leftarrow S(s) + G_t$
- Value estimate is $V(s) = \frac{S(s)}{N(s)}$

Value estimate converge to $v_\pi(s) = \mathbb{E}[G_t | S_t = s]$

`lecture_10_mc_action_value_first_one_state.py`

Control

First-Visit Monte-Carlo **action-value** estimation



Simulate an episode of experience $s_0, a_0, r_1, s_1, a_1, r_2, \dots, r_T$ using π

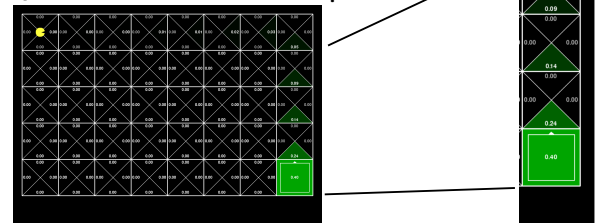
- **First** step t we visit a pair (s, a)
- Increment number of times (s, a) visited $N(s, a) \leftarrow N(s, a) + 1$
- Increment total return $S(s, a) \leftarrow S(s, a) + G_t$
- Action-value estimate is $Q(s, a) = \frac{S(s, a)}{N(s, a)}$

Action-value estimate converge to $q_\pi(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a]$

`lecture_10_mc_q_estimation.py` (first-visit)

Control

Quiz: Control and incremental updates

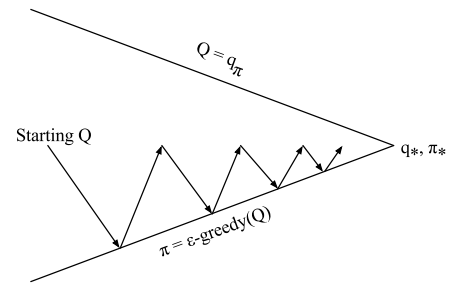


A first-visit Monte-Carlo agent (with incremental updates) is trained for one episode (terminal reward of +1). What was the discount factor γ ?

- $\gamma = 0.5$
- $\gamma = 0.4$
- $\gamma = 0.6$
- $\gamma = 0.3$
- Don't know.

Policy improvement, ϵ -greedy version

For any ϵ -greedy policy π , the ϵ -greedy policy π' with respect to q_π is an improvement: $v_{\pi'}(s) \geq v_\pi(s)$.



Repeat for every episode

- **Policy evaluation:** Monte-Carlo policy evaluation to approximate $q_\pi \approx Q$
- **Policy improvement:** ϵ -greedy policy improvement on Q

- To implement this, store Q -values in `self.Q[s,a]` in the `TabularAgent` class
- Note we already have implemented the epsilon-greedy exploration method

On-policy first-visit MC control (for ϵ -soft policies), estimates $\pi \approx \pi_*$

Initialize:

$\pi \leftarrow$ an arbitrary ϵ -soft policy
 $Q(s, a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$
 $Returns(s, a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

Generate an episode following π : $S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1, \dots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$A^* \leftarrow \arg \max_a Q(S_t, a)$

(with ties broken arbitrarily)

For all $a \in \mathcal{A}(S_t)$:

$\pi(a|S_t) \leftarrow \begin{cases} 1 - \epsilon + \epsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \epsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$

`lecture_10_mc_control.py` ($\epsilon = 0.15$, $\gamma = 0.9$).

Greedy in limit of infinite exploration (GLIE)

GLIE means that

- All state-action pairs are explored infinitely often

$$\lim_{k \rightarrow \infty} N_k(s, a) = \infty$$

- The exploration rate ϵ decays to zero

$$\lim_{k \rightarrow \infty} \pi_k(a = a^* | s) = 1, \quad a^* = \arg \max_a Q_k(s, a')$$

- One way to ensure GLIE is letting $\epsilon_k = \frac{1}{k}$
- Assuming GLIE then MC control will converge to the optimal policy.

Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning: An Introduction.
 The MIT Press, second edition, 2018.
 (Freely available online).