Approximation of linear-quadratic regulators for delay differential equations with distributed time delays

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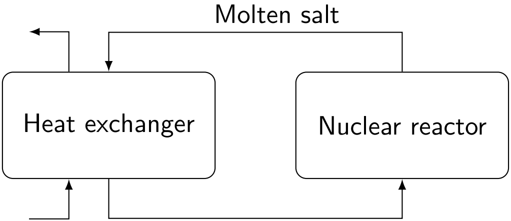
Overview

What are distributed time delays?

Why do we care about them?

How do we design controllers for them?

Molten salt nuclear reactor



- Copenhagen Atomics A/S
- Saltfoss Energy ApS (Seaborg Technologies)

Nuclear fission

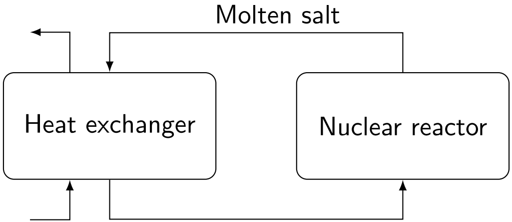
$$ {}^{1}_{0}\mathrm{n} + {}^{235}_{92}\mathrm{U} \xrightarrow{\substack{\text{neutron}\\ \text{absorp.}}} {}^{236}_{92}\mathrm{U} \xrightarrow{\text{fission}} \sum_{i=1}^{N_p} \nu_i \, {}^{a_i}_{z_i}\mathrm{E}_i + \nu_0 \, {}^{1}_{0}\mathrm{n} \tag{1} $$

Negative beta decay and neutron emission

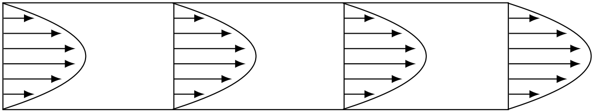
$$ {}^{a_i}_{z_i}\mathrm{E}_i \xrightarrow{\substack{\text{neg. beta}\\ \text{decay}}} {}^{a_i}_{z_i+1}\mathrm{E}_i^{*} \xrightarrow{\substack{\text{neutron}\\ \text{emission}}} {}^{a_i-1}_{z_i+1}\mathrm{E}_i^{*} + {}^{1}_{0}\mathrm{n} \tag{2} $$

Labrie-Cleary, L., Stubsgaard, A., Jørgensen, J.B., Ritschel, T.K.S., 2026. A Nonlinear Dynamic Model for the Aircraft Reactor Experiment: Experimental Validation. Annals of Nuclear Energy 236, pp. 112371. DOI: 10.1016/j.anucene.2026.112371.

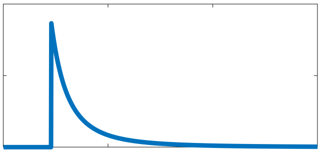
Nonuniform flow in pipes



Non-uniform velocity profile



Kernel for Hagen-Poiseuille flow (quadratic velocity profile)



Ritschel, T.K.S., 2025. *Numerical Optimal Control for Distributed Delay Differential Equations: A Simultaneous Approach based on Linearization of the Delayed Variables*. **In:** Proceedings of the 2025 European Control Conference (ECC), pp. 759–764, June 24-27, Thessaloniki, Greece. DOI: 10.23919/ECC65951.2025.11187183.

Molten salt nuclear fission reactor

Neutron concentration (states, memory states, and manipulated inputs)

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \lambda C_p(t) \tag{3}$$

Precursor concentration

$$\dot{C}_p(t) = \frac{\beta}{\Lambda} C_n(t) - \lambda C_p(t) + (C_{p,in}(t) - C_p(t))D \tag{4}$$

Reactor inlet precursor concentration

$$C_{p,in}(t) = \int_{-\infty}^{t} \alpha(t-s) C_p(s) \, \mathrm{d}s, \tag{5a}$$

$$\alpha(t) = \begin{cases} 2\frac{\tau_0^2}{t^3} e^{-\lambda t}, & t \in [\tau_0, \infty), \\ 0, & \text{otherwise} \end{cases} \tag{5b}$$

Linear differential equations with distributed time delays

System and kernel approximation

System

$$\dot{x}(t) = Ax(t) + Bu(t) + G\int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s \tag{6}$$

Erlang mixture approximation

$$\hat{\alpha}(t) = \sum_{m=0}^{M} c_m \ell_m(t), \qquad \ell_m(t) = b_m t^m e^{-at}, \qquad b_m = \frac{a^{m+1}}{m!} \tag{7}$$

$$\boxed{\text{We know how to choose the coefficients } c_m}$$

Erlang ODE approximation

Approximate system of ordinary differential equations

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{Z}_0(t) \\ \dot{Z}_1(t) \\ \vdots \\ \dot{Z}_M(t) \end{bmatrix} = \begin{bmatrix} A & c_0 G & c_1 G & \cdots & c_M G \\ aI & -aI & & & \\ & aI & -aI & & \\ & & \ddots & \ddots & \\ & & & aI & -aI \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ Z_0(t) \\ Z_1(t) \\ \vdots \\ Z_M(t) \end{bmatrix} + \begin{bmatrix} B \\ \\ \\ \\ \\ \end{bmatrix} u(t) \quad (8)$$

Auxiliary memory state

$$Z_m(t) = \int_{-\infty}^{t} \ell_m(t-s)\hat{x}(s)\,\mathrm{d}s \quad (9)$$

General form

$$\dot{\hat{X}}(t) = \hat{A}\hat{X}(t) + \hat{B}u(t) \quad (10)$$

Ritschel, T.K.S., 2026. On Erlang ODE Approximations of Differential Equations with Distributed Time Delays. arXiv: 2502.12984.

Ritschel, T.K.S., Wyller, J., 2025. An Algorithm for Distributed Time Delay Identification without A Priori Knowledge of the Kernel. Automatica 178, pp. 112382. DOI: 10.1016/j.automatica.2025.112382.

Can we use the ODE approximation
to design controllers?

Linear-quadratic regulator

Optimal control problem

$$\begin{aligned} \min_u \quad & \int_0^\infty \hat{X}^T(t)\hat{Q}\hat{X}(t) + u^T(t)Ru(t)\,\mathrm{d}t, && \text{(11a)} \\ \text{subject to} \quad & \hat{X}(0) = X_0, && \text{(11b)} \\ & \dot{\hat{X}}(t) = \hat{A}\hat{X}(t) + \hat{B}u(t) && \text{(11c)} \end{aligned}$$

Linear-quadratic regulator

$$u(t) = -L\hat{X}(t), \qquad L = R^{-1}\hat{B}^T S \tag{12}$$

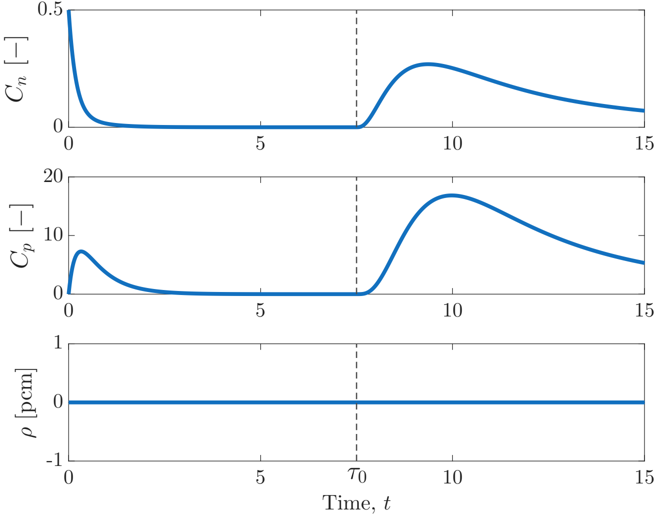
Continuous-time algebraic Riccati equation (CARE)

$$\hat{A}^T S + S\hat{A} + Q - S\hat{B}R^{-1}\hat{B}^T S = 0 \tag{13}$$

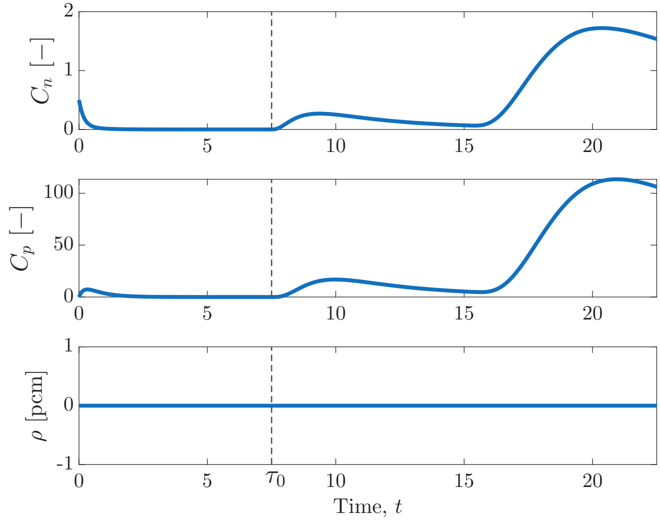
Control law in integral form

$$u(t) = -L_x\hat{x}(t) - \int_{-\infty}^t \alpha_u(t-s)\hat{x}(s)\,\mathrm{d}s, \quad \alpha_u(t) = \sum_{m=0}^M L_m \ell_m(t) \tag{14}$$

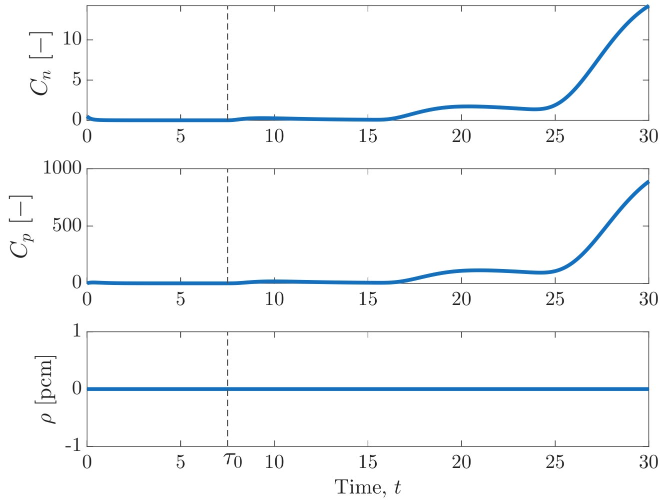
Numerical example – open-loop



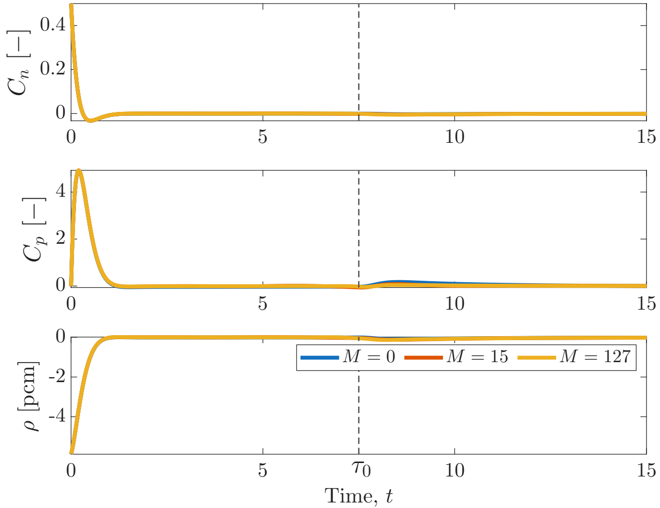
Numerical example – open-loop



Numerical example – open-loop



Numerical example – LQR based on $\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}u$



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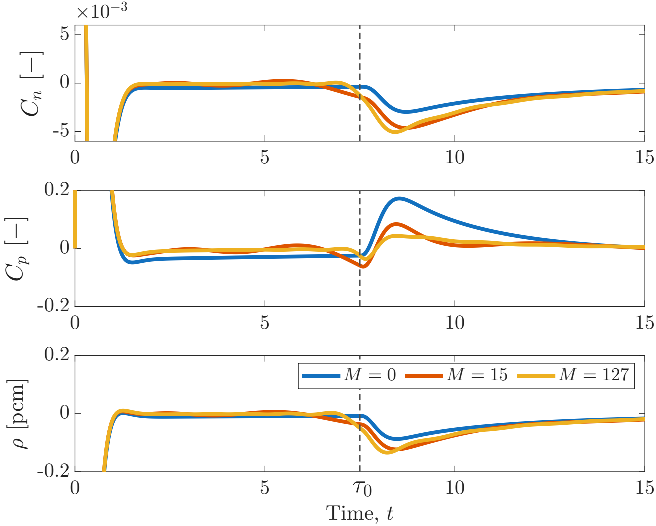
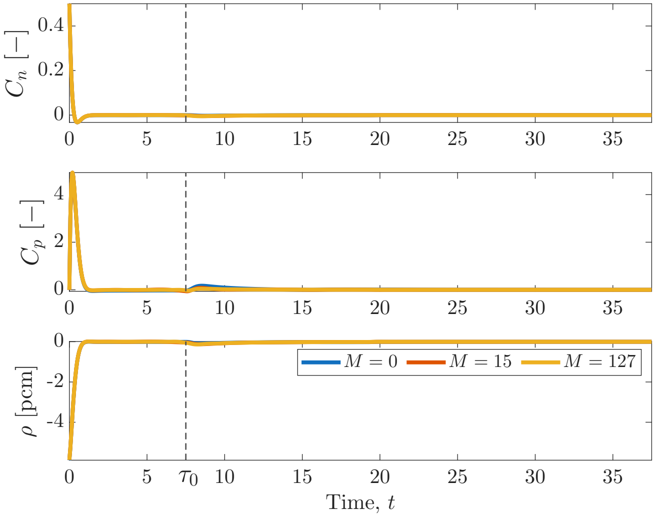
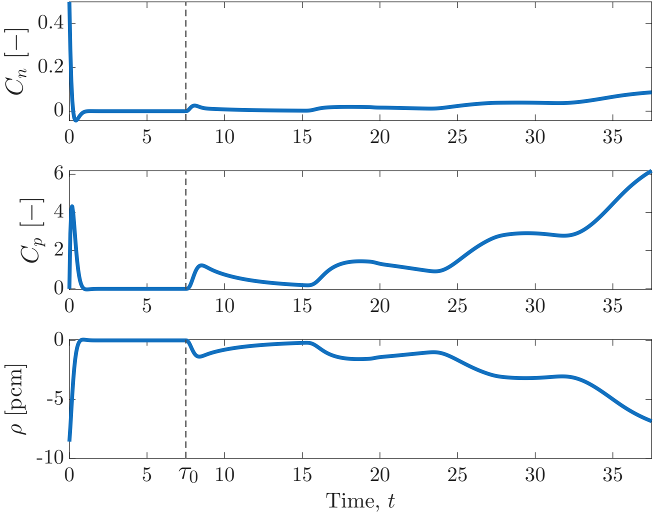


Numerical example – LQR based on $\dot{\hat{X}} = \hat{A}\hat{X} + \hat{B}u$



Numerical example – LQR based on $\dot{x} = (A + kG)x + Bu$

Approximate $x(s)$ by $x(t)$ in the integral



Summary

We can use the Erlang ODE approximation to develop LQRs
for DDEs with distributed time delays

Can we prove that it's stabilizing?
(will be explored with Jan Heiland, TU Ilmenau)

Can we use it to control molten salt reactors?
(will be explored in a PhD project with KAIST, South Korea)

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Thank you!

Molten salt nuclear fission reactor – steady state

Steady state precursor concentration ($\bar{C}_n$ is given)

$$0 = \frac{\beta}{\Lambda}\bar{C}_n + \left(-\lambda + \left(\int_0^\infty \alpha(s)\,\mathrm{d}s - 1\right) D\right)\bar{C}_p$$

$$\Rightarrow \bar{C}_p = \frac{1}{\lambda - \left(\int_0^\infty \alpha(s)\,\mathrm{d}s - 1\right) D}\frac{\beta}{\Lambda}\bar{C}_n \tag{15}$$

Steady state reactivity

$$0 = \frac{\bar{\rho} - \beta}{\Lambda}\bar{C}_n + \lambda\bar{C}_p \quad \Rightarrow \quad 0 = \frac{\bar{\rho} - \beta}{\Lambda} + \frac{\lambda}{\lambda - \left(\int_0^\infty \alpha(s)\,\mathrm{d}s - 1\right) D}\frac{\beta}{\Lambda}$$

$$\Rightarrow \quad \bar{\rho} = \left(1 - \frac{\lambda}{\lambda - \left(\int_0^\infty \alpha(s)\,\mathrm{d}s - 1\right) D}\right)\beta \tag{16}$$

Integral of kernel

$$\int_0^\infty \alpha(t)\,\mathrm{d}t = (\lambda\tau_0)^2 \int_{\lambda\tau_0}^\infty e^{-t}/t\,\mathrm{d}t + (1 - \lambda\tau_0)e^{-\lambda\tau_0} \tag{17}$$

The integral can be evaluated with Matlab's `expint`.

System

System

$$\dot{x}(t) = Ax(t) + Bu(t) + G(\alpha * x)(t) \tag{18}$$

Convolution

$$(\alpha * x)(t) = \int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s \tag{19}$$

Stability criterion: $\operatorname{Re}\lambda < 0$ for $\lambda \in \mathbb{C}$ satisfying

$$P(\lambda) = \det(A - \lambda I + G\int_{0}^{\infty} \alpha(s)e^{-\lambda s}\,\mathrm{d}s) = 0 \tag{20}$$

Approximation

Approximation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(\hat{\alpha} * \hat{x})(t) \tag{21}$$

Errors

$$E_x(t) = x(t) - \hat{x}(t), \qquad E_\alpha(t) = \alpha(t) - \hat{\alpha}(t) \tag{22}$$

Error system

$$\dot{E}_x(t) = AE_x(t) + (\alpha * E_x)(t) + (E_\alpha * \hat{x})(t) \tag{23}$$

Augmented system

Augmented system

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{E}_x(t) \end{bmatrix} = \begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ E_x(t) \end{bmatrix} + \begin{bmatrix} B \\ \end{bmatrix} u(t) + \left(\begin{bmatrix} \hat{\alpha} & \\ E_\alpha & \alpha \end{bmatrix} * \begin{bmatrix} \hat{x} \\ E_x \end{bmatrix} \right)(t) \tag{24}$$

Stability criterion

$$\begin{aligned} Q(\lambda) &= \det \left(\begin{bmatrix} A & \\ & A \end{bmatrix} - \lambda \begin{bmatrix} I & \\ & I \end{bmatrix} + \int_0^\infty \begin{bmatrix} \hat{\alpha}(s) & \\ E_\alpha(s) & \alpha(s) \end{bmatrix} e^{-\lambda s} \, \mathrm{d}s \right) \\ &= \det(A - \lambda I + \int_0^\infty \hat{\alpha}(s) e^{-\lambda s} \, \mathrm{d}s) \\ &\quad \cdot \det(A - \lambda I + \int_0^\infty \alpha(s) e^{-\lambda s} \, \mathrm{d}s) \\ &= \hat{P}(\lambda) P(\lambda) = 0 \end{aligned} \tag{25}$$

Conclusion: The error dynamics are stable if the original and approximate systems are both stable.

Erlang mixture approximation of memory kernel, α

Erlang mixture approximation

$$\hat{\alpha}(t) = \sum_{m=0}^{M} c_m \ell_m(t) \tag{26}$$

Basis function

$$\ell_m(t) = b_m t^m e^{-at}, \qquad b_m = \frac{a^{m+1}}{m!} \tag{27}$$

Theoretical coefficients

$$c_m = \int_{s_m}^{s_{m+1}} \alpha(s)\,\mathrm{d}s, \qquad s_m = m\Delta s, \qquad \Delta s = \frac{1}{a} \tag{28}$$

Alternative: Least-squares estimates

$$\min_{\{c_m\}_{m=0}^{M}} \quad \phi = \frac{1}{2}\sum_{k=0}^{N-1}(\hat{\alpha}(t_k) - \alpha(t_k))^2 \Delta t + \frac{1}{2}\sum_{m=0}^{M} c_m^2 \tag{29a}$$

$$\text{subject to} \quad \sum_{m=0}^{M} c_m = \int_0^\infty \alpha(t)\,\mathrm{d}t \tag{29b}$$

Linear chain trick

Auxiliary memory state

$$Z_m(t) = \int_{-\infty}^{t} \ell_m(t-s)\hat{x}(s)\,\mathrm{d}s \tag{30}$$

Memory state

$$\begin{aligned}\hat{z}(t) &= \int_{-\infty}^{t} \hat{\alpha}(t-s)\hat{x}(s)\,\mathrm{d}s = \int_{-\infty}^{t} \sum_{m=0}^{M} c_m \ell_m(t-s)\hat{x}(s)\,\mathrm{d}s \\ &= \sum_{m=0}^{M} c_m Z_m(t) \end{aligned} \tag{31}$$

Linear chain trick

Basis function derivatives

$$\dot{\ell}_0(t) = -a^2 e^{-at} = -a\ell_0(t), \tag{32a}$$

$$\begin{aligned} \dot{\ell}_m(t) &= m b_m t^{m-1} e^{-at} - a b_m t^m e^{-at}, \qquad m b_m = a b_{m-1}, \\ &= a b_{m-1} t^{m-1} e^{-at} - a b_m t^m e^{-at} \\ &= a(\ell_{m-1}(t) - \ell_m(t)), \qquad m = 1, \dots, M \end{aligned} \tag{32b}$$

Auxiliary memory state ($z_{-1} = 0$)

$$\begin{aligned} \dot{z}_m(t) &= \ell_m(0)\hat{x}(t) + \int_{-\infty}^{t} \dot{\ell}_m(t-s)\hat{x}(s)\,\mathrm{d}s \\ &= \ell_m(0)\hat{x}(t) + a\int_{-\infty}^{t} (\ell_{m-1}(t) - \ell_m(t))\hat{x}(s)\,\mathrm{d}s \\ &= \ell_m(0)\hat{x}(t) + a(z_{m-1}(t) - z_m(t)) \end{aligned} \tag{33}$$

System of ordinary differential equations ($\ell_m(0) = a\delta_{m0}$)

$$\dot{z}_0(t) = a(\hat{x}(t) - z_0(t)), \tag{34a}$$

$$\dot{z}_m(t) = a(z_{m-1}(t) - z_m(t)), \qquad m = 1, \dots, M \tag{34b}$$

Stability of condensed form

Condensed form

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{Z}(t) \end{bmatrix} = \begin{bmatrix} A & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ Z(t) \end{bmatrix} + \begin{bmatrix} B \\ \ \end{bmatrix} u(t), \quad A_{12} = \begin{bmatrix} G_0 & G_1 & \cdots & G_M \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} aI \\ \\ \\ \\ \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -aI & & & \\ aI & -aI & & \\ & \ddots & \ddots & \\ & & aI & -aI \end{bmatrix} \tag{35}$$

Schur's determinant formula

$$\begin{aligned} &\det\left(\begin{bmatrix} A - \lambda I & A_{12} \\ A_{21} & A_{22} - \lambda I \end{bmatrix}\right) \\ &= \det(A - \lambda I - A_{12}(A_{22} - \lambda I)^{-1} A_{21}) \det(A_{22} - \lambda I) \\ &= \det(A_{22} - \lambda I - A_{21}(A - \lambda I)^{-1} A_{12}) \det(A - \lambda I) \end{aligned} \tag{36}$$

Matrices

Matrix

$$A_{22} - \lambda I = \begin{bmatrix} -(a+\lambda)I & & & \\ aI & -(a+\lambda)I & & \\ & \ddots & \ddots & \\ & & aI & -(a+\lambda)I \end{bmatrix}$$

$$= -(a+\lambda) \begin{bmatrix} I & & & \\ \frac{-a}{a+\lambda}I & I & & \\ & \ddots & \ddots & \\ & & \frac{-a}{a+\lambda}I & I \end{bmatrix} \tag{37}$$

Inverse

$$(A_{22} - \lambda I)^{-1} = -\frac{1}{a+\lambda} \begin{bmatrix} I & & & \\ \frac{a}{a+\lambda}I & I & & \\ \vdots & \ddots & \ddots & \\ \left(\frac{a}{a+\lambda}\right)^M I & \cdots & \frac{a}{a+\lambda}I & I \end{bmatrix} \tag{38}$$

Matrices

Product involving the inverse

$$A_{12}(A_{22}-\lambda I)^{-1}A_{21} = -\sum_{m=0}^{M} G_m \left(\frac{a}{a+\lambda}\right)^{m+1} \tag{39}$$

Characteristic equation

$$\begin{aligned} &\det\left(\begin{bmatrix} A-\lambda I & A_{12} \\ A_{21} & A_{22}-\lambda I \end{bmatrix}\right) \\ &= \det\left(A-\lambda I + \sum_{m=0}^{M} G_m \left(\frac{a}{a+\lambda}\right)^{m+1}\right)\det(A_{22}-\lambda I) \end{aligned} \tag{40}$$

Integral form of the LQR

Control law

$$u(t) = -L\hat{X} = \begin{bmatrix} L_x & L_0 & \cdots & L_M \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ Z_0(t) \\ \vdots \\ Z_M(t) \end{bmatrix} = -L_x\hat{x}(t) - \sum_{m=0}^{M} L_m Z_m$$

$$= -L_x\hat{x}(t) - \sum_{m=0}^{M} L_m \int_{-\infty}^{t} \ell_m(t-s)\hat{x}(s)\,\mathrm{d}s$$

$$= -L_x\hat{x}(t) - \int_{-\infty}^{t} \sum_{m=0}^{M} L_m \ell_m(t-s)\hat{x}(s)\,\mathrm{d}s$$

$$= -L_x\hat{x}(t) - \int_{-\infty}^{t} \alpha_u(t-s)\hat{x}(s)\,\mathrm{d}s \tag{41}$$

Controller memory kernel

$$\alpha_u(t) = \sum_{m=0}^{M} L_m \ell_m(t) \tag{42}$$