

On mixture approximations for differential equations with distributed time delays

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Outline

1. What are mixture approximations?
 - ▶ Three different types of mixture approximations
 - ▶ Convergence
 - ▶ Practical implementation
 - ▶ What can you do with them?
2. Mixture approximations for delay differential equations (DDEs)
 - ▶ Examples of DDEs with distributed time delays
 - ▶ The linear chain trick (DDEs $\rightarrow$ ODEs)
 - ▶ Convergence
 - ▶ Simulation example (bifurcation analysis)

Mixture approximations

Problem statement

Approximate the $L^1(\mathcal{D})$ function α with the following properties

1. Non-negative

$$\alpha(t) \geq 0, \quad t \in \mathcal{D}$$

2. Integrates to one

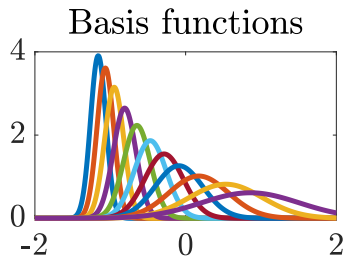
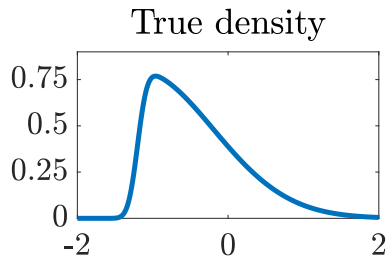
$$\int_{\mathcal{D}} \alpha(t) \, dt = 1$$

3. Continuous

The domain may be

- ▶ the real numbers, $\mathcal{D} = (-\infty, \infty)$,
- ▶ the non-negative real numbers, $\mathcal{D} = [0, \infty)$, or
- ▶ an interval, $\mathcal{D} = [a, b]$

Example for $\mathcal{D} = (-\infty, \infty)$



Mixture approximation

Approximate α by a sum of basis functions

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^M c_m \ell_m(t)$$

Weight constraints

$$\sum_{m=0}^M c_m = 1, \quad 0 \leq c_m \leq 1, \quad m = 0, \dots, M$$

Non-negative basis functions

$$\ell_m(t) \geq 0, \quad m = 0, \dots, M$$

Basis functions integrate to one

$$\int_{\mathcal{D}} \ell_m(t) \, dt = 1, \quad m = 0, \dots, M$$

Gaussian, Erlang, and beta mixture approximations

Gaussian basis functions, $t \in \mathcal{D} = (-\infty, \infty)$

$$\ell_m(t) = b_m \exp \left(-\frac{1}{2} \left(\frac{t - \mu_m}{\sigma_m} \right)^2 \right), \quad b_m = \frac{1}{\sqrt{2\pi\sigma_m^2}}$$

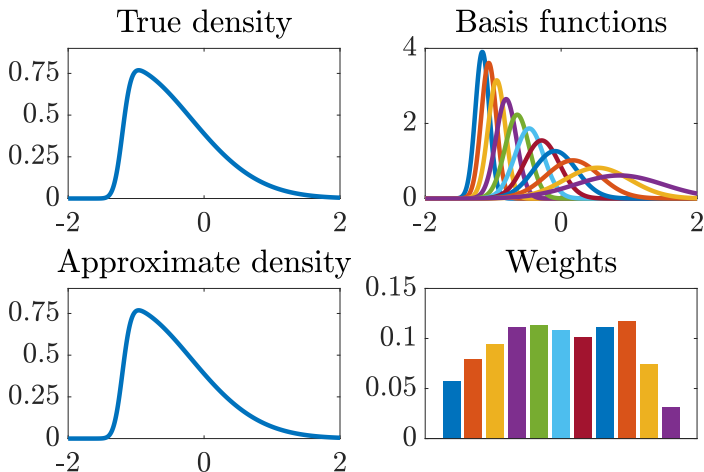
Erlang basis functions, $t \in \mathcal{D} = [0, \infty)$

$$\ell_m(t) = b_m t^m e^{-at}, \quad b_m = \frac{a^{m+1}}{m!}$$

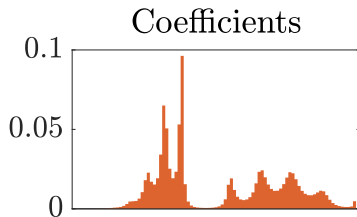
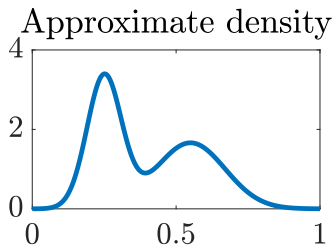
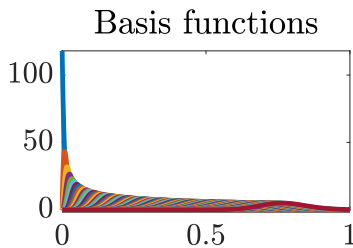
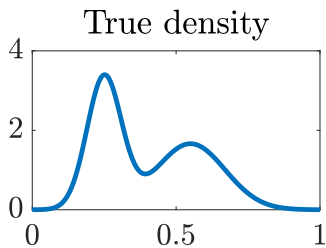
Beta basis functions, $t \in \mathcal{D} = [0, \Delta t]$

$$\ell_m(t) = b_m t^m (\Delta t - t)^{M-m}, \quad b_m = \frac{1}{\Delta t^{M+1}} \frac{(M+1)!}{m!(M-m)!}$$

Gaussian mixture approximation ($M = 10$)



Example: Erlang mixture approximation ($M = 90$)



Convergence of Erlang mixture approximations ($M = \infty$)

Weak convergence (Tijms [1, 2])

$$\int_0^t \hat{\alpha}(s) \, ds \rightarrow \int_0^t \alpha(s) \, ds \quad \text{as } a \rightarrow \infty$$

Pointwise convergence (based on delta families [3])

$$\hat{\alpha}(t) \rightarrow \alpha(t) \quad \text{as } a \rightarrow \infty$$

Vitali's convergence lemma

$$\int_0^\infty |\hat{\alpha}(s) - \alpha(s)| \, ds \rightarrow 0 \quad \text{as } a \rightarrow \infty$$

if

- ▶ $\hat{\alpha}$ converges pointwise ✓
- ▶ $\hat{\alpha}$ is tight ✓
- ▶ $\hat{\alpha}$ is uniformly integrable ✓

Extend proof to Gaussian basis functions for $\mathcal{D} = (-\infty, \infty)$

Is convergence guaranteed?

1. Beta mixture approximations, $\mathcal{D} = [0, 1]$

- ▶ ℓ_m is a Bernstein polynomial scaled by $M + 1$
- ▶ pointwise convergence is guaranteed by Weierstrass' approximation theorem [4] as $M \rightarrow \infty$

$$c_m = \frac{1}{M+1} \alpha(s_m), \quad s_m = m\Delta s, \quad \Delta s = 1/M,$$
$$c_m = \int_{s_m}^{s_{m+1}} \alpha(s) \, ds, \quad s_m = m\Delta s, \quad \Delta s = 1/(M+1)$$

2. Gaussian mixture approximations, $\mathcal{D} = (-\infty, \infty)$

- ▶ Error analysis by Maz'ya and Schmidt [5]
- ▶ Convergence proof by Sorenson and Alspach [6]

Optimal coefficients and rate parameter

Time domain ($k = 0, \dots, N - 1$)

$$\int_0^{t_h} \alpha(s) \, ds = 1 - \epsilon, \quad t_k = k/t_h$$

Least-squares optimization problem

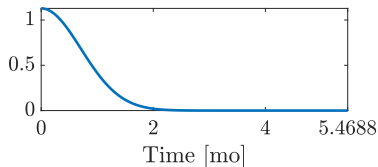
$$\begin{aligned} \min_{\{c_m\}_{m=0}^M, a} \quad & \phi = \frac{1}{2} \sum_{k=0}^{N-1} (\alpha(t_k) - \hat{\alpha}(t_k))^2 \Delta t, \\ \text{subject to} \quad & \sum_{m=0}^M c_m = 1, \\ & 0 \leq c_m \leq 1, \quad m = 0, \dots, M, \\ & a_{\min} \leq a \end{aligned}$$

- Use Matlab's `fmincon`
- Provide analytical first- and second-order derivatives

Example: Gaussian kernel

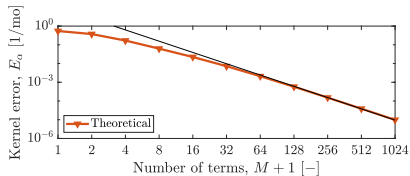
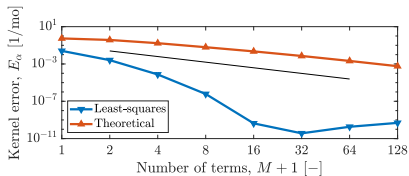
Kernel

$$\alpha(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$$



Error

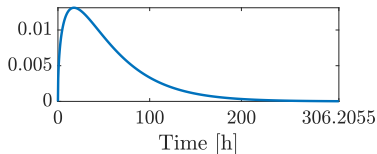
$$E_{\alpha} = \sum_{k=0}^{N_{\alpha}-1} (\hat{\alpha}(t_k) - \alpha(t_k))^2 \Delta t$$



Example: Gamma kernel

Kernel ($\lambda = -0.46$, $\kappa = 0.0263$)

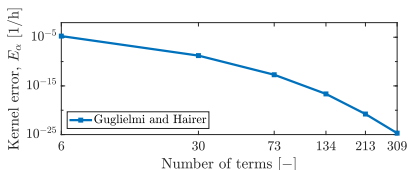
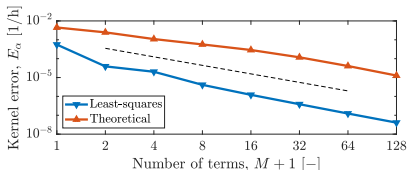
$$\alpha(t) = bt^{-\lambda}e^{-\kappa t}, \quad b = \frac{\kappa^{1-\lambda}}{\Gamma(1-\lambda)}$$



Exponential mixture approximation by Guglielmi and Hairer [7]

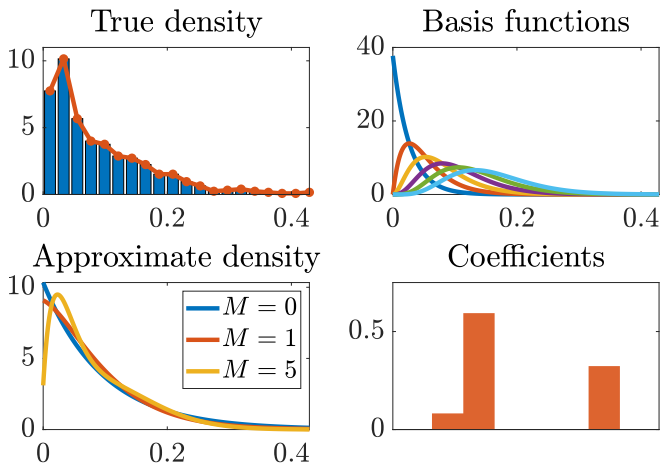
$$\hat{\alpha}(t) = t \sum_{m=M}^{N-1} c_m \ell_m(t), \quad \ell_m(t) = b_m e^{-a_m t}, \quad b_m = a_m,$$

$$c_m = \frac{h}{\Gamma(\lambda + 1)} e^{\lambda m h}, \quad a_m = e^{m h}$$



What can you do with mixture approximations?

Model of concavity from breast cancer data set¹



¹Link to data set on Kaggle.

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Differential equations with distributed time delays

Delay differential equations

Logistic equation

$$\dot{x}(t) = x(t) (1 - x(t))$$

General form (ODE)

$$\dot{x}(t) = f(x(t))$$

Logistic equation w. absolute time delay

$$\dot{x}(t) = x(t) (1 - x(t - \tau))$$

General form (DDE)

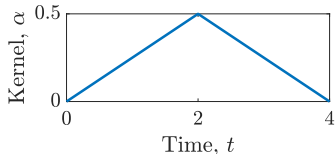
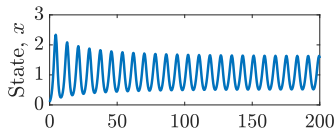
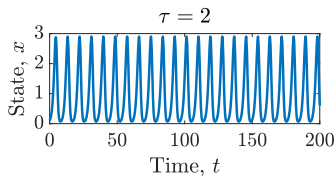
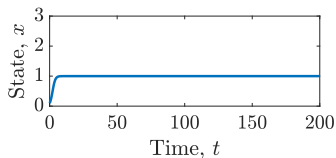
$$\dot{x}(t) = f(x(t), x(t - \tau))$$

Logistic equation w. distributed time delay

$$\dot{x}(t) = x(t) \left(1 - \int_{-\infty}^t \alpha(t-s)x(s) ds \right)$$

General form (DDDE)

$$\dot{x}(t) = f(x(t), \int_{-\infty}^t \alpha(t-s)x(s) ds)$$



Epidemiology

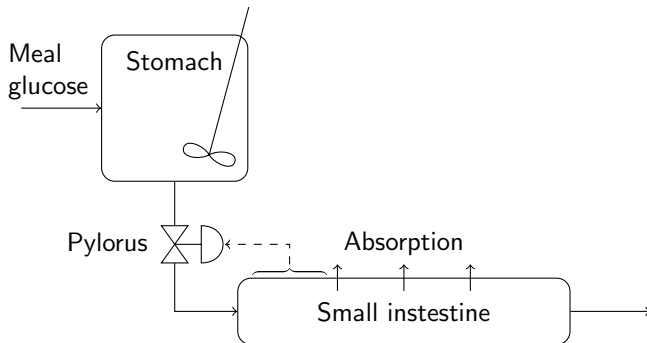
SIR model without delays

$$\begin{aligned}\dot{S}(t) &= -\beta S(t)I(t), \\ \dot{I}(t) &= \beta S(t)I(t) - \eta I(t), \\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

SIR model with distributed time delay

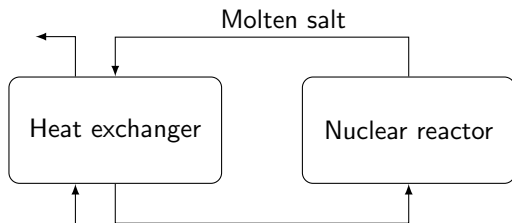
$$\begin{aligned}\dot{S}(t) &= -\beta S(t) \int_{-\infty}^t \alpha(t-s)I(s) \, ds, \\ \dot{I}(t) &= \beta S(t) \int_{-\infty}^t \alpha(t-s)I(s) \, ds - \eta I(t), \\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

Diabetes

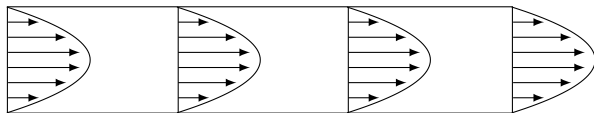


Ritschel, T.K.S., Reenberg, A.T., Carstensen, P.E., Bendtsen, J., Jørgensen, J.B., 2023. Mathematical Meal Models for Simulation of Human Metabolism. arXiv: 2307.16444.

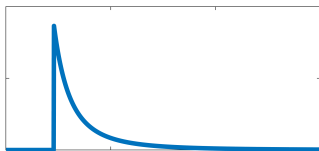
Molten salt nuclear reactor and nonuniform flow in pipes



Non-uniform velocity profile



Kernel for Hagen-Poiseuille flow (quadratic velocity profile)



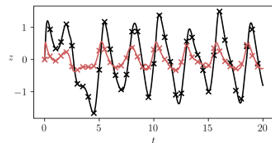
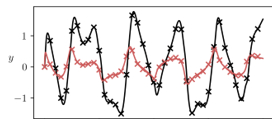
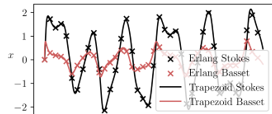
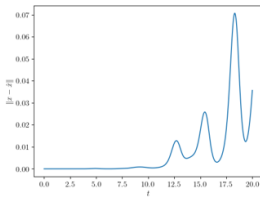
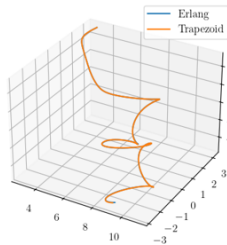
Ritschel, T.K.S., 2025. Numerical Optimal Control for Distributed Delay Differential Equations: A Simultaneous Approach based on Linearization of the Delayed Variables. In: Proceedings of the 2025 European Control Conference (ECC), June 24-27, Thessaloniki, Greece. arXiv: 2410.15083.

Particle flow in velocity field

Particle subject to Stoke's drag force and Basset history force

$$\dot{x}_p = u_p,$$

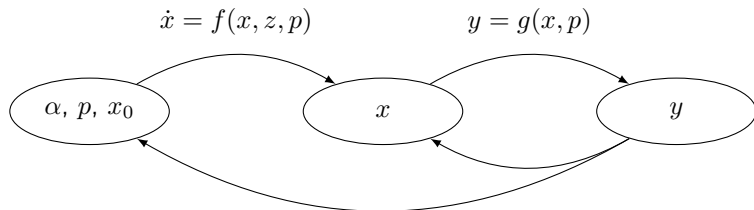
$$\dot{u}_p = \underbrace{\frac{1}{St} \mathcal{F}(a)(u_p - u_f)}_{\text{Nonlinear drag}} + \underbrace{C \int_0^t \frac{1}{\sqrt{t-s}} (\dot{u}_p - \dot{u}_f) ds}_{\text{Basset history force}}$$



Collaboration with PhD candidate Zejian You, Asst. Prof. Qi Wang, and Prof. Gustaaf Jacobs from San Diego State University.

Kernel identification (with John Wyller, NMBU, Norway)

Forward problem



Inverse problem

Ritschel, T.K.S., Wyller, J., 2025. An Algorithm for Distributed Time Delay Identification without A Priori Knowledge of the Kernel. *Automatica* 178, pp. 112382. DOI: 10.1016/j.automatica.2025.112382.

Linear chain trick (DDE $\rightarrow$ ODE)

Linear chain trick

DDE with distributed time delay

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds$$

Approximate kernel

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^M c_m \ell_m(t)$$

Substitute into integral

$$z(t) \approx \hat{z}(t) = \sum_{m=0}^M c_m \int_{-\infty}^t \ell_m(t-s)x(s) \, ds = \sum_{m=0}^M c_m z_m(t)$$

Differentiate z_m ($\dot{\ell}_0 = -a\ell_0(t)$, $\dot{\ell}_m(t) = a(\ell_{m-1}(t) - \ell_m(t))$)

$$\begin{aligned} \dot{z}_m(t) &= \ell_m(0)x(t) + \int_{-\infty}^t \dot{\ell}_m(t-s)x(s) \, ds \\ &= \begin{cases} a(x(t) - z_0(t)), & m = 0, \\ a(z_{m-1}(t) - z_m(t)), & m > 0 \end{cases} \end{aligned}$$

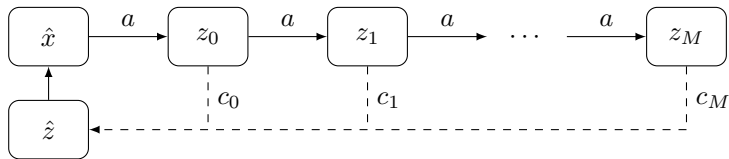
Approximate ODEs

System

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \quad \hat{z}(t) = \sum_{m=0}^M c_m z_m(t)$$

Auxiliary memory states

$$\begin{aligned} \dot{z}_0(t) &= a(\hat{x}(t) - z_0(t)), \\ \dot{z}_m(t) &= a(z_{m-1}(t) - z_m(t)), \quad m = 1, \dots, M \end{aligned}$$



Approximate ODEs

Approximate ODEs

$$\begin{aligned}\dot{\hat{x}}(t) &= f(\hat{x}(t), \hat{z}(t)), \\ \dot{Z}(t) &= AZ(t) + B\hat{x}(t), \\ \hat{z}(t) &= CZ(t)\end{aligned}$$

Matrices

$$A = a \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \quad B = a \begin{bmatrix} 1 \\ \\ \\ \end{bmatrix},$$
$$C = [c_0 \quad c_1 \quad \cdots \quad c_M]$$

Does the approximate state converge ($M = \infty$)?

Bound on state error [7, Thm. 1] when f is Lipschitz and x is bounded

$$\int_0^\infty |\hat{\alpha}(t) - \alpha(t)| dt < \epsilon \quad \Rightarrow \quad |\hat{x}(t) - x(t)| \leq \epsilon w(t) \leq \epsilon w(t_f)$$

Auxiliary function

$$w(t_0) = 0,$$

$$\dot{w}(t) = L_z K_x + L_x w(t) + L_z \int_0^t \hat{\alpha}(t-s) w(s) ds, \quad t \in [t_0, t_f]$$

- ▶ The steady state of the approximate ODEs and the DDEs is the same
- ▶ The eigenvalues converge to the roots for the DDEs (Hurwitz' convergence theorem)
- ▶ Error dynamics are locally stable if the DDEs and ODEs are locally stable

Ritschel, T.K.S., 2025. On Erlang Mixture Approximations for Differential Equations with Distributed Time Delays. arXiv: 2502.12984. In submission.

Numerical example: Simulation

Numerical example

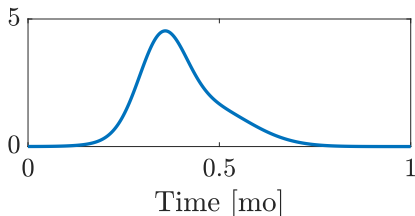
► System

$$\dot{N}(t) = \kappa N(t) \left(1 - \int_{-\infty}^t \alpha(t-s) N(s) \, ds \right)$$

► Kernel

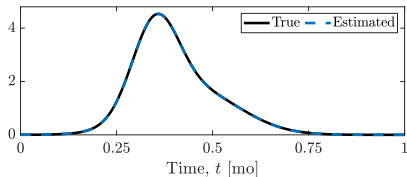
$$\alpha(t) = \gamma_1 F(t; \mu_1, \sigma_1) + \gamma_2 F(t; \mu_2, \sigma_2),$$

$$F(t; \mu, \sigma) = \frac{\exp\left(-\frac{1}{2} \left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2} \left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$

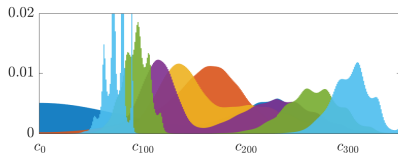
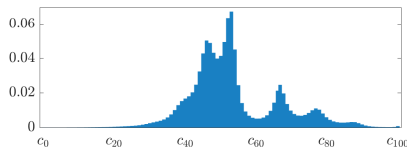
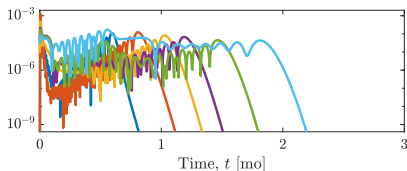
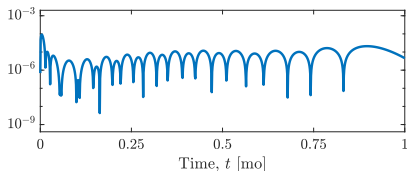
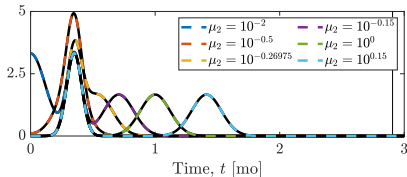


Numerical example: Bifurcation analysis

Model parameter

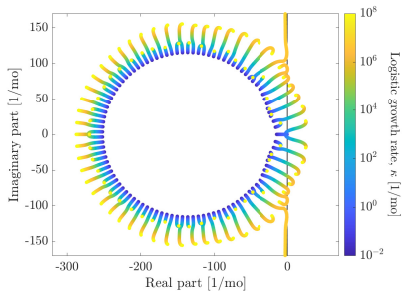


Kernel parameter

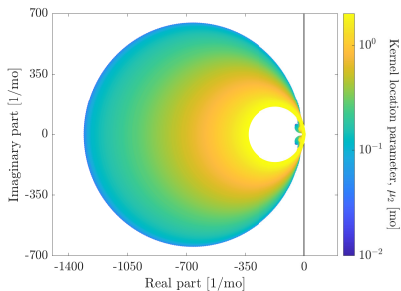


Numerical example: Bifurcation analysis

Model parameter

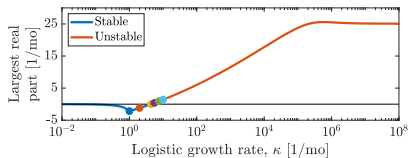


Kernel parameter

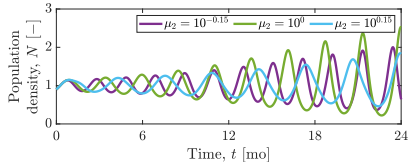
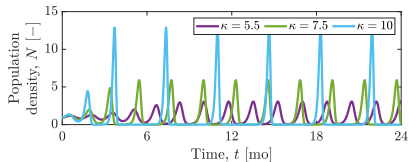
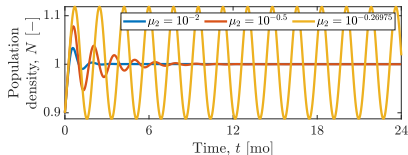
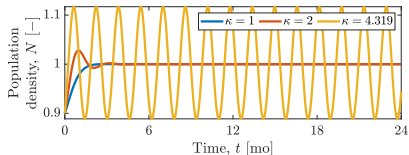
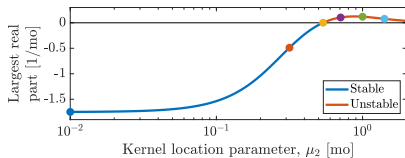


Numerical example: Bifurcation analysis

Model parameter



Kernel parameter



The numerical simulations are obtained with Euler's implicit method and a right rectangle rule for approximating the integral.

Future work: Stochastic differential equations

Stochastic differential equation

$$dx(t) = f(x(t)) dt + g(x(t)) dw(t)$$

Solution (time-varying probability density function)

$$p(t, x) = ?$$

Fokker-Planck equation

$$\partial_t p(t, x) = -\partial_x (f(x)p(t, x)) + \frac{1}{2} \partial_{xx} (g^2(x)p(t, x))$$

Mixture approximation

$$p(t, x) \approx \hat{p}(t, x) = \sum_{m=0}^M c_m(t) \ell_m(x)$$

Substitute into Fokker-Planck equation

$$\begin{aligned} R(t, x) &= \sum_{m=0}^M \left(\ell_m(x) \dot{c}_m(t) + \left[\partial_x (f(x) \ell_m(x)) - \frac{1}{2} \partial_{xx} (g^2(x) \ell_m(x)) \right] c_m(t) \right) \\ &= \sum_{m=0}^M \left(\ell_m(x) \dot{c}_m(t) + A_m(x) c_m(t) \right) \end{aligned}$$

Summary

1. Approximate $L^1(\mathcal{D})$ functions

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^M c_m \ell_m(t), \quad t \in \mathcal{D}$$

2. Approximate solution to DDEs with distributed time delays

$$\dot{x}(t) = f(x(t), \int_{-\infty}^t \alpha(t-s)x(s) \, ds)$$

3. Many different applications/possible extensions
 - ▶ Optimal/feedback/adaptive control for DDEs
 - ▶ Controllers with memory for ODEs (PID, LQR, MPC, etc.)
 - ▶ Stochastic differential equations
 - ▶ Continuous thermodynamics (chemical phase/reaction equilibria)

Perspectivation

Thank you!

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Pointwise convergence based on delta families

Identity based on Dirac's delta, δ

$$\alpha(t) = \int_{\mathcal{D}} \delta(t-s)\alpha(s) \, ds$$

Express approximations in terms of delta family, δ_a

$$\hat{\alpha}(t) = \int_{\mathcal{D}} \delta_a(t-s)\alpha(s) \, ds$$

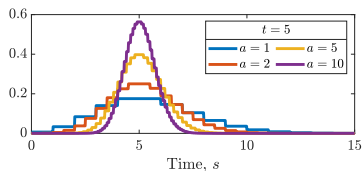
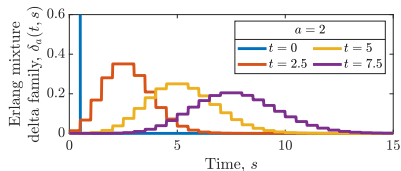
If δ_a “converges to” δ , then $\hat{\alpha}(t)$ converges to $\alpha(t)$

See, e.g., the book by Korevaar [3]

What should the delta family be?

Erlang mixture delta family

$$\delta_a(t, s) = \ell_m(t), \quad s \in [s_m, s_{m+1}), \quad s_m = m\Delta s, \quad \Delta s = 1/a,$$
$$\ell_m(t) = b_m t^m e^{-at}$$



Opportunities for model reduction?

Linear system

$$\begin{aligned}\dot{x}(t) &= Fx(t) + Ez(t) + Gu(t), \\ y(t) &= Hx(t)\end{aligned}$$

Approximation of integral term $z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds$

$$\begin{aligned}\dot{Z}(t) &= AZ(t) + Bx(t), \\ z(t) &= CZ(t)\end{aligned}$$

Can we reduce the combined system when Z is large?

Can we use model reduction to determine better basis functions, ℓ_m ?

Stochastic differential equations

Stochastic differential equations

Stochastic differential equation

$$dx(t) = f(x(t)) dt + g(x(t)) dw(t)$$

Solution (time-varying probability density function)

$$p(t, x) = ?$$

Fokker-Planck equation

$$\partial_t p(t, x) = -\partial_x (f(x)p(t, x)) + \frac{1}{2} \partial_{xx} (g^2(x)p(t, x))$$

Mixture approximation

$$p(t, x) \approx \hat{p}(t, x) = \sum_{m=0}^M c_m(t) \ell_m(x)$$

Substitute into Fokker-Planck equation

$$\begin{aligned} R(t, x) &= \sum_{m=0}^M \left(\ell_m(x) \dot{c}_m(t) + \left[\partial_x (f(x) \ell_m(x)) - \frac{1}{2} \partial_{xx} (g^2(x) \ell_m(x)) \right] c_m(t) \right) \\ &= \sum_{m=0}^M \left(\ell_m(x) \dot{c}_m(t) + A_m(x) c_m(t) \right) \end{aligned}$$

Minimize residual function

Least-squares approach

$$\min_{\{\dot{c}_m\}_{m=0}^M} \phi = \frac{1}{2} \int_{\mathcal{D}} R^2(t, x) w(x) \, dx$$

First-order optimality conditions

$$\begin{aligned} \frac{\partial \phi}{\partial \dot{c}_j} = \sum_{m=0}^M \left(\int_{\mathcal{D}} \ell_m(x) \ell_j(x) w(x) \, dx \, \dot{c}_m(t) \right. \\ \left. + \int_{\mathcal{D}} \ell_j(x) A_m(x) w(x) \, dx \, c_m(t) \right) = 0 \end{aligned}$$

If the basis functions are orthonormal wrt. the weight function w

$$\dot{c}_j(t) = - \sum_{m=0}^M \int_{\mathcal{D}} \ell_j(x) A_m(x) w(x) \, dx \, c_m(t) = \sum_{m=0}^M w_{jm} c_m(t)$$

How do we compute w_{jm} and ensure that $0 \leq c_m$ and $\sum_{m=0}^M c_m(t) = 1$?