ODE approximations of delay differential equations with distributed time delays

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Key questions

What are distributed time delays?

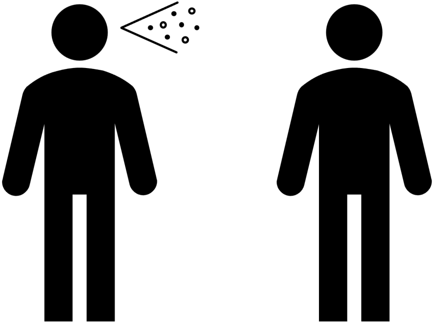
Why are they important?

How can we work with them?

What are distributed time delays?

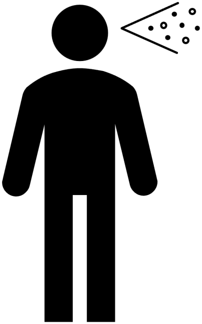
What are distributed time delays?

Infectious Susceptible



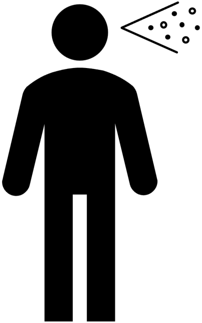
Shortly after . . .

Infectious   Infected

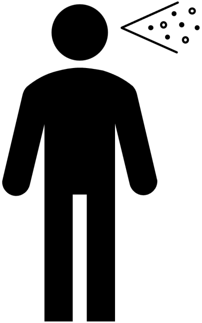


A while later (incubation period)

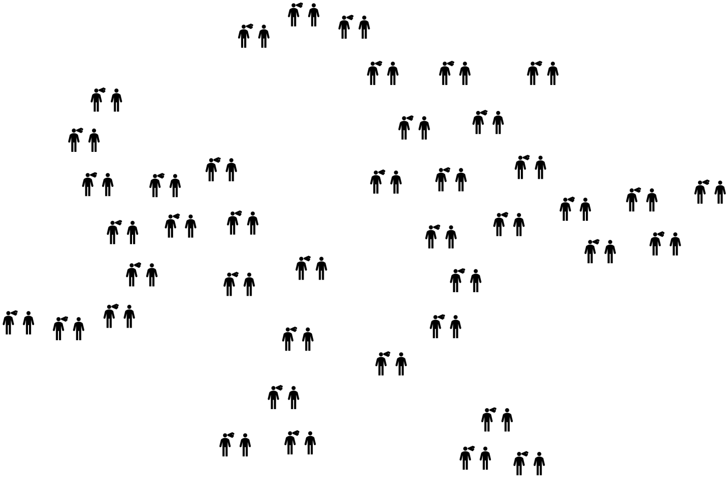
Infectious



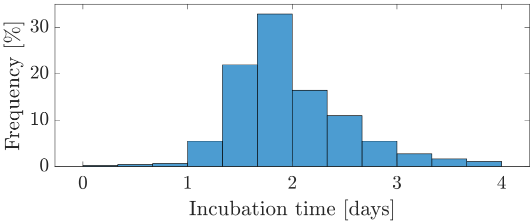
Infectious



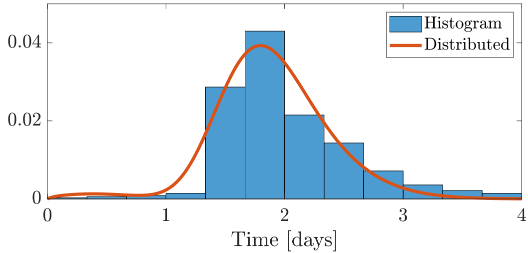
The incubation period varies across the population



Distribution of incubation periods



Distribution of incubation periods



People becoming infectious

$$\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s) \,\mathrm{d}s$$

Susceptible-infectious-recovered model

SIR model without delays

$$\begin{aligned}\dot{S}(t) &= -\beta S(t) I(t),\\ \dot{I}(t) &= \beta S(t) I(t) - \eta I(t),\\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

SIR model with distributed time delay

$$\begin{aligned}\dot{S}(t) &= -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s,\\ \dot{I}(t) &= \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s - \eta I(t),\\ \dot{R}(t) &= \eta I(t)\end{aligned}$$

Population dynamics (modified logistic equation)

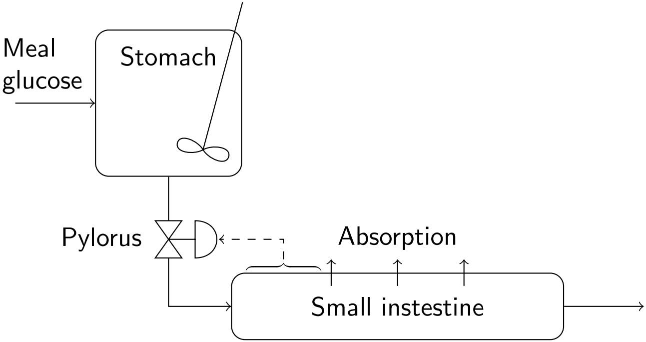
Logistic equation

$$\dot{x}(t) = \sigma x(t) \left(1 - \frac{x(t)}{K}\right)$$

Logistic equation w. distributed time delay

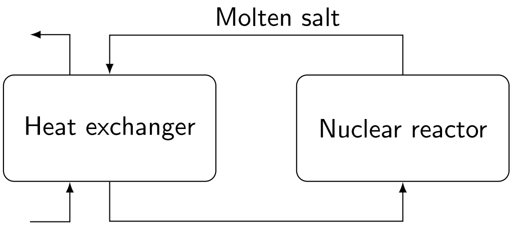
$$\dot{x}(t) = \sigma x(t) \left(1 - \frac{1}{K} \int_{-\infty}^{t} \alpha(t-s) x(s) \, \mathrm{d}s\right)$$

Diabetes (or physiology in general)

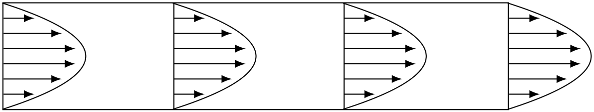


Ritschel, T.K.S., Reenberg, A.T., Carstensen, P.E., Bendsen, J., Jørgensen, J.B., 2023. Mathematical Meal Models for Simulation of Human Metabolism. arXiv: 2307.16444.

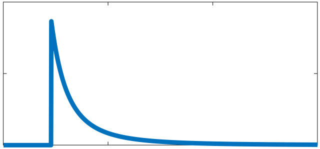
Molten salt nuclear reactor and nonuniform flow in pipes



Non-uniform velocity profile



Kernel for Hagen-Poiseuille flow (quadratic velocity profile)



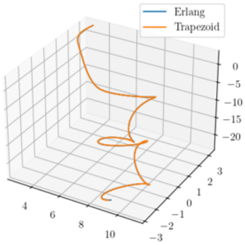
Ritschel, T.K.S., 2025. Numerical Optimal Control for Distributed Delay Differential Equations: A Simultaneous Approach based on Linearization of the Delayed Variables. In: Proceedings of the 2025 European Control Conference (ECC), June 24-27, Thessaloniki, Greece. arXiv: 2410.15083.

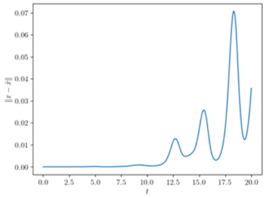
Particle flow in velocity field

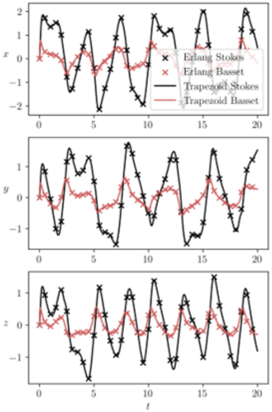
Particle subject to Stoke's drag force and Basset history force

$$\dot{x}_p = u_p,$$

$$\dot{u}_p = \underbrace{\frac{1}{\mathrm{St}}\mathcal{F}(a)(u_p - u_f)}_{\text{Nonlinear drag}} + \underbrace{C\int_0^t \frac{1}{\sqrt{t-s}}(\dot{u}_p - \dot{u}_f)\,\mathrm{d}s}_{\text{Basset history force}}$$



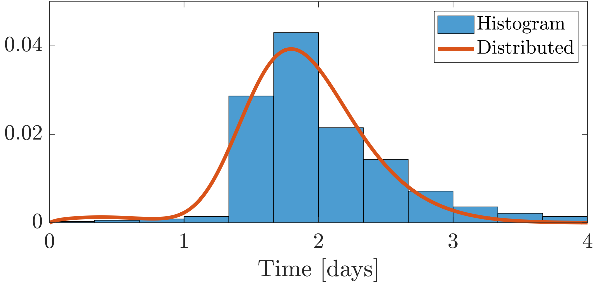




Collaboration with PhD candidate Zejian You, Asst. Prof. Qi Wang, and Prof. Gustaaf Jacobs from San Diego State University.

Why are distributed time delays important?

Why are distributed time delays important?

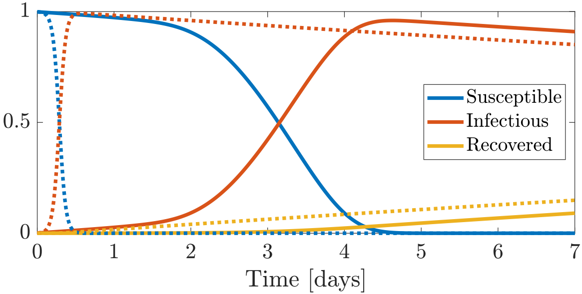


Why are distributed time delays important?

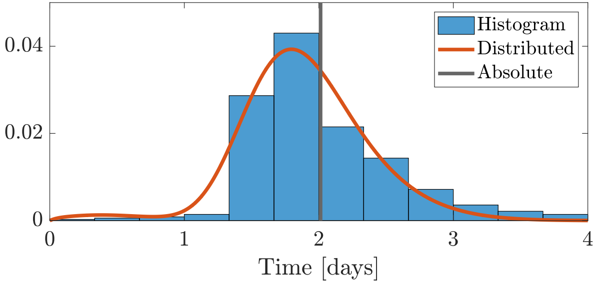
$$\dot{S}(t) = -\beta S(t) I(t), \qquad \dot{S}(t) = -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s,$$

$$\dot{I}(t) = \beta S(t) I(t) - \eta I(t), \qquad \dot{I}(t) = \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s - \eta I(t),$$

$$\dot{R}(t) = \eta I(t), \qquad \dot{R}(t) = \eta I(t)$$



Why are distributed time delays important?

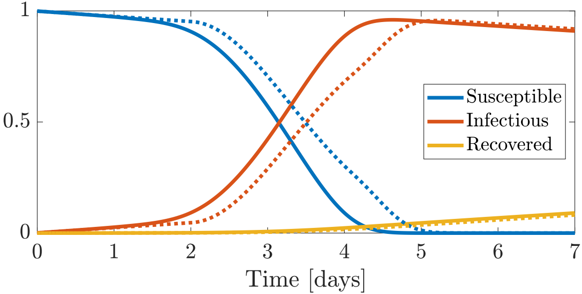


Why are distributed time delays important?

$$\dot{S}(t) = -\beta S(t) I(t-\tau), \qquad \dot{S}(t) = -\beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s,$$

$$\dot{I}(t) = \beta S(t) I(t-\tau) - \eta I(t), \qquad \dot{I}(t) = \beta S(t) \int_{-\infty}^{t} \alpha(t-s) I(s)\,\mathrm{d}s - \eta I(t),$$

$$\dot{R}(t) = \eta I(t), \qquad \dot{R}(t) = \eta I(t)$$



Why are distributed time delays important?

Stability criterion: $\mathrm{Re}\,\lambda < 0$ for all λ solving the characteristic equation

Characteristic equation (ODEs)

$$\det\left(A - \lambda I\right) = 0$$

Characteristic equation (DDEs w. absolute time delay)

$$\det\left(A + Be^{-\lambda\tau} - \lambda I\right) = 0$$

Characteristic equation (DDEs w. distributed time delay)

$$\det\left(A + B\int_0^\infty e^{-\lambda s}\alpha(s)\,\mathrm{d}s - \lambda I\right) = 0$$

ODE approximation

ODE approximation through kernel approximation

DDE with distributed time delay

$$\dot{x}(t) = f(x(t), z(t)), \qquad z(t) = \int_{-\infty}^{t} \alpha(t-s)x(s)\,\mathrm{d}s$$

Approximate kernel

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^{M} c_m \ell_m(t), \qquad \ell_m(t) = \frac{a^{m+1}}{m!} t^m e^{-at}$$

Derivatives of basis functions

$$\dot{\ell}_m(t) = a(\ell_{m-1}(t) - \ell_m(t)), \qquad \dot{\ell}_0 = -a\ell_0(t)$$

Substitute into integral

$$z(t) \approx \hat{z}(t) = \sum_{m=0}^{M} c_m \int_{-\infty}^{t} \ell_m(t-s)x(s)\,\mathrm{d}s = \sum_{m=0}^{M} c_m z_m(t)$$

Differentiate z_m

$$\dot{z}_m(t) = \ell_m(0)x(t) + \int_{-\infty}^{t} \dot{\ell}_m(t-s)x(s)\,\mathrm{d}s = \begin{cases} a(x(t) - z_0(t)), & m = 0, \\ a(z_{m-1}(t) - z_m(t)), & m \geq 1 \end{cases}$$

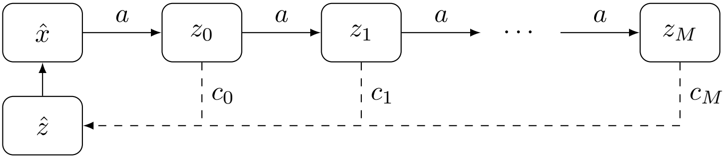
Approximate ODEs

Approximate system

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \qquad \hat{z}(t) = \sum_{m=0}^{M} c_m z_m(t)$$

Auxiliary memory states

$$\begin{aligned} \dot{z}_0(t) &= a(\hat{x}(t) - z_0(t)), \\ \dot{z}_m(t) &= a(z_{m-1}(t) - z_m(t)), \qquad m = 1, \ldots, M \end{aligned}$$



Approximate ODEs

Approximate ODEs

$$\begin{aligned}
\dot{\hat{x}}(t) &= f(\hat{x}(t), \hat{z}(t)), \\
\dot{Z}(t) &= AZ(t) + B\hat{x}(t), \\
\hat{z}(t) &= CZ(t)
\end{aligned}$$

Matrices

$$A = a \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \qquad B = a \begin{bmatrix} 1 \\ \\ \\ \\ \end{bmatrix},$$

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_M \end{bmatrix}$$

Does it work?

Choose the coefficients

$$c_m = \int_{s_m}^{s_{m+1}} \alpha(s)\,\mathrm{d}s, \qquad s_m = m\Delta s, \qquad \Delta s = 1/a \tag{1}$$

As $a \to \infty$ and $M/a \to \infty$, the kernel approximation converges

- uniformly

$$\hat{\alpha}(t) \to \alpha(t), \qquad t \in [0, \infty)$$

- weakly

$$\int_0^t \hat{\alpha}(s)\,\mathrm{d}s \to \int_0^t \alpha(s)\,\mathrm{d}s,$$

- in L_1-norm

$$\int_0^\infty |\hat{\alpha}(s) - \alpha(s)|\,\mathrm{d}s \to 0,$$

if α is continuous and bounded

Does it work?

If α is also exponentially bounded,

- The state and memory state converge uniformly

$$\hat{x}(t) \to x(t), \qquad \hat{z}(t) \to z(t), \qquad t \in [t_0, t_f]$$

- The steady state converges

$$\hat{x}_s \to x_s$$

- For every isolated root, there are roots of the characteristic equation that converge*

Numerical example: Bifurcation analysis

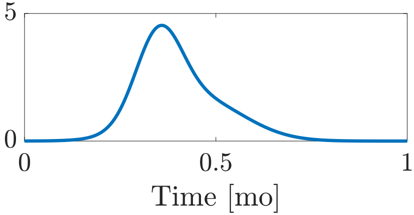
Numerical example

- System

$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{1}{K} \int_{-\infty}^{t} \alpha(t-s) N(s)\, \mathrm{d}s\right)$$

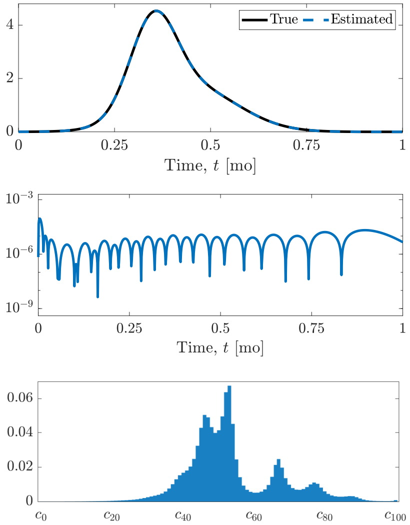
- Kernel

$$\alpha(t) = \gamma_1 F(t; \mu_1, \sigma_1) + \gamma_2 F(t; \mu_2, \sigma_2),$$
$$F(t; \mu, \sigma) = \frac{\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2}\left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$

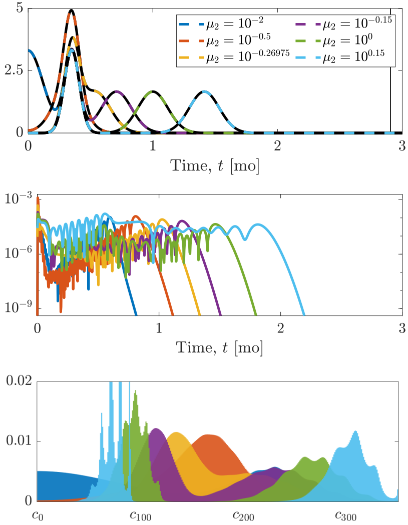


Numerical example: Bifurcation analysis

Model parameter

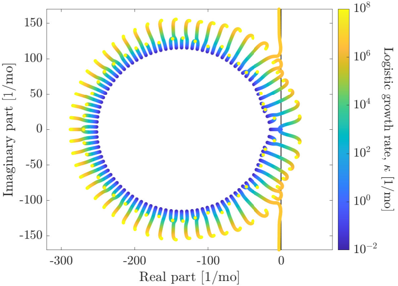


Kernel parameter

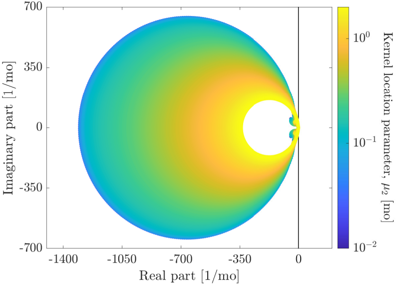


Numerical example: Bifurcation analysis

Model parameter

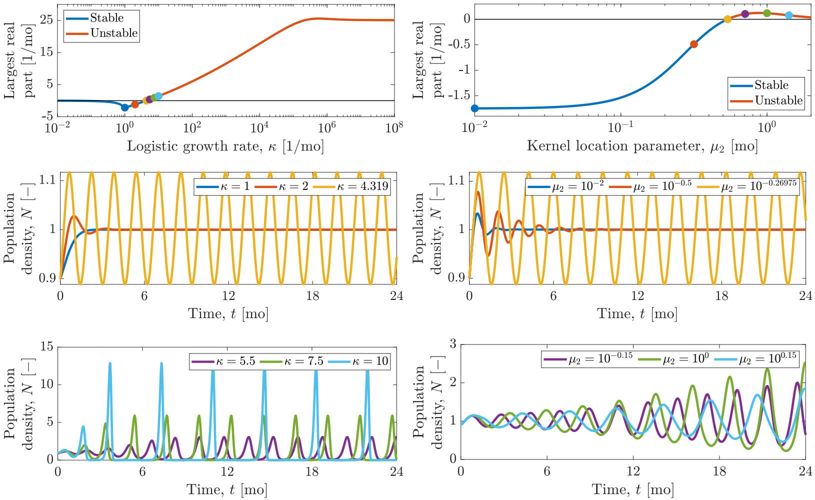


Kernel parameter



Numerical example: Bifurcation analysis

Model parameter | Kernel parameter

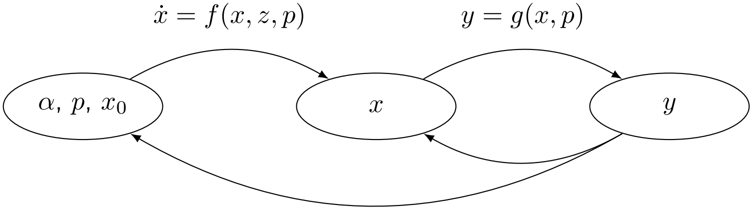


The numerical simulations are obtained with Euler's implicit method and a right rectangle rule for approximating the integral.

Kernel identification

Kernel identification (with John Wyller, NMBU, Norway)

Forward problem



Inverse problem

Ritschel, T.K.S., Wyller, J., 2025. An Algorithm for Distributed Time Delay Identification without A Priori Knowledge of the Kernel. Automatica 178, pp. 112382. DOI: 10.1016/j.automatica.2025.112382.

Example

- System

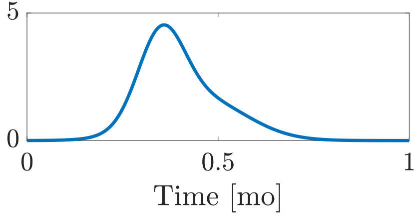
$$\dot{N}(t) = \kappa N(t)\left(1 - \frac{1}{K(t)}\int_{-\infty}^{t}\alpha(t-s)N(s)\,\mathrm{d}s\right),$$

$$K(t) = (1 + A_1\sin(2\pi\omega_1 t) + A_2\sin(2\pi\omega_2 t))\bar{K}$$

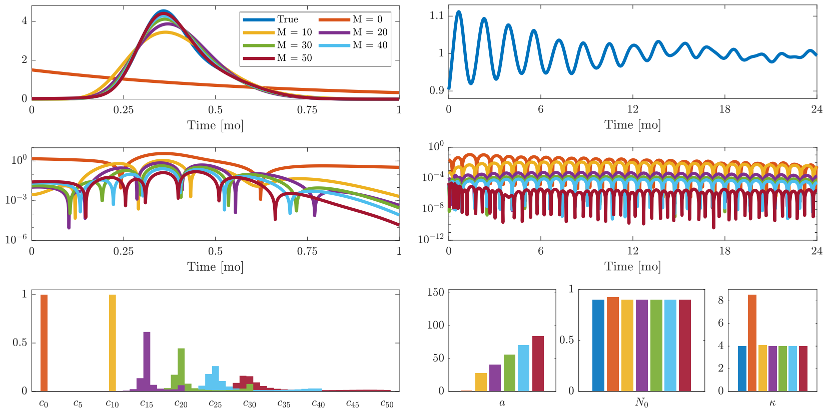
- Kernel

$$\alpha(t) = \gamma_1 F(t;\mu_1,\sigma_1) + \gamma_2 F(t;\mu_2,\sigma_2),$$

$$F(t;\mu,\sigma) = \frac{\exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2}\left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$



Example


True
M = 0
M = 10
M = 20
M = 30
M = 40
M = 50
Time [mo]
Time [mo]
Time [mo]
Time [mo]
c_0 c_5 c_{10} c_{15} c_{20} c_{25} c_{30} c_{35} c_{40} c_{45} c_{50}
a
N_0
κ

Summary

Summary – distributed time delays

- Distributed time delays arise from a collection of individual delays
  - Epidemiology
  - Population dynamics
  - Diabetes (physiology)
  - Flow in pipes
  - Flow in velocity field
- They can significantly affect transient dynamics and stability
- We can approximate them by ODEs
  - Simulation
  - Bifurcation analysis
  - Uncertainty quantification
- We can estimate the kernel based on data

Future work

- Approximate systems with finite memory
- Approximate solution of optimal control problems
- Approximate feedback control laws
- Approximate delay-adaptive control laws

Bibliography

[1] T. K. S. Ritschel, "Numerical optimal control for distributed delay differential equations: A simultaneous approach based on linearization of the delayed variables." arXiv:2410.15083, 2024. Preprint.

[2] T. K. S. Ritschel and J. Wyller, "An algorithm for distributed time delay identification without a priori knowledge of the kernel," *Automatica*, vol. 178, p. 112382, 2025.

[3] T. K. S. Ritschel, "On Erlang mixture approximations for differential equations with distributed time delays." arXiv:2502.12984, 2026. In submission.

[4] N. Guglielmi and E. Hairer, "Applying stiff integrators for ordinary differential equations and delay differential equations to problems with distributed delays," *SIAM Journal on Scientific Computing*, vol. 47, no. 1, pp. A102–A123, 2025.

`https://tobiasritschel.github.io/mixbox`

Thank you!