01622 Advanced Dynamical Systems: Applications in Science and Engineering

Week 7: Feedback control and numerical optimal control

Tobias K. S. Ritschel,
Assoc. Prof. in Stochastic Adaptive Control

Department of Applied Mathematics and Computer Science,
Technical University of Denmark

Last updated on July 13, 2026

Feedback control

Feedback control – Linear systems

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

Control law

$$u(t) = -Lx(t) \tag{2}$$

Closed-loop system

$$\dot{x}(t) = Ax(t) - BLx(t) = \underbrace{(A - BL)}_{\bar{A}}\,x(t) \tag{3}$$

Question: Is the closed-loop system stable?

Question: Are the real parts of the eigenvalues of $\bar{A}$ negative?

Feedback control – Nonlinear systems

Nonlinear system

$$\dot{x}(t) = f(x(t), u(t), d(t), p) \tag{4}$$

Control law

$$u(t) = \mu(x(t)) \tag{5}$$

Closed-loop system

$$\dot{x}(t) = f(x(t), \mu(x(t)), d(t), p) = F(x(t), d(t), p) \tag{6}$$

Typical objectives

- In steady state, $x(t) = \bar{x}$, for some given setpoint $\bar{x}$
- The steady state should be reached quickly
- The steady state should be stable
- Minimal variation in the state, x
- Robustness/adaptation to changes/uncertainty in f, d, and p
- The manipulated inputs should be used "sparingly"
- Optimize the economy of operation

Closed-loop stability

Closed-loop system

$$\dot{x}(t) = f(x(t), \mu(x(t)), d(t), p) = F(x(t), d(t), p) \tag{7}$$

Steady state

$$0 = f(x_s, \mu(x_s), d_s, p) = F(x_s, d_s, p) \tag{8}$$

Jacobian matrix

$$A = \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial \mu}{\partial x} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \tag{9}$$

State and output feedback

State feedback: All states are known to the controller

$$u(t) = \mu(x(t)) \tag{10}$$

Output equation (variables of interest/variables that are measurable)

$$y(t) = g(x(t), p) \tag{11}$$

Output feedback: Only the outputs are known

$$u(t) = \mu(y(t)) \tag{12}$$

Rule of thumb: You can only control as many states/outputs as you have manipulated inputs

Proportional-integral-derivative (PID) control

Error

$$e(t) = x(t) - \bar{x} \tag{13}$$

Proportional controller

$$u(t) = K_p e(t) \tag{14}$$

Proportional-integral controller

$$u(t) = K_p e(t) + K_i \int_0^t e(s)\,\mathrm{d}s \tag{15}$$

Proportional-integral-derivative controller ($\bar{x}$ constant)

$$u(t) = K_p e(t) + K_i \int_0^t e(s)\,\mathrm{d}s + K_d \dot{e}(t) \tag{16}$$

Proportional-integral-derivative control with nominal input

$$u(t) = \bar{u} + K_p e(t) + K_i \int_0^t e(s)\,\mathrm{d}s + K_d \dot{e}(t) \tag{17}$$

Closed-loop simulation of PID-controlled system

Integral

$$w(t) = \int_{t_0}^{t} e(s)\,\mathrm{d}s \tag{18}$$

Formulate integral as initial value problem

$$\dot{w}(t) = e(t), \qquad\qquad w(t_0) = 0 \tag{19}$$

Closed-loop system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), d(t), p), && \text{(20a)} \\ \dot{w}(t) &= e(t), && \text{(20b)} \\ u(t) &= K_p e(t) + K_i w(t) + K_d \dot{e}(t) && \text{(20c)} \end{aligned}$$

This is a set of *implicit* differential equations

Use, e.g., Matlab's ode15i to simulate (20)

Discrete-time PID control

Integral (right rectangle rule)

$$w_k = w_{k-1} + e_k \Delta t, \qquad w_0 = 0 \tag{21}$$

Derivative term

$$K_d \frac{x_k - x_{k-1}}{\Delta t} \tag{22}$$

Discrete-time PID controller ($\bar{x}$ constant)

$$u_k = \bar{u} + K_p e_k + K_i w_k + K_d \frac{x_k - x_{k-1}}{\Delta t} \tag{23}$$

Optimal control

Optimal control problem

$$\min_{u} \quad \phi(x, u) = \int_{t_0}^{t_f} \Phi(x(t), u(t), d(t), p)\, \mathrm{d}t, \tag{24a}$$

subject to

$$x(t_0) = x_0, \tag{24b}$$

$$\dot{x}(t) = f(x(t), u(t), d(t), p), \qquad t \in [t_0, t_f], \tag{24c}$$

$$u_{\min} \leq u(t) \leq u_{\max}, \qquad t \in [t_0, t_f] \tag{24d}$$

Zero-order-hold parametrization

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}[, \tag{25a}$$

$$d(t) = d_k, \qquad t \in [t_k, t_{k+1}[\tag{25b}$$

Numerical optimal control – Single-shooting

Objective function containing initial value problem [1]

$$\psi(\{u_k, d_k\}_{k=0}^{N-1}, x_0) = \left\{ \phi(x, u) \middle| \begin{aligned} \dot{x}(t) &= f(x(t), u(t), d(t), p), && \text{(26a)} \\ x(t_0) &= x_0, && \text{(26b)} \\ u(t) &= u_k, \quad t \in [t_k, t_{k+1}[&& \text{(26c)} \\ d(t) &= d_k, \quad t \in [t_k, t_{k+1}[&& \text{(26d)} \end{aligned} \right\}$$

for $k = 0, \ldots, N-1$

Nonlinear program

$$\min_{\{u_k\}_{k=0}^{N-1}} \quad \psi(\{u_k, d_k\}_{k=0}^{N-1}, x_0) \tag{27a}$$

subject to

$$u_{\min} \leq u_k \leq u_{\max}, \qquad k = 0, \ldots, N-1 \tag{27b}$$

Numerical optimal control - transform objective function

Objective function

$$\phi(x,u) = \int_{t_0}^{t_f} \Phi(x(t), u(t), d(t), p)\,\mathrm{d}t \tag{28}$$

Auxiliary variable

$$w(t) = \int_{t_0}^{t} \Phi(x(s), u(s), d(s), p)\,\mathrm{d}s \tag{29}$$

Initial value problem

$$w(t_0) = 0, \tag{30a}$$

$$\dot{w}(t) = \Phi(x(t), u(t), d(t), p), \qquad t \in [t_0, t_f] \tag{30b}$$

Augmented system

$$\dot{X}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} f(x(t), u(t), d(t), p) \\ \Phi(x(t), u(t), d(t), p) \end{bmatrix} = F(X(t), u(t), d(t), p) \tag{31}$$

Optimal control – Linear quadratic regulator

Optimal control problem

$$\min_{u} \quad \phi(x,u) = \int_{t_0}^{\infty} x^T(t)Qx(t) + u^T(t)Ru(t)\,\mathrm{d}t, \tag{32a}$$

subject to

$$x(t_0) = x_0, \tag{32b}$$
$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \in [t_0, t_f] \tag{32c}$$

Riccati equation

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \tag{33}$$

Optimal controller [2, Section 5.5], [3, Thm. 14.3 and 14.4]

$$u(t) = -Lx(t), \qquad L = R^{-1}B^T S \tag{34}$$

Use, e.g., Matlab's icare or lqr

Deviation variables

Linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{35}$$

Desired steady state

$$0 = A\bar{x} + B\bar{u}, \qquad \bar{x} = -A^{-1}B\bar{u} \qquad \text{if } A \text{ is invertible} \tag{36}$$

Deviation variables

$$X(t) = x(t) - \bar{x}, \qquad U(t) = u(t) - \bar{u} \tag{37}$$

Linear system for deviation variables

$$\begin{aligned}\dot{X}(t) = \dot{x}(t) - \dot{\bar{x}} &= Ax(t) + Bu(t) \\ &= A(X(t) + \bar{x}) + B(U(t) + \bar{u}) = AX(t) + BU(t) + \underbrace{A\bar{x} + B\bar{u}}_{=0} \\ &= AX(t) + BU(t)\end{aligned} \tag{38}$$

Control law

$$U(t) = -LX(t), \qquad u(t) = \bar{u} + U(t) = \bar{u} - L(x(t) - \bar{x}) \tag{39}$$

Closed-loop simulation

Closed-loop simulation

Nonlinear system

$$\dot{x}(t) = f(x(t), u(t), d(t), p), \qquad t \in [t_0, t_f], \tag{40a}$$

$$y_k = g(x(t_k), p), \qquad k = 0, \ldots, N \tag{40b}$$

Zero-order-hold parametrization

$$u(t) = u_k, \qquad t \in [t_k, t_{k+1}[, \tag{41a}$$

$$d(t) = d_k, \qquad t \in [t_k, t_{k+1}[\tag{41b}$$

Control law

$$u_k = \mu(y_k) \tag{42}$$

Closed-loop simulation

Initial value problems ($\{d_k\}_{k=0}^{N-1}$ are given)

$$x_k(t_k) = \begin{cases} x_0, & k = 0, \\ x_{k-1}(t_k), & k = 1, \dots, N-1, \end{cases} \tag{43a}$$

$$y_k = g(x_k(t_k), p), \tag{43b}$$

$$u_k = \mu(y_k), \tag{43c}$$

$$\dot{x}_k(t) = f(x_k(t), u_k, d_k, p), \quad t \in [t_k, t_{k+1}], \quad k = 0, \dots, N-1 \tag{43d}$$

Closed-loop simulation

1. Determine the k'th initial state from (43a)
2. Compute the k'th measurement, y_k, from (43b)
3. Compute the k'th manipulated input, u_k, from (43c)
4. Solve the initial value problem (43d)

Nuclear reactor model

Nuclear reactor model 6 – Model 5 revisited

Reactivity and thermal reactivity

$$\rho(t) = \rho_{th}(t) + \rho_{ext}(t), \qquad \dot{\rho}_{th}(t) = -\kappa \dot{T}_r(t) \tag{44}$$

Mass balance equations

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \sum_{i=1}^{m} \lambda_i C_i(t), \tag{45a}$$

$$\dot{C}_i(t) = \frac{\beta_i}{\Lambda} C_n(t) - \lambda_i C_i(t), \qquad i = 1, \ldots, m \tag{45b}$$

Energy balance equations (reactor core and heat exchanger)

$$\dot{T}_r(t) = \frac{f(t)}{n_r}(T_{hx}(t) - T_r(t)) + \frac{Q_g(t)}{n_r c_P}, \tag{46a}$$

$$\dot{T}_{hx}(t) = \frac{f(t)}{n_{hx}}(T_r(t) - T_{hx}(t)) - \frac{k_{hx}}{n_{hx} c_P}(T_{hx}(t) - T_c) \tag{46b}$$

Questions?

Bibliography I

[1] T. Binder, L. Blank, H. G. Bock, R. Bulirsch, W. Dahmen, M. Diehl, T. Kronseder, W. Marquardt, J. P. Schlöder, and O. von Stryk, "Introduction to model based optimization of chemical processes on moving horizons," in *Online Optimization of Large Scale Systems* (M. Grötschel, S. O. Krumke, and J. Rambau, eds.), Springer, 2001.

[2] E. Hendricks, O. Jannerup, and P. H. Sørensen, *Linear systems control: Deterministic and stochastic methods*. Springer, 2008.

[3] C. M. Kellett and P. Braun, *Introduction to nonlinear control: Stability, control design, and estimation*. Princeton University Press, 2023.

[4] D. E. Seborg, T. F. Edgar, D. A. Mellichamp, and F. J. Doyle III, *Process dynamics and control*. Wiley, 3rd ed., 2011.

[5] A. Ilka and N. Murgovski, "Novel results on output-feedback LQR design," *IEEE Transaction on Automatic Control*, vol. 68, no. 9, pp. 5187–5200, 2023.

Bibliography II

[6] T. K. S. Ritschel and S. Stange, "Numerical optimal control for delay differential equations: A simultaneous approach based on linearization of the delayed state," in *Proceedings of the 2025 European Control Conference (ECC)*, (Thessaloniki, Greece), pp. 777–782, June 2025.

[7] T. K. S. Ritschel, "Numerical optimal control for distributed delay differential equations: A simultaneous approach based on linearization of the delayed variables," in *Proceedings of the 2025 European Control Conference (ECC)*, (Thessaloniki, Greece), pp. 759–764, June 2025.

[8] T. K. S. Ritschel and J. Wyller, "An algorithm for distributed time delay identification without a priori knowledge of the kernel," *Automatica*, vol. 178, p. 112382, 2025.