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Uncertainty Quantification in Computational Science

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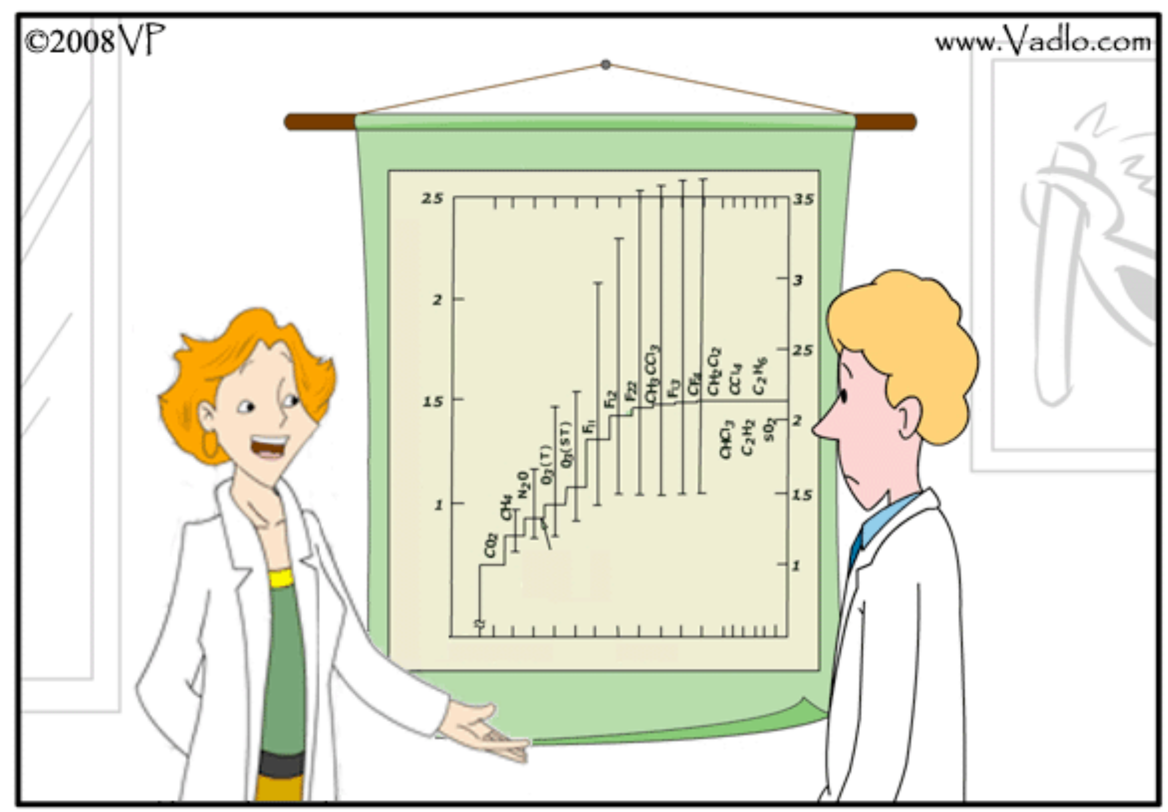


Did you really have to show the error bars?

The global picture

- Lecture I - Introduction to UQ

  Motivation, terminology, background, Wiener chaos expansions.

- Lecture II - Stochastic Galerkin methods

  Formulation, extensions, polynomial chaos, and examples.

- Lecture III - Stochastic Collocation methods

  Motivation, formulation, high-d integration, and examples.

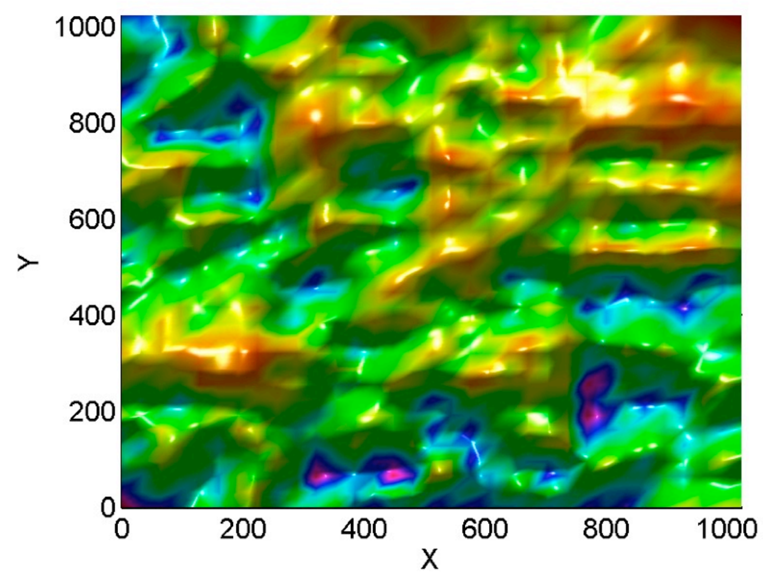
- Lecture IV - Extensions, challenges, and open questions

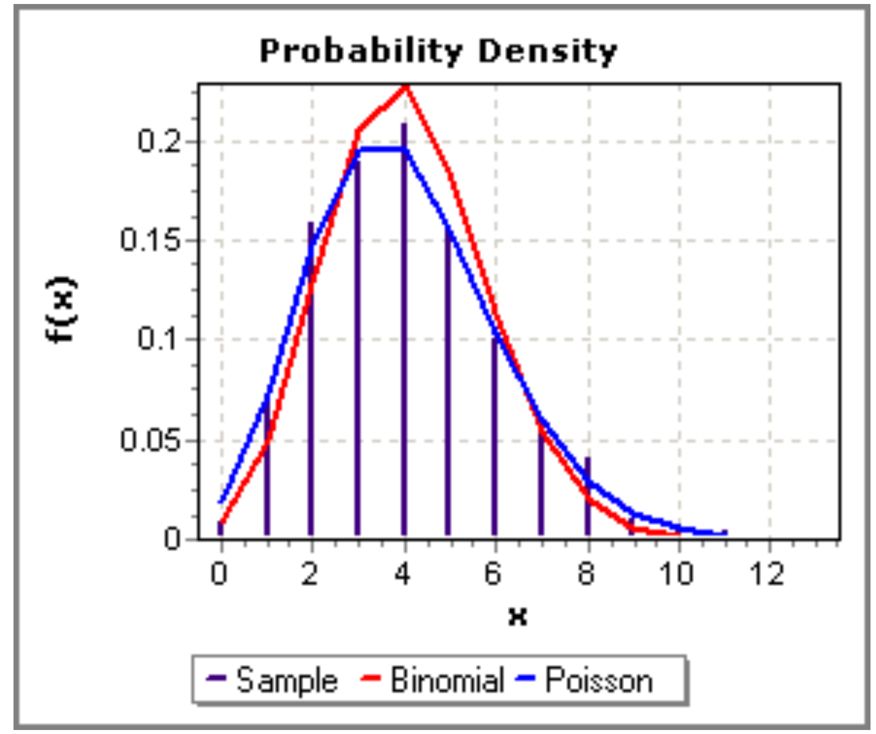
  Geometric uncertainty, ANOVA expansions, and discussion of open questions.

On smoothness

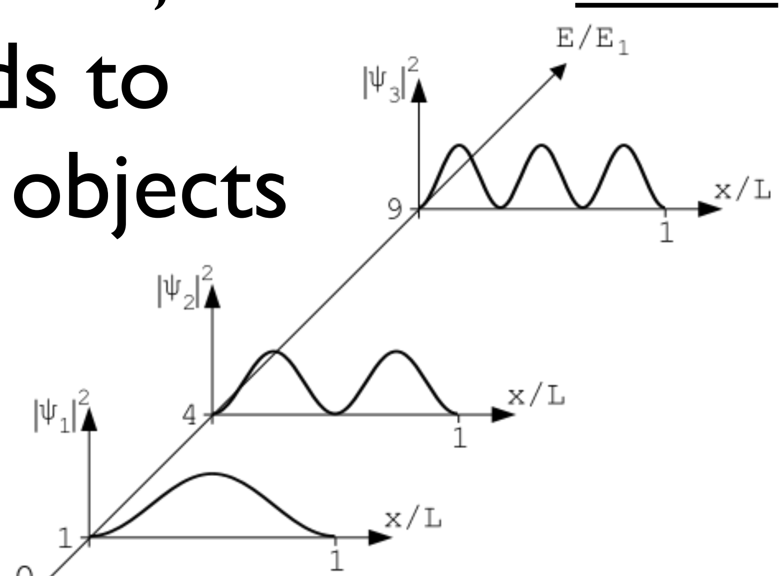
The assumption of smoothness is on the random variable - not on the solution - in MC something similar is natural

Imagine an experiment

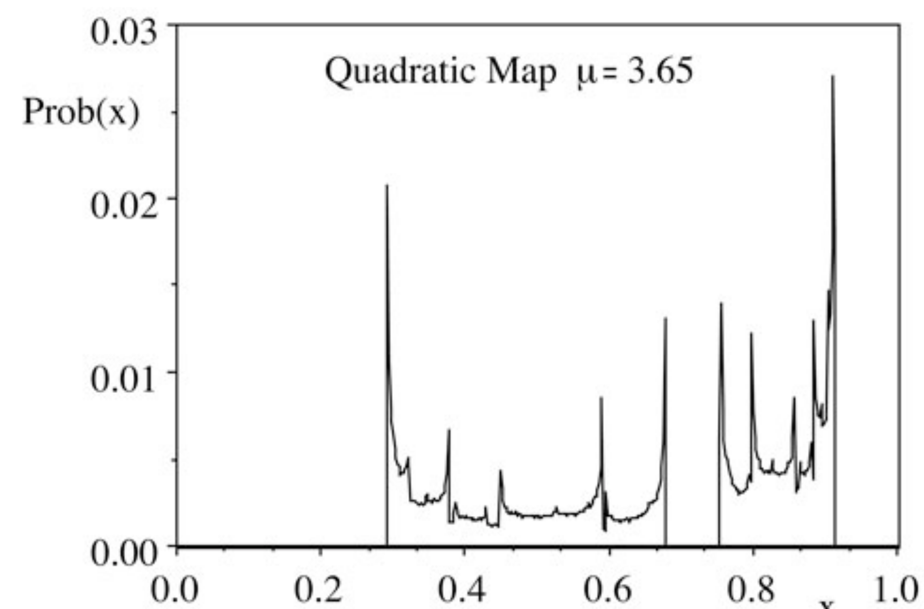




After the fit, evolution often also leads to smooth objects

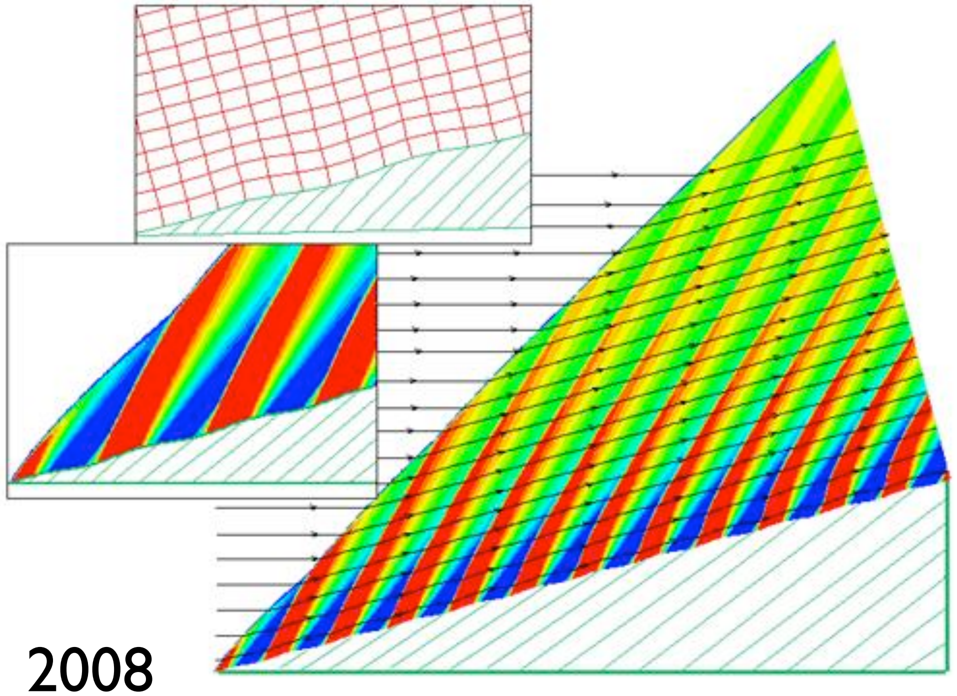


In this case, we are all out of luck



The local picture

- A brief reminder
- Stochastic Galerkin methods for ODEs
- Stochastic Galerkin methods for PDEs
- Extensions to non-Gaussian variables
- Summary



Lin et al, 2008

A brief reminder

Through a series of arguments we realized

- We need to be able to quantify with the impact of uncertainty in modeling of complex systems.

- While MC is tested and tried, its cost is problematic for complex systems and/or high accuracy requirements

- For many systems the random variables have smooth densities and this we should explore

- We introduced the Wiener Chaos expansion for this purpose

A brief reminder

We introduced the homogeneous Chaos expansion

$$f_N(\mathbf{X}) = \sum_{|i|=0}^{N} \hat{f}_i \Phi_i(\mathbf{X}) \in \mathsf{P}_N^d \qquad \dim \mathsf{P}_N^d = \left(\begin{array}{c} N+d \\ N \end{array} \right) = \frac{(N+d)!}{N!d!}$$

to represent functions of d-dimensional random vectors

$$\mathbf{X} = (X_1, \ldots, X_d)$$

$$F_{X_i}(x_i) = P(X_i \leq x_i)$$

$$F_X = F_{X_1} \times \ldots \times F_{X_d}$$

Here we defined the Chaos polynomial

$$\Phi_i(\mathbf{X}) = \Phi_{i_1}(X_1) \times \ldots \times \Phi_{i_d}(X_d)$$

$$\mathsf{E}[\Phi_i(\mathbf{X})\Phi_j(\mathbf{X})] = \int \Phi_i(\mathbf{x})\Phi_j(\mathbf{x})\, dF_X(x) = \gamma_i \delta_{ij} \qquad \gamma_i = \mathsf{E}[\Phi_i^2]$$

For Gaussian variables, these are known as Hermite Poly.

Stochastic Galerkin for ODEs

Let us now see how we can use these development

We consider again the simple ODE

$$\frac{du}{dt}(t, Z) = -\alpha(Z)u, \qquad u(t=0, Z) = \beta,$$

Let us assume that $\alpha \sim \mathrm{N}(\mu, \sigma^2)$ and express it as

$$\alpha_N(Z) = \sum_{i=0}^{N} a_i H_i(Z), \qquad a_0 = \mu, \qquad a_1 = \sigma, \qquad a_i = 0, \quad i > 1.$$

Similarly for the deterministic initial condition

$$\beta_N = \sum_{i=0}^{N} b_i H_i(Z), \qquad b_0 = \beta, \qquad b_i = 0, \quad i > 0,$$

Note: This is very simple for illustration only !

Stochastic Galerkin for ODEs

We now seek solutions of the form

$$v_N(t,Z) = \sum_{i=0}^{N} \hat{v}_i H_i(Z)$$

To find the N+1 unknown we apply the Galerkin procedure

$$\mathsf{E}\left[\frac{dv_N}{dt} H_k\right] = -\mathsf{E}[\alpha_N v_N H_k], \quad \forall k = 0, \dots, N$$

Yielding

$$\frac{d\hat{v}_k}{dt} = -\frac{1}{\gamma_k} \sum_{i=0}^{N} \sum_{j=0}^{N} a_i \hat{v}_j e_{ijk} \qquad \forall k = 0, \dots, N, \qquad e_{ijk} = \mathsf{E}[H_i H_j H_k]$$

$$\hat{v}_k(0) = b_k, \qquad 0 \le k \le N.$$

Can now be solved using a standard method

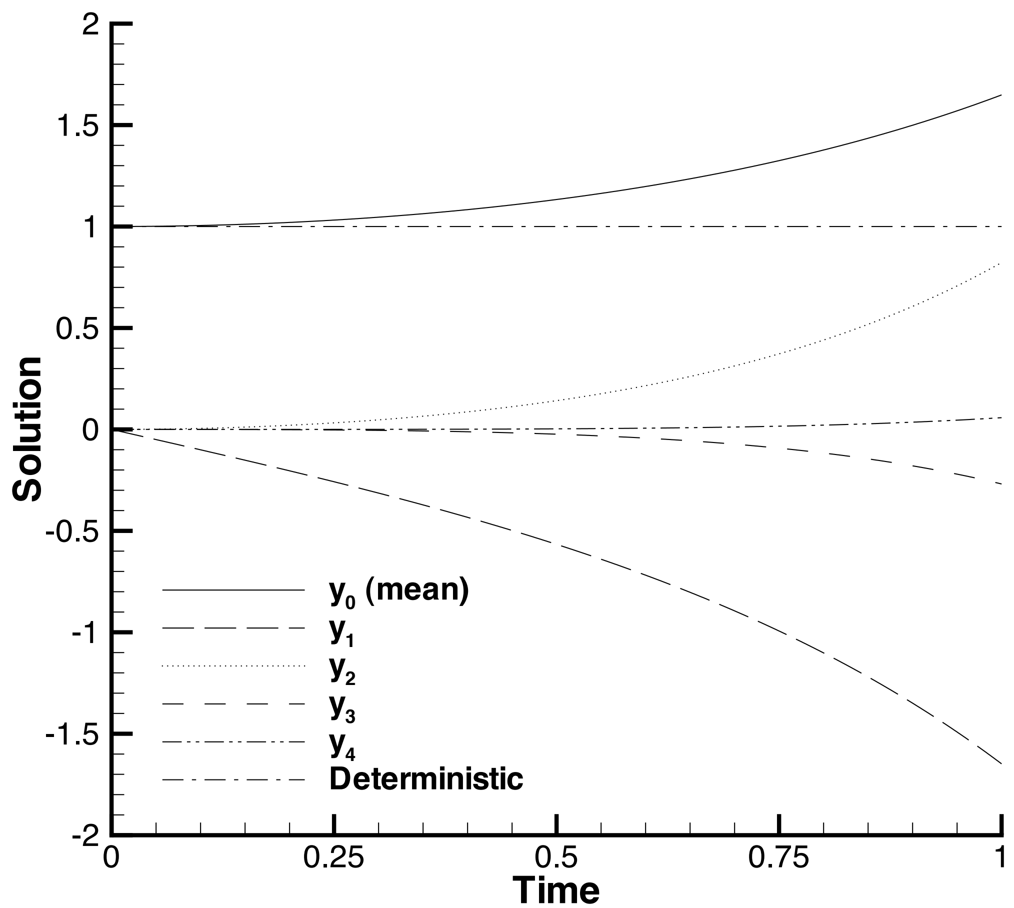
Stochastic Galerkin for ODEs

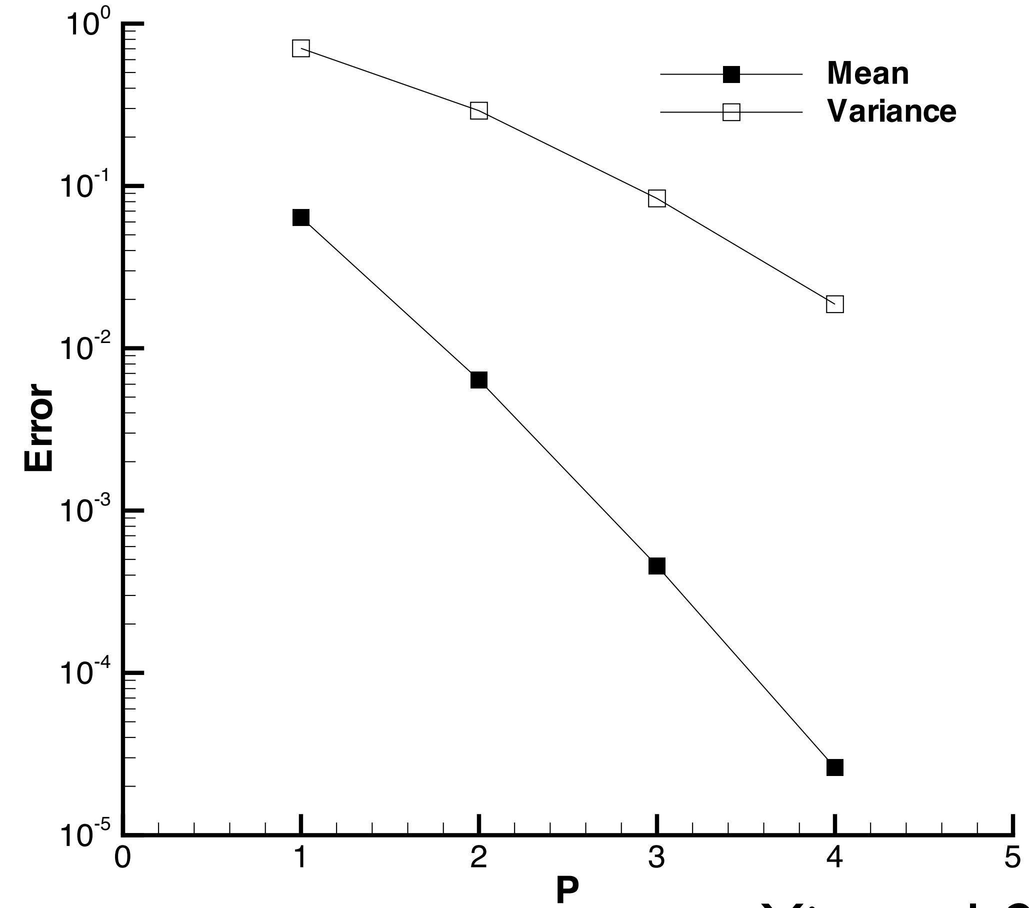
Define (N+1)x(N+1) matrix

$$A_{jk} = -\frac{1}{\gamma_k} \sum_{i=0}^{N} a_i e_{ijk},$$

to recover the system

$$\frac{d\mathbf{v}}{dt}(t) = \mathbf{A}^T \mathbf{v}, \qquad \mathbf{v}(0) = \mathbf{b},$$





Xiu et al, 2002

Stochastic Galerkin for ODEs

A few observations are worth making

- Solving with the mean coefficients is not sufficient
- A stochastic scalar problem becomes a deterministic system
- Some work is needed to derive system and matrix entries $e_{ijk} = \mathsf{E}[H_i H_j H_k]$
- System is only coupled with multiplicative randomness
- Spectral convergence is clear, i.e., we have recovered the benefits of global expansions from PDE solvers

Stochastic Galerkin for ODEs

Let us briefly discuss the generalization to general SDEs

$$\frac{du(\mathbf{X},t)}{dt} = f(\alpha(\mathbf{X}),u,t) + g(\beta(\mathbf{X}),t), \quad u(\mathbf{X},0) = h(\gamma(\mathbf{X}))$$

Parameters depends on d-dimensional random space

$$\alpha(\mathbf{X}) = (\alpha_1(\mathbf{X}),\dots,\alpha_k(\mathbf{X}))$$

$$\beta(\mathbf{X}) = (\beta_1(\mathbf{X}),\dots,\beta_l(\mathbf{X}))$$

$$\gamma(\mathbf{X}) = (\gamma_1(\mathbf{X}),\dots,\gamma_l(\mathbf{X}))$$

$$\mathbf{X} = (X_1,\dots,X_d)$$

As in the simple case they are expanded in Chaos expansion

$$\alpha_N(\mathbf{X}) = \sum_{|i|=0}^{N} \hat{\alpha}_i \Phi_i(\mathbf{X}) \quad \beta_N(\mathbf{X}) = \sum_{|i|=0}^{N} \hat{\beta}_i \Phi_i(\mathbf{X}) \quad \gamma_N(\mathbf{X}) = \sum_{|i|=0}^{N} \hat{\gamma}_i \Phi_i(\mathbf{X})$$

$$\hat{\alpha}_i = \frac{1}{\gamma_i}\mathsf{E}[\alpha(\mathbf{X})\Phi_i(\mathbf{X})] \quad \hat{\beta}_i = \frac{1}{\gamma_i}\mathsf{E}[\beta(\mathbf{X})\Phi_i(\mathbf{X})] \quad \hat{\gamma}_i = \frac{1}{\gamma_i}\mathsf{E}[\gamma(\mathbf{X})\Phi_i(\mathbf{X})]$$

Stochastic Galerkin for ODEs

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Now proceed and express the solution as

$$u_N(\mathbf{X},t) = \sum_{|i|=0}^{N} \hat{u}_i(t)\Phi_i(\mathbf{X})$$

Applying a Galerkin approach yields the system to solve

$$\frac{d}{dt}\mathsf{E}[u\,\Phi_k] = \frac{d\hat{u}_k}{dt} = \mathsf{E}[f(\alpha_N, u_N)\,\Phi_k] + \mathsf{E}[g(\beta_N)\,\Phi_k], \quad \forall |k| = 0,\ldots,N$$

$$\hat{u}_k(0) = \frac{1}{\gamma_k}\mathsf{E}[h(\gamma_N)\,\Phi_k], \quad |k| = 0,\ldots,N$$

- Total number of variables $\frac{(N+d)!}{N!d!} \sim \frac{N^d}{d!}$
- Generally terms $\mathsf{E}[f(\alpha_N, u_N)\,\Phi_k]$ and $\mathsf{E}[g(\beta_N)\,\Phi_k]$ must be evaluated through quadrature

Stochastic Galerkin for ODEs

Consider a biological cell-signal problem

$$\frac{de_{1p}}{dt} = \frac{I(t)}{1+G_4 e_{3p}} \frac{V_{\max,1}(1-e_{1p})}{K_{m,1}+(1-e_{1p})} - \frac{V_{\max,2}e_{1p}}{K_{m,2}+e_{1p}},$$

$$\frac{de_{2p}}{dt} = \frac{V_{\max,3}e_{1p}(1-e_{2p})}{K_{m,3}+(1-e_{2p})} - \frac{V_{\max,4}e_{2p}}{K_{m,4}+e_{2p}},$$

$$\frac{de_{3p}}{dt} = \frac{V_{\max,5}e_{2p}(1-e_{3p})}{K_{m,5}+(1-e_{3p})} - \frac{V_{\max,6}e_{3p}}{K_{m,6}+e_{3p}}.$$

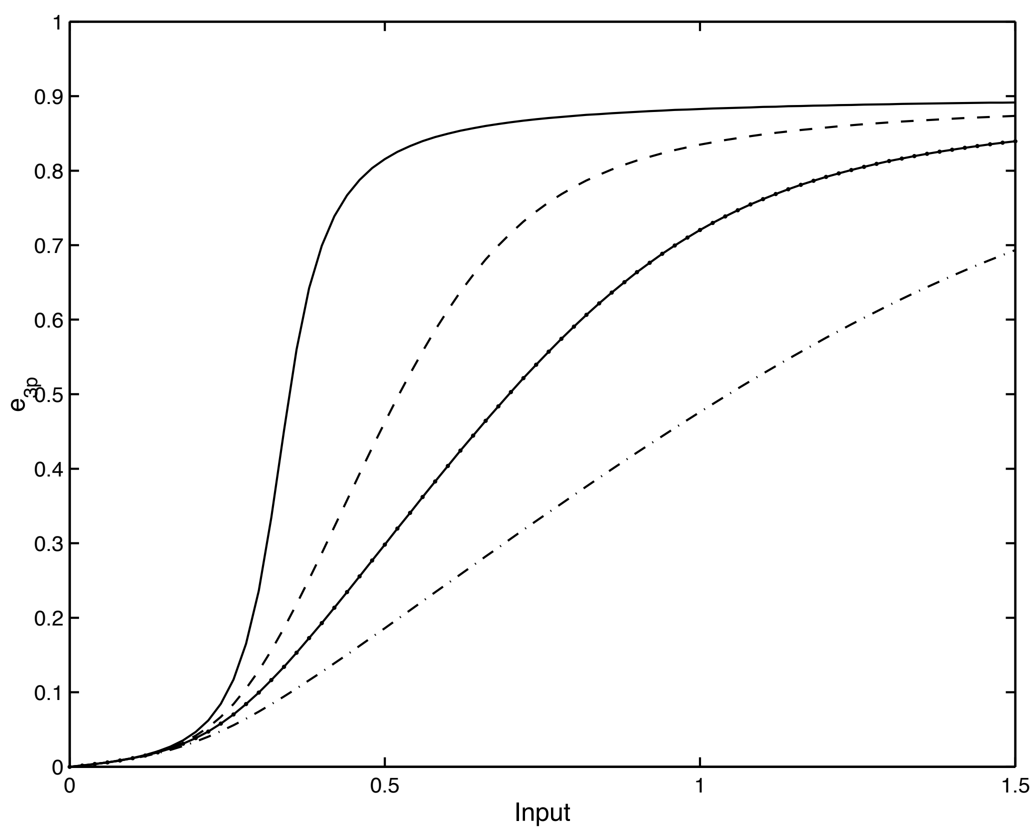
$$K_{m,1-6} = 0.2$$

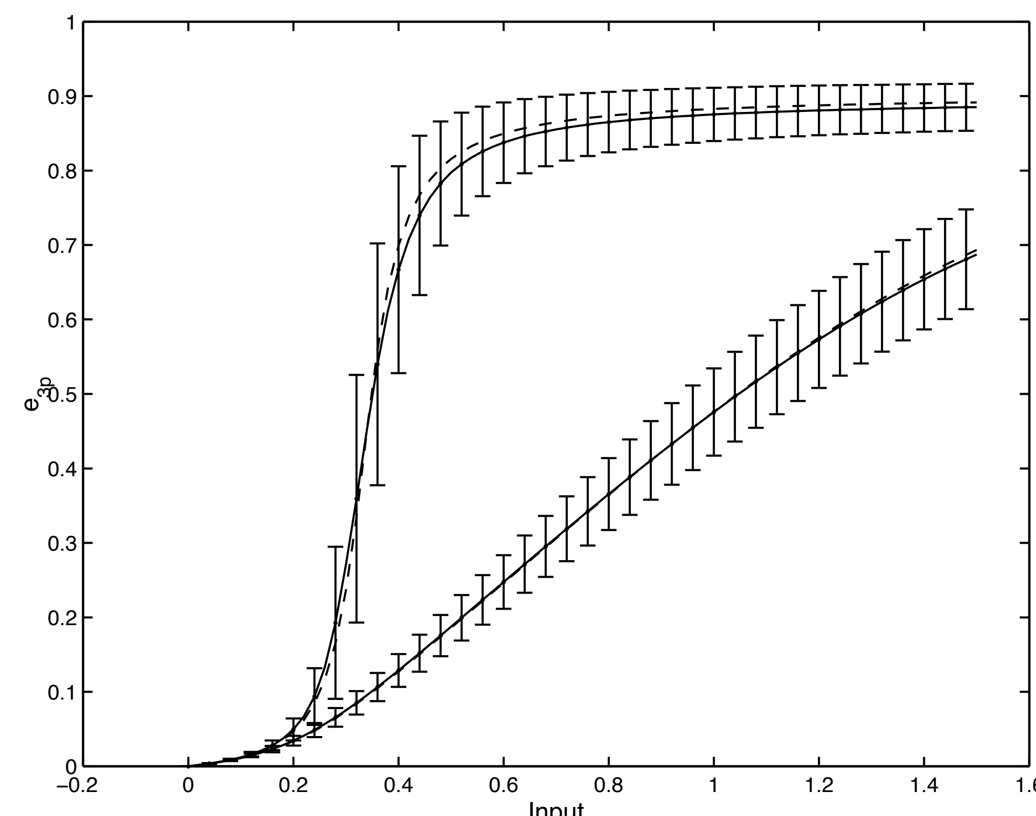
$$< V_{max} >= (0.5, 0.15, 0.15, 0.15, 0.25, 0.05)$$

I(t) is a control parameter

Model $\quad V_{max,i} =< V_{max} > (1+\sigma X_i) \quad \sigma = 0.1 \quad f_{X_i} = U(-1,1)$

N=2





Xiu, 2007

Stochastic Galerkin for ODEs

Consider as example a genetic toggle switch

$$\frac{du}{dt} = \frac{\alpha_1}{1+v^\beta} - u,$$

$$\frac{dv}{dt} = \frac{\alpha_2}{1+\omega^\gamma} - v,$$

$$\omega = \frac{u}{(1+[IPTG]/\mathcal{K})^\eta}$$

$$\alpha = (\alpha_1, \cdots, \alpha_6) = (\alpha_1, \alpha_2, \beta, \gamma, \eta, \mathcal{K})$$

IPTG is a control parameter

<u>Model</u>

$$\alpha(\mathbf{X}) = <\alpha> (1+\sigma\mathbf{X})$$

$$f_{X_i} = U(-1,1)$$

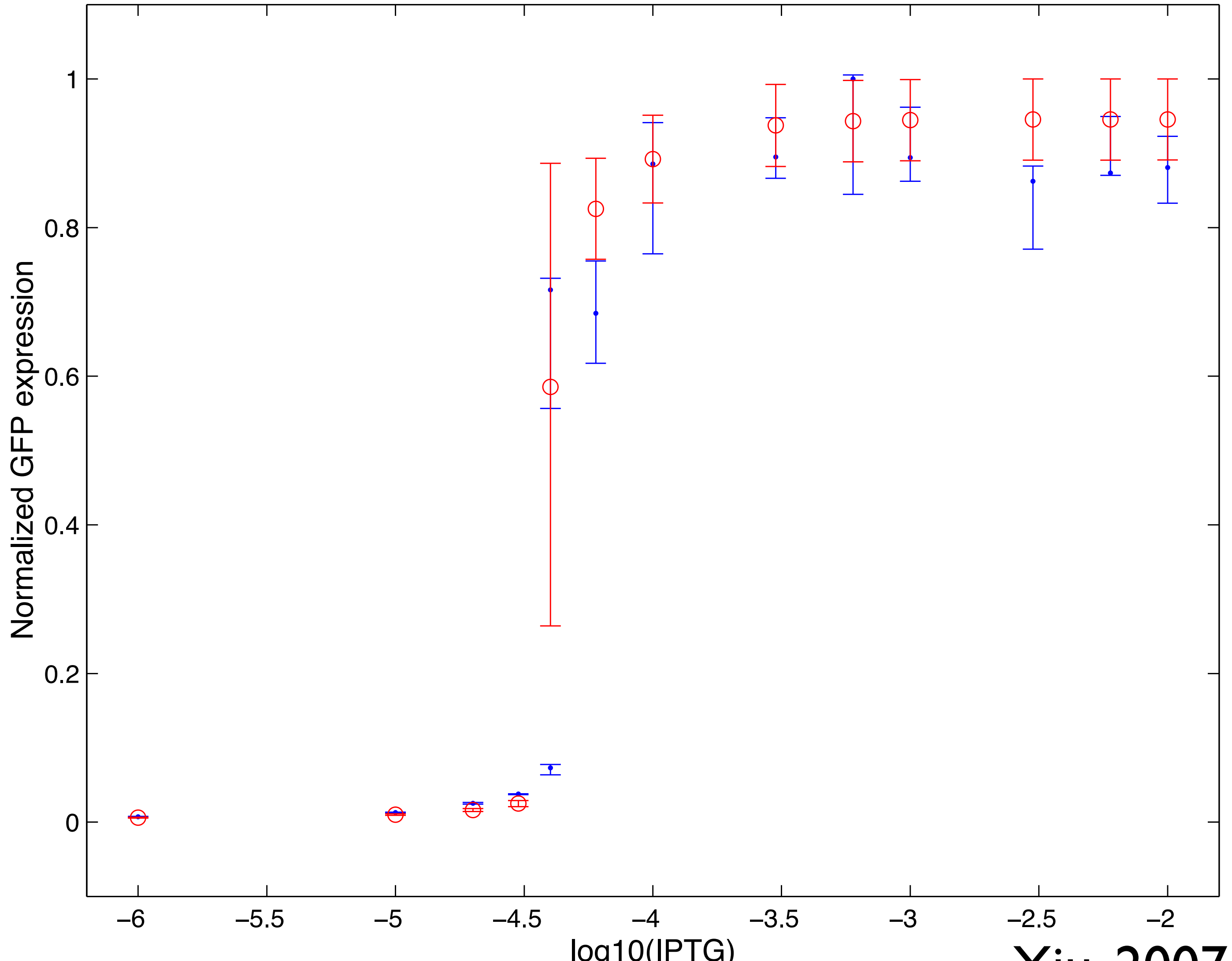
$$\langle\alpha\rangle = (156.25, 15.6, 2.5, 1, 2.0015, 2.9618\times 10^{-5})$$

$$\sigma = 0.1$$

Experimental data

Computation results

N=2



Xiu, 2007

Stochastic Galerkin for ODEs

Lets summarize our results for SDEs

- Approach is systematic
- SDE scalar problems leads to deterministic coupled systems of ODEs
- Results for both linear and non-linear are convincing and the potential for savings significant.

What changes for SPDEs ?

Stochastic Galerkin for PDEs

Consider the general SPDE

$$\begin{cases} u_t(x,t,\omega) = \mathcal{L}(u), & D \times (0,T] \times \Omega, \\ \mathcal{B}(u) = 0, & \partial D \times [0,T] \times \Omega, \\ u = u_0, & D \times \{t=0\} \times \Omega, \end{cases}$$

Assume that the uncertainty can be represented by

$$Z = (Z_1, \dots, Z_d)$$

to recover the recognizable formulation

$$\begin{cases} u_t(x,t,Z) = \mathcal{L}(u), & D \times (0,T] \times \mathbb{R}^d, \\ \mathcal{B}(u) = 0, & \partial D \times [0,T] \times \mathbb{R}^d, \\ u = u_0, & D \times \{t=0\} \times \mathbb{R}^d. \end{cases}$$

Stochastic Galerkin for PDEs

Let us first consider the elliptic problem

$$\begin{cases} \nabla \cdot [\kappa(x;\omega)\nabla u(x;\omega)] = f(x;\omega), & (x;\omega) \in D \times \Omega \\ u(x;\omega) = g(x;\omega), & (x;\omega) \in \partial D \times \Omega \end{cases}$$

We continue as before

$$\kappa_N(x, Z(\omega)) = \sum_{n=0}^{N} \hat{\kappa}_n(x)\Phi_n(Z)$$

$$f_N(x, Z(\omega)) = \sum_{n=0}^{N} \hat{f}_n(x)\Phi_n(Z) \qquad g_N(x, Z(\omega)) = \sum_{n=0}^{N} \hat{g}_n(x)\Phi_n(Z)$$

and seek solutions of the form

$$u_N(x, Z(\omega)) = \sum_{n=0}^{N} \hat{u}_n(x)\Phi_n(Z)$$

Stochastic Galerkin for PDEs

Inserting this into the PDE yields

$$\sum_{n=0}^{N}\sum_{m=0}^{N}\left[\nabla\cdot(\hat{\kappa}_n\nabla\hat{u}_m)\right]\Phi_n\Phi_m = \sum_{n=0}^{N}\hat{f}_n\Phi_n$$

Applying the Galerkin procedure yields

$$\sum_{n=0}^{N}\sum_{m=0}^{N}\left[\nabla\cdot(\hat{\kappa}_n\nabla\hat{u}_m)\right]e_{mnk} = \hat{f}_k\mathsf{E}[\Phi_k^2]$$

$$e_{mnk} = \mathsf{E}[\Phi_m\,\Phi_n\,\Phi_k]$$

with boundary conditions

$$\hat{u}_n = \mathsf{E}[g_N\,\Phi_n]$$

Essentially the same as for the SDE

Stochastic Galerkin for PDEs

- Requires the solution of N+1 coupled of the form

$$\sum_{m=0}^{N} \nabla \cdot (\tilde{\kappa}_{mk} \nabla \hat{u}_m) = \hat{f}_k \gamma_k \qquad \tilde{\kappa}_{mk} = \sum_{n=0}^{N} \hat{\kappa}_n e_{mnk}$$

- In space you can discretize as you prefer to recover

$$\mathbf{Au} = \mathbf{f}$$

where (u,f) are (N+1)xDOF long vectors.

- Procedure requires solvers to be rewritten.

Stochastic Galerkin for PDEs

Lets consider a couple of examples

$$\frac{d}{dx}\left[\kappa(x;\omega)\frac{du}{dx}(x;\omega)\right] = 0, \qquad x \in [0,1], \qquad u(0;\omega) = 0, \qquad u(1;\omega) = 1.$$

Diffusivity is assumed random

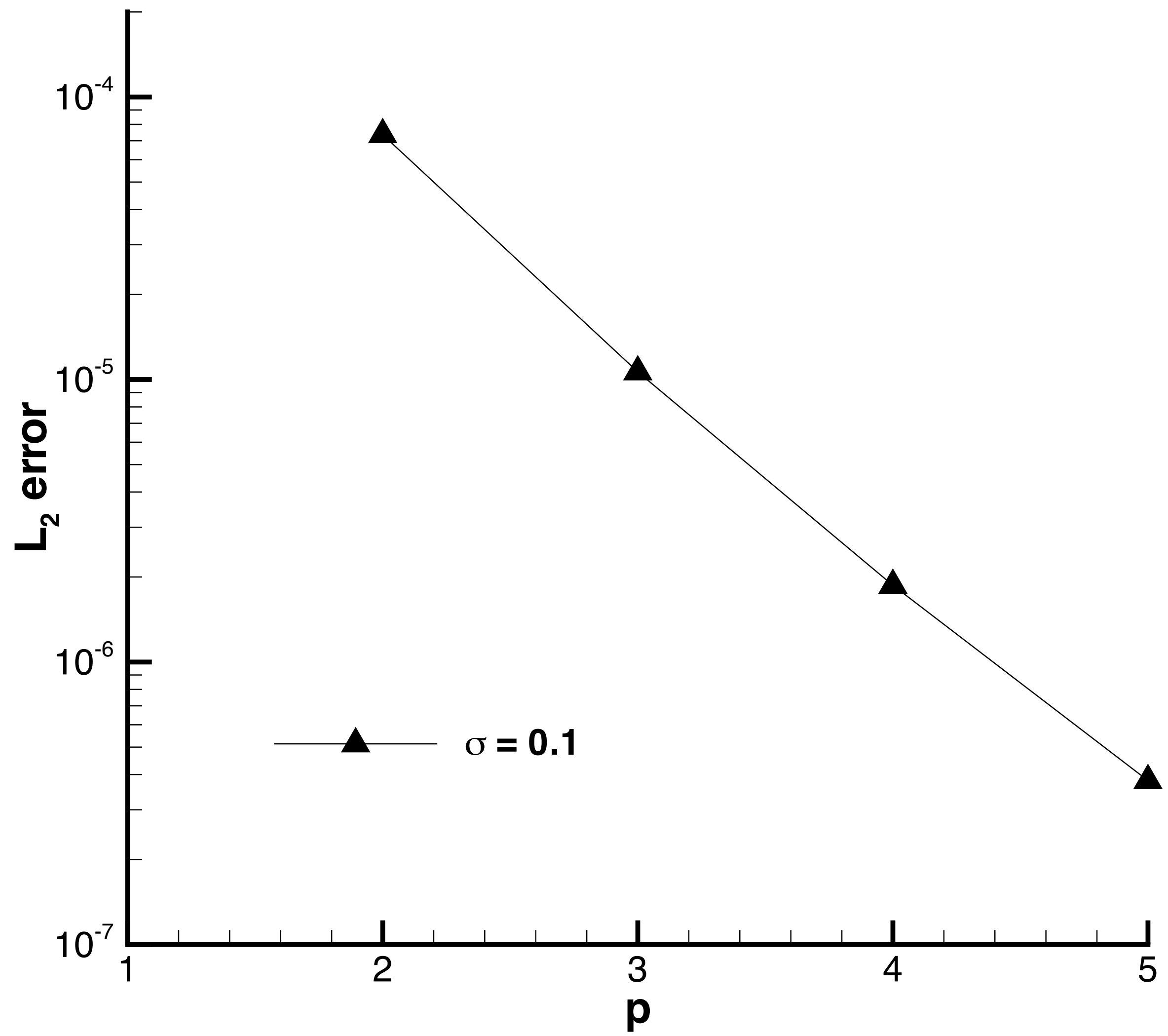
$$\kappa(x;\omega) = 1 + \epsilon(\omega)x,$$

<u>Model</u>

$$\epsilon(\omega) = \sigma X$$

$$f_X = \mathrm{N}(0,1)$$

$$\sigma = 0.1$$



Xiu, 2004

Stochastic Galerkin for PDEs

Consider a 2nd example

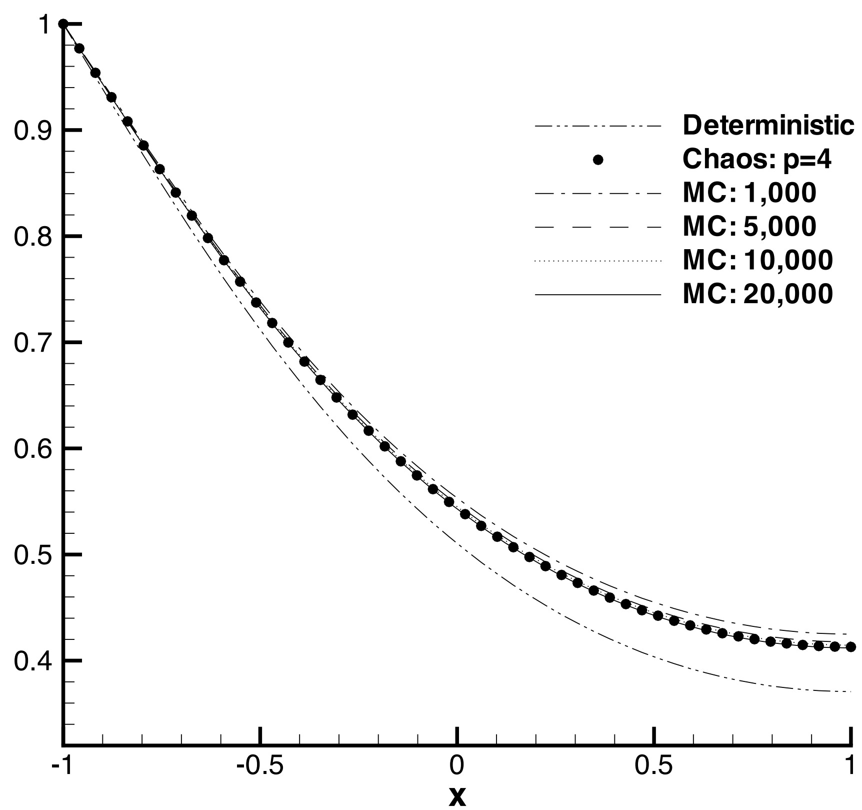
$$\nabla \cdot [\kappa(x,y;\omega)\nabla u(x,y;\omega)] = f(x,y;\omega), \qquad (x,y) \in [-1,1]\times[-1,1]$$

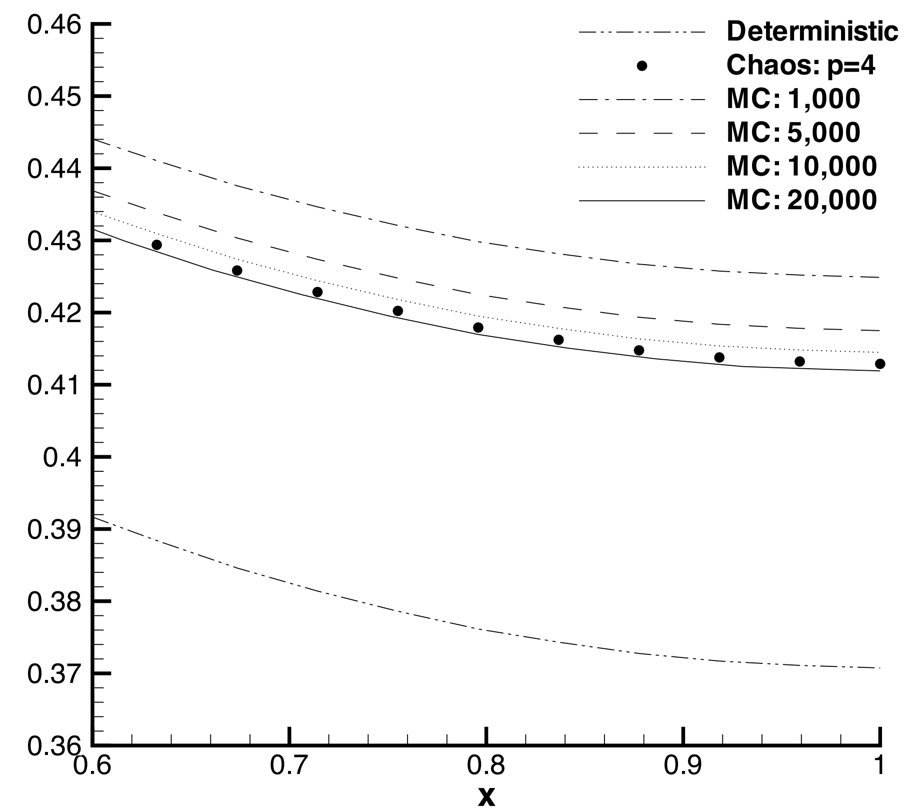
$$u(-1,y;\omega)=1, \qquad \frac{\partial u}{\partial x}(1,y;\omega)=0, \qquad u(x,-1;\omega)=0, \qquad \frac{\partial u}{\partial y}(x,1;\omega)=0.$$

<u>Model</u> $\quad f(x,y;X_1)=\sigma_f X_1 \qquad \kappa(x,y;X_2)=(1+\sigma_\kappa)X_2 \qquad f_{X_i}=\mathrm{N}[0,1]$

$\sigma_f=\sigma_\kappa=0.2$

N=4





Xiu, 2004

Stochastic Galerkin for PDEs

Let us also consider time dependent problems

$$\frac{\partial u(x,t,Z)}{\partial t} = c(Z)\frac{\partial u(x,t,Z)}{\partial x}, \quad x \in (-1,1), \quad t > 0,$$

$$u(x,0,Z) = u_0(x,Z). \qquad \begin{aligned} u(1,t,Z) &= u_R(t,Z), \quad & c(Z) > 0, \\ u(-1,t,Z) &= u_L(t,Z), \quad & c(Z) < 0. \end{aligned}$$

We proceed as before

$$v_N(x,t,Z) = \sum_{i=0}^{N} \hat{v}_i(x,t)\Phi_i(Z)$$

Applying the Galerkin procedure results in

$$\frac{\partial \hat{v}_k(x,t)}{\partial t} = \sum_{i=0}^{N} a_{ik}\frac{\partial \hat{v}_i(x,t)}{\partial x}, \qquad k = 0,\ldots,N, \qquad a_{ik} = \mathsf{E}[c(Z)\,\Phi_i(Z)\,\Phi_k(Z)]$$

or the system

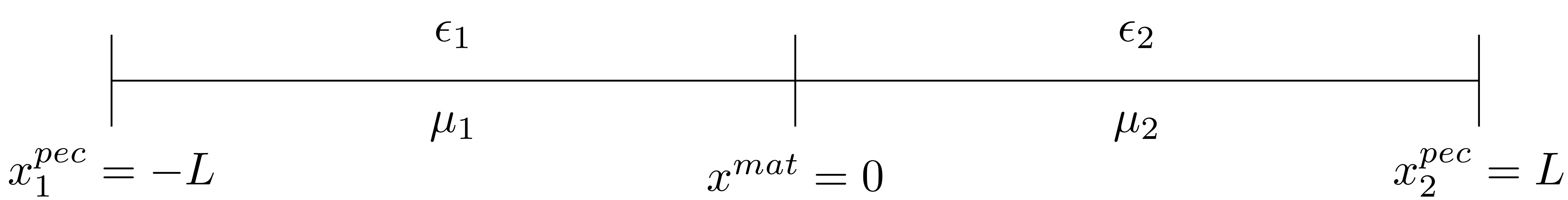
$$\frac{\partial \mathbf{v}(x,t)}{\partial t} = \mathbf{A}\frac{\partial \mathbf{v}(x,t)}{\partial x}.$$

Stochastic Galerkin for PDEs

Consider Maxwell's equations

$$\epsilon \frac{\partial \mathbf{E}^s}{\partial t} = \nabla \times \mathbf{H}^s + \sigma \mathbf{E}^s + \mathbf{S}^E,$$
$$\mu \frac{\partial \mathbf{H}^s}{\partial t} = -\nabla \times \mathbf{E}^s + \mathbf{S}^H,$$

We first consider a 1D cavity problem



$$\varepsilon_1 = \mu_1 = \mu_2 = 1 \qquad \varepsilon_2(x, Z) = 2.25\left(1 + \sigma \frac{Z^2}{1 + Z^2}\right)$$

We wish to estimate sensitivity of eigenfrequencies

Stochastic Galerkin for PDEs

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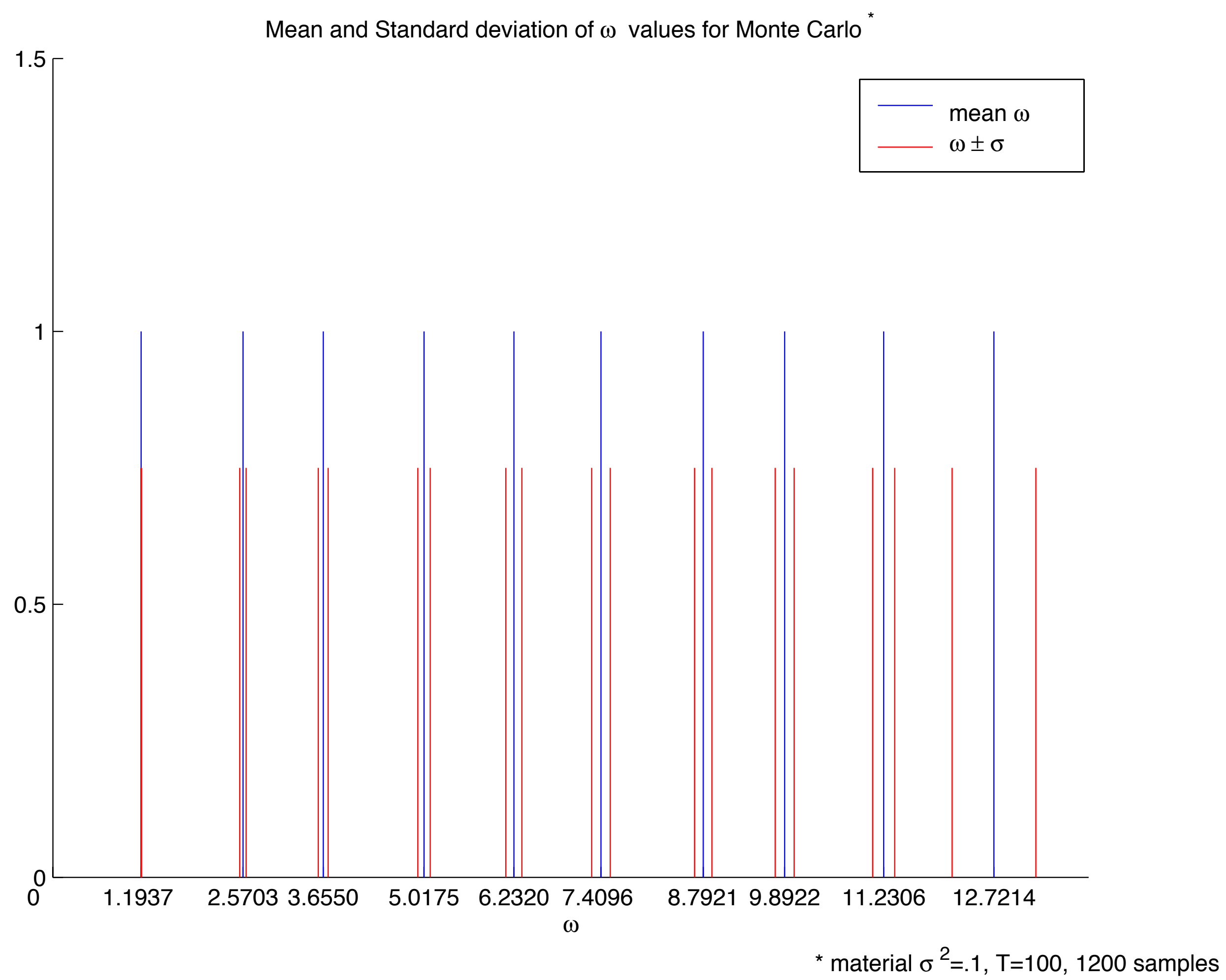
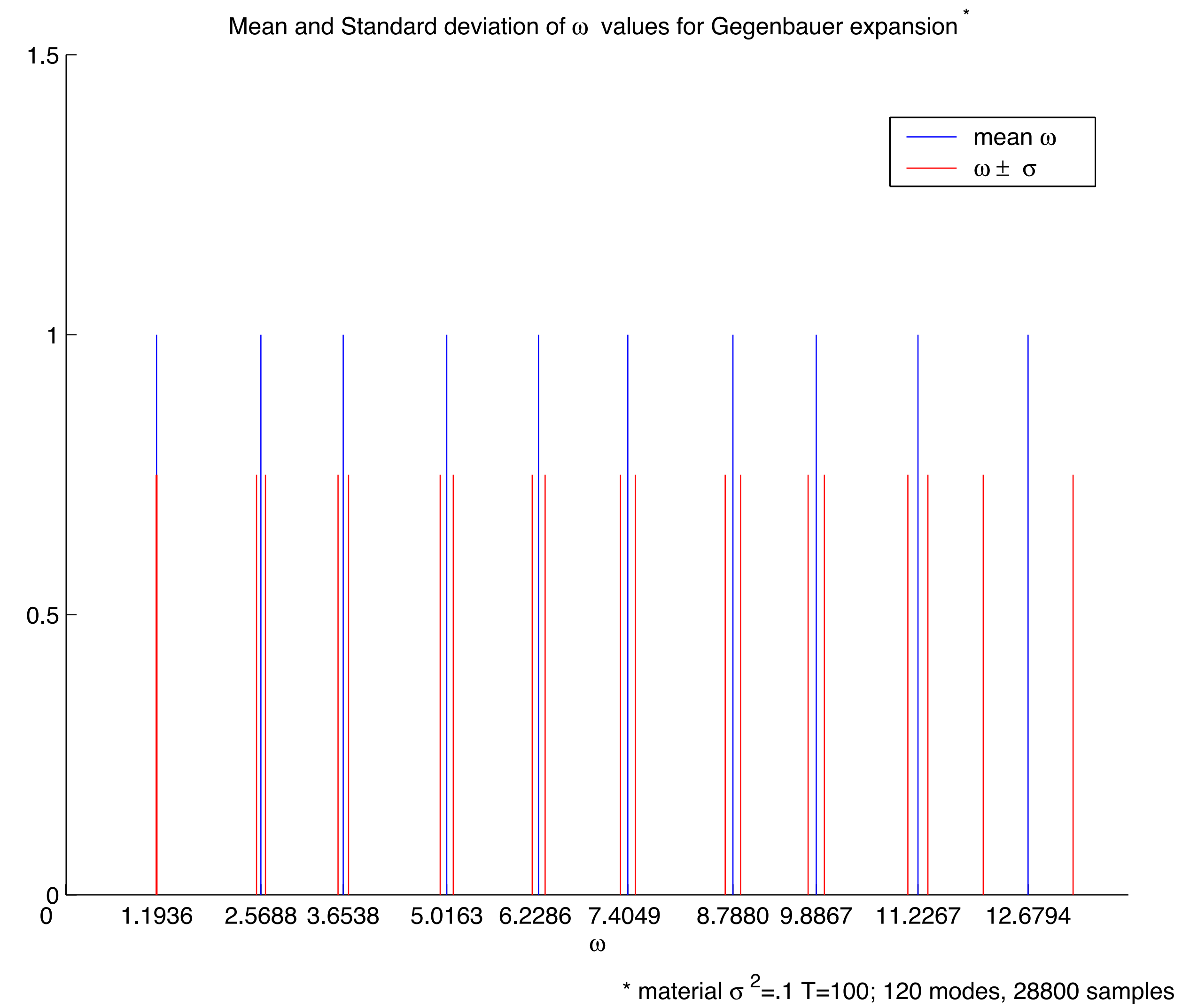
Direct comparison with MC



1200 samples

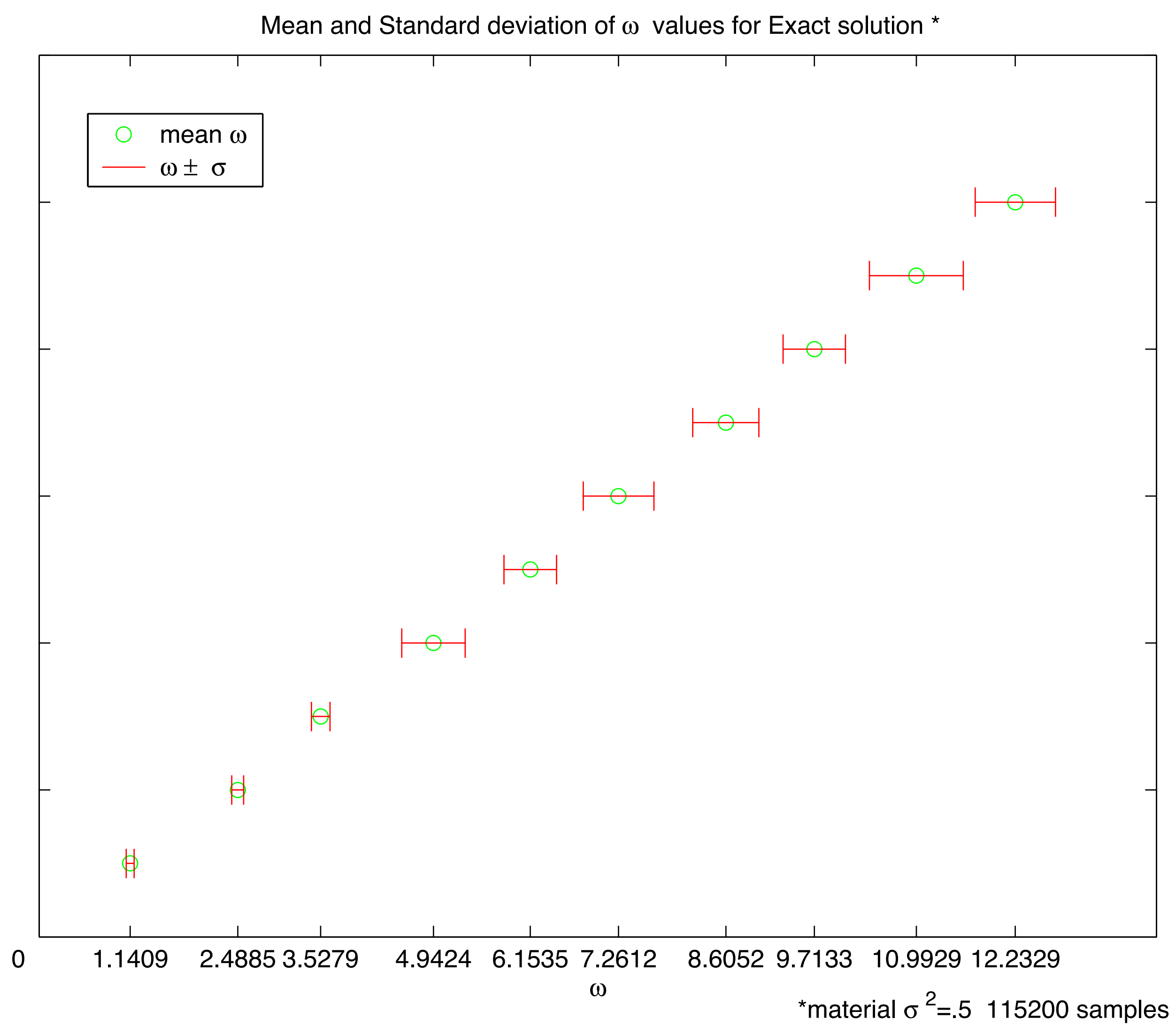


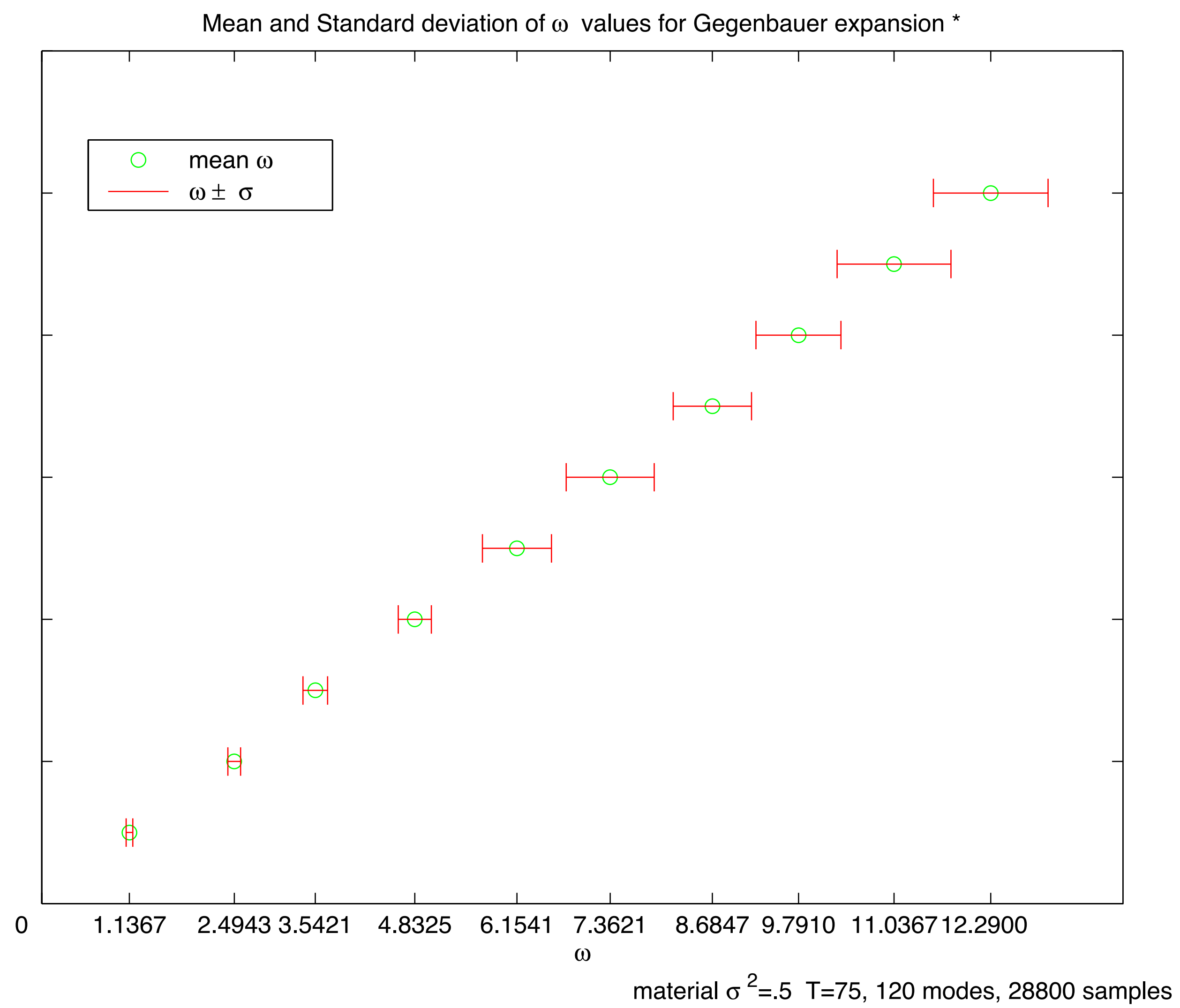
N=120

Stochastic Galerkin for PDEs

Computation for large variation

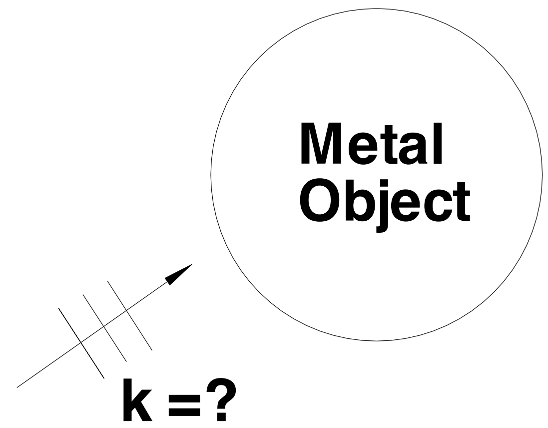
$$\sigma = 0.25$$

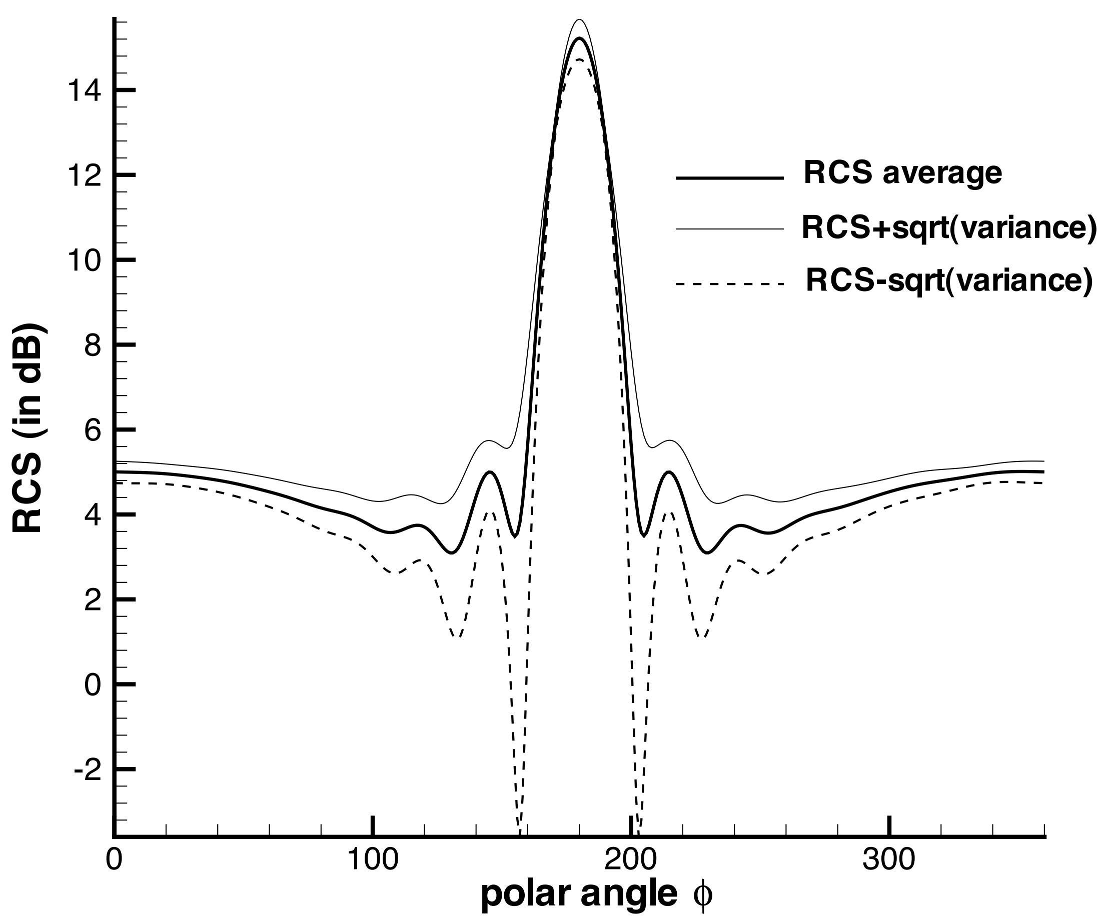


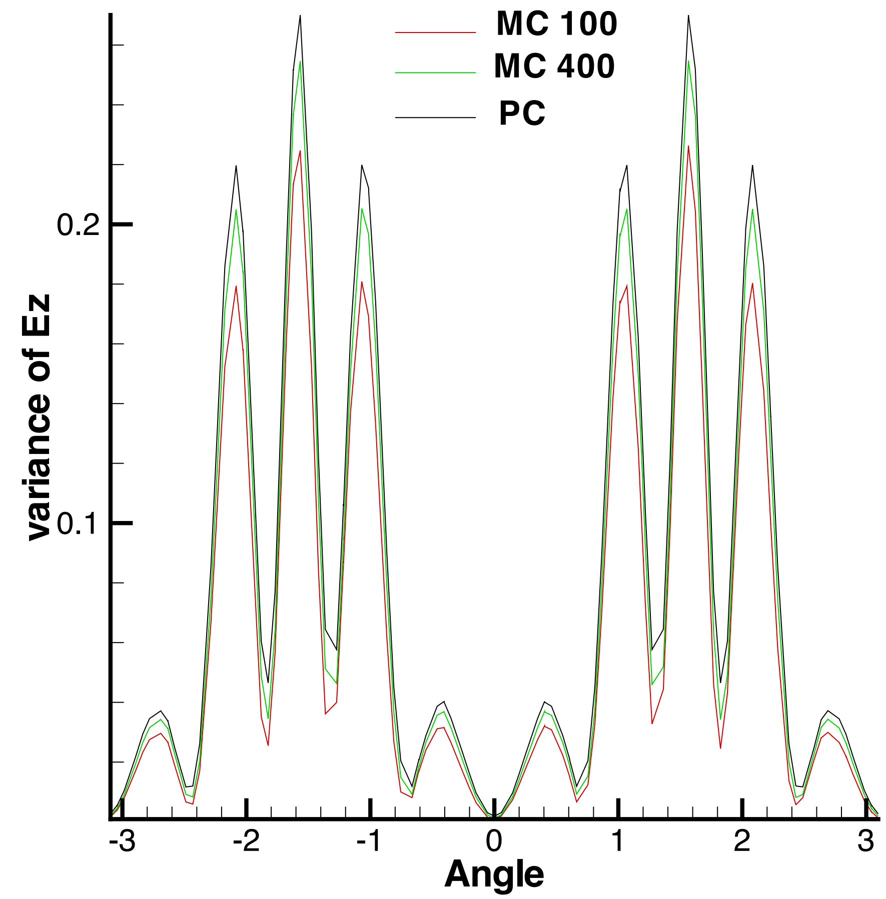


Stochastic Galerkin for PDEs

Let us also consider a 2D scattering problem





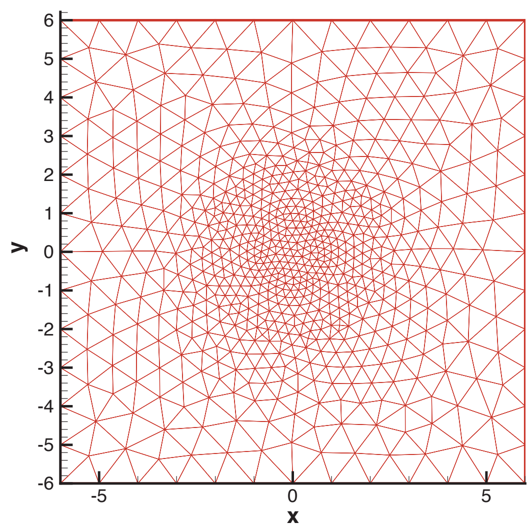


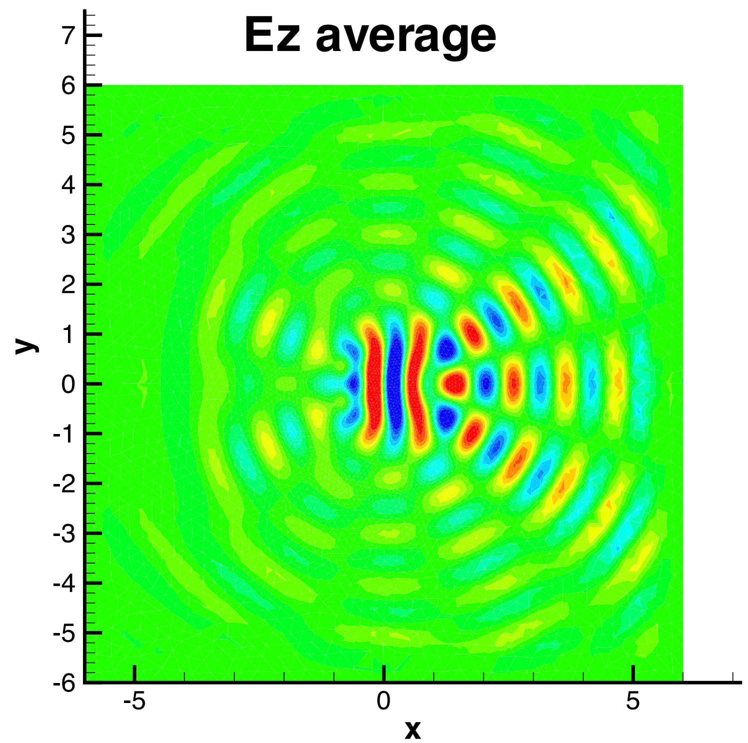
N=4

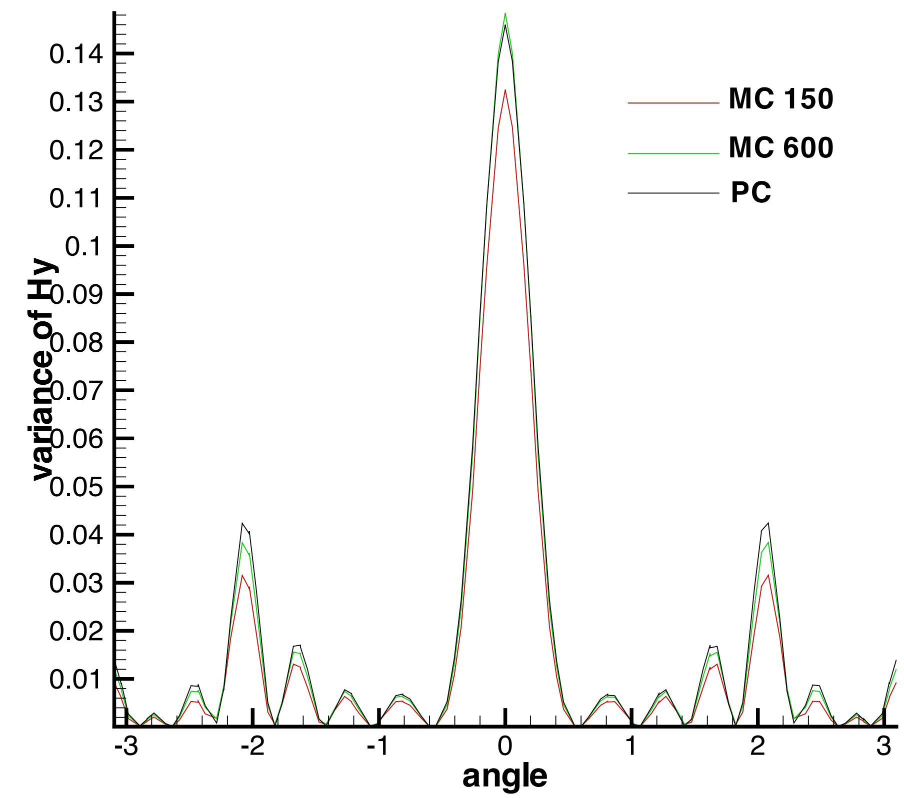
$$\mathbf{k}(Z) = [\cos(0.1Z), \sin(0.1Z)]$$

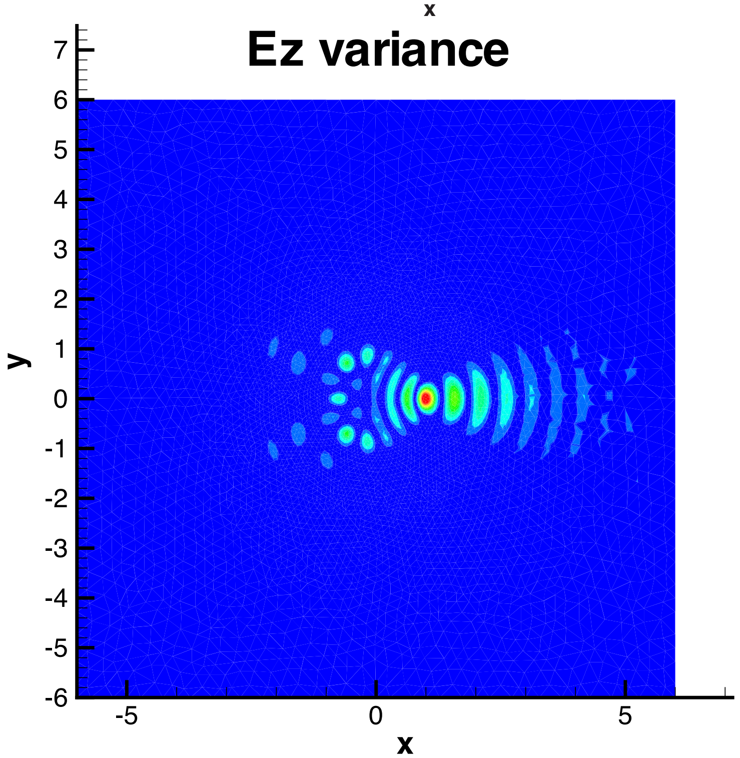
Stochastic Galerkin for PDEs

Consider a 2D material scattering problem

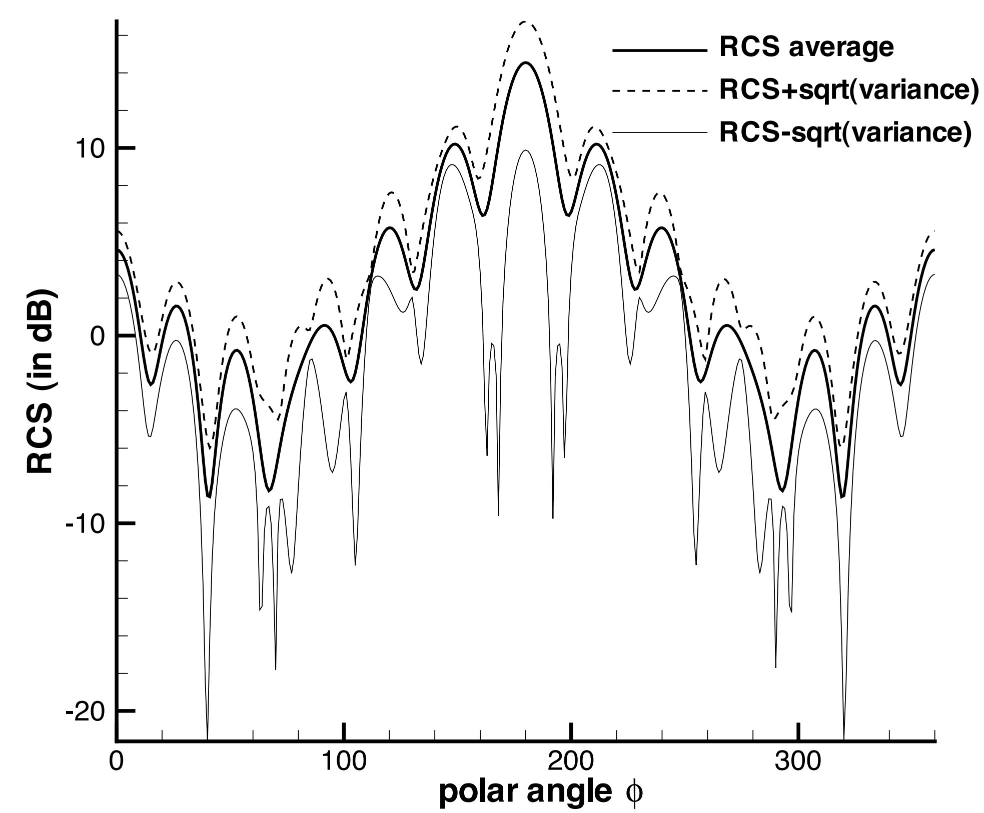








N=8



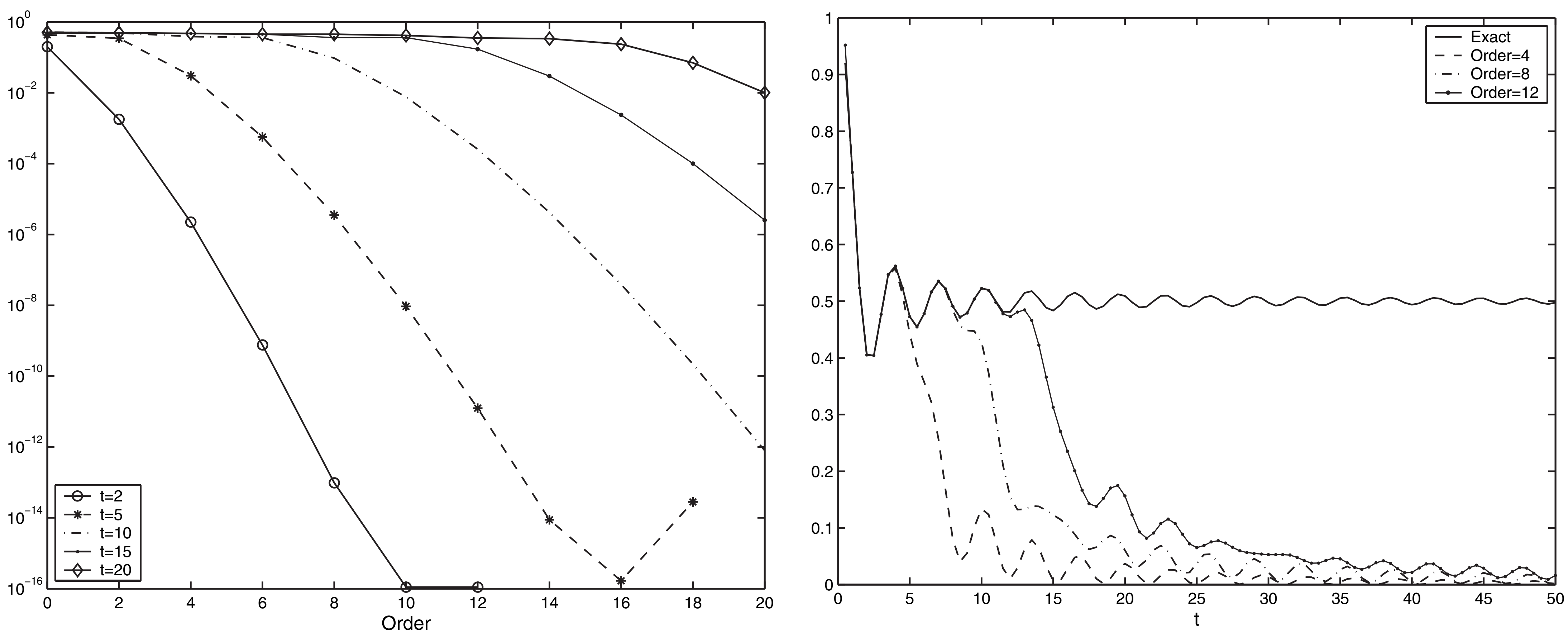
$$\varepsilon_2(x, Z) = 2.25\left(1 + \sigma\frac{Z^2}{1+Z^2}\right)$$

Stochastic Galerkin for PDEs

Let us again consider

$$\frac{\partial u(x,t,Z)}{\partial t} = c(Z)\frac{\partial u(x,t,Z)}{\partial x}, \quad x \in (-1,1), \quad t > 0,$$

Xiu, 2010



- Spectral convergence at fixed time as expected
- Resolution requirement is time-dependent

Stochastic Galerkin for PDEs

One easily proves the following result

$$\mathsf{E}[\|u - u_N\|^2] \leq \frac{C}{N^{2m-1}} t$$

m depends on smoothness in Z

Assume periodicity in x and write the solution as

$$\hat{u}_n(t, Z) = \hat{u}_n(0) \exp(inc(Z)t)$$

Shows that in Z the wavenumber to resolve is t-dependent

It is a property of the equation -- worst case scenario.

This remains a major practical challenge

Stochastic Galerkin for PDEs

Let us briefly consider nonlinear problems

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & x \in [-1, 1], \\ u(-1) = 1 + \delta(Z), & u(1) = -1, \end{cases}$$

Assume again

$$v_N(x, t, Z) = \sum_{i=0}^{N} \hat{v}_i(x, t)\Phi_i(Z)$$

Following the same procedure yields

$$e_{mnk} = \mathsf{E}[\Phi_m \, \Phi_n \, \Phi_k]$$

$$\frac{\partial \hat{v}_k}{\partial t} + \frac{1}{\gamma_k} \sum_{i=0}^{N} \sum_{j=0}^{N} \hat{v}_i \frac{\partial \hat{v}_j}{\partial x} e_{ijk} = \nu \frac{\partial^2 \hat{v}_k}{\partial x^2}, \qquad k = 0, \dots, N,$$

More complex non-linearities become problematic

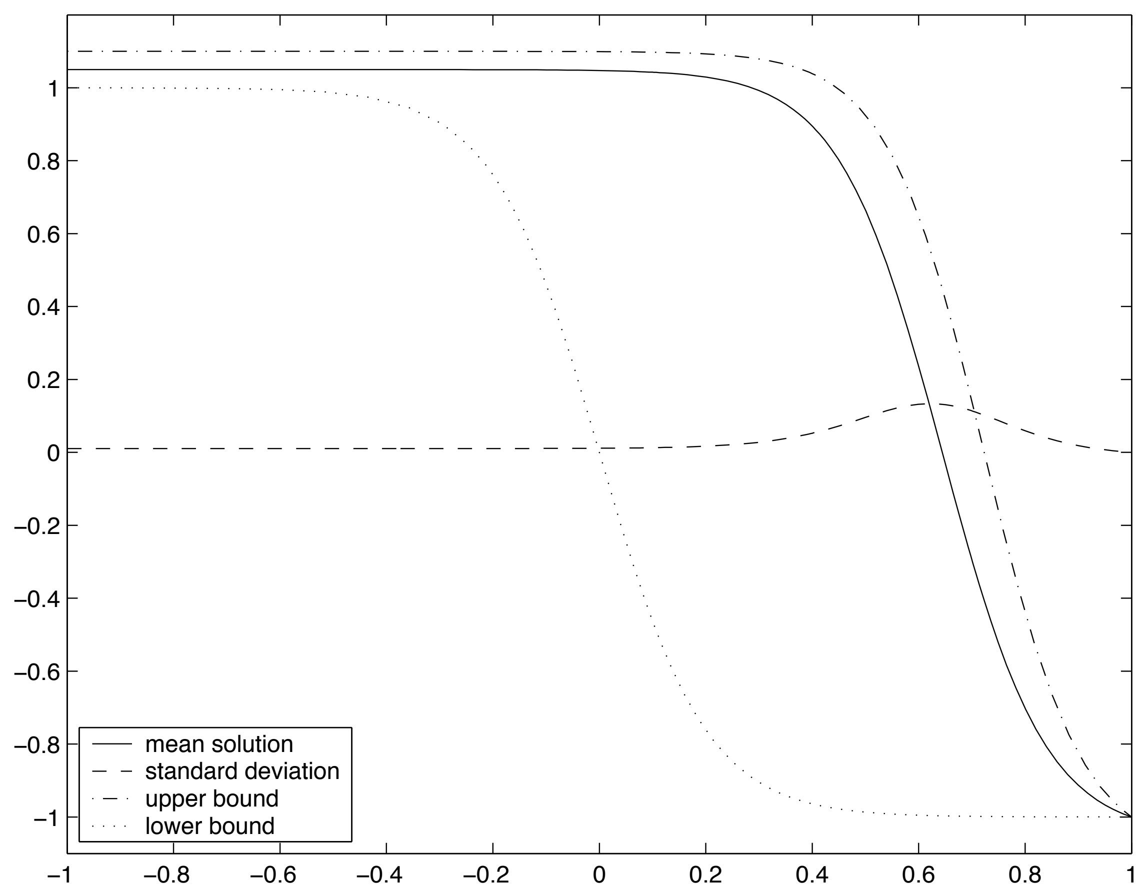
$$e^u \Rightarrow \qquad \mathbb{E}\left[e^{v_N} \Phi_k\right] = \int e^{\sum_i \hat{v}_i \Phi_i(z)} \Phi_k(z) dF_Z(z),$$

Stochastic Galerkin for PDEs

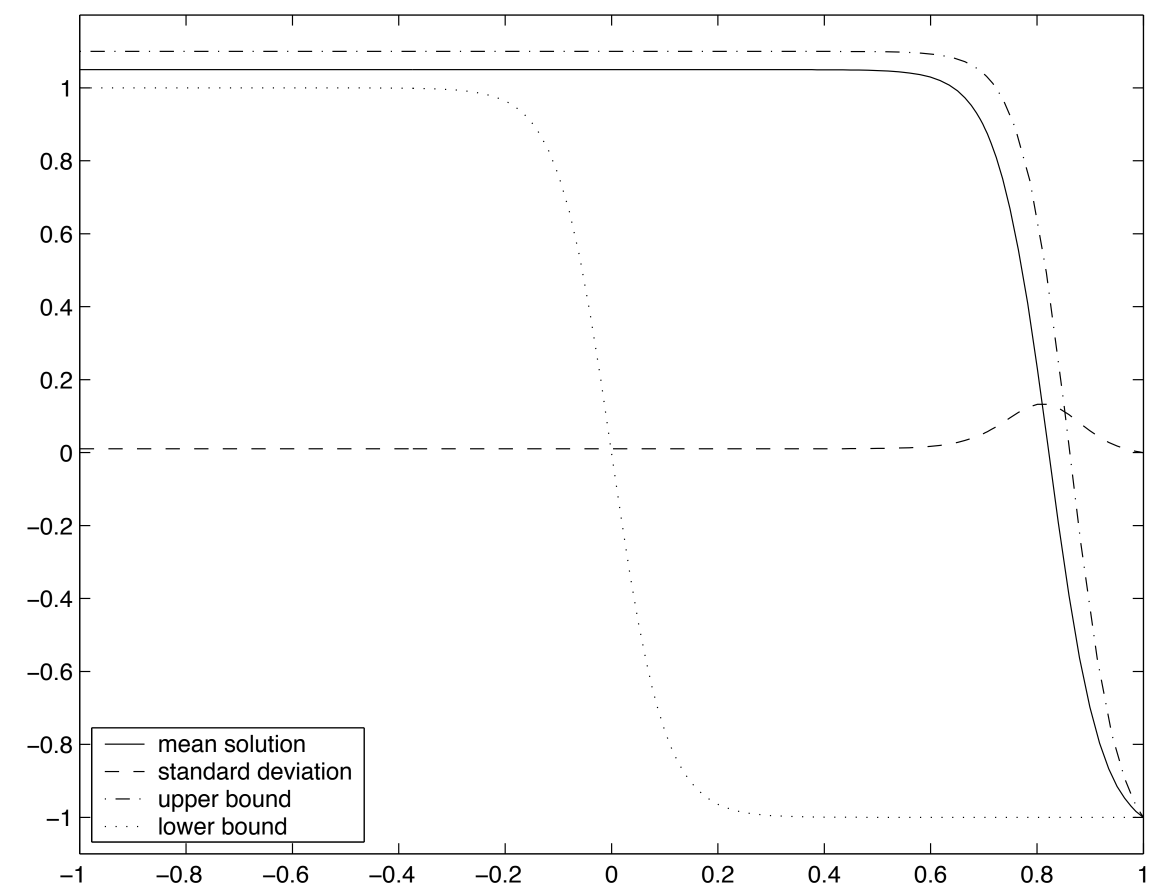
Consider the Burgers problem

Xiu, 2010

$\nu = 0.1$



$\nu = 0.05$



$$\delta = (1 + 0.1Z)$$

$$f_Z = \mathrm{N}[0, 1]$$

Referred to supersensitivity

Stochastic Galerkin for PDEs

- Galerkin approach reformulates SPDE to larger system of deterministic PDE
- The approach is systematic and applicable to general systems of SPDE's.
- Standard PDE solvers need to be rewritten but standard methods are applicable
- Main issue with nonlinear problems is cost.
- Advection dominated problems have special challenges

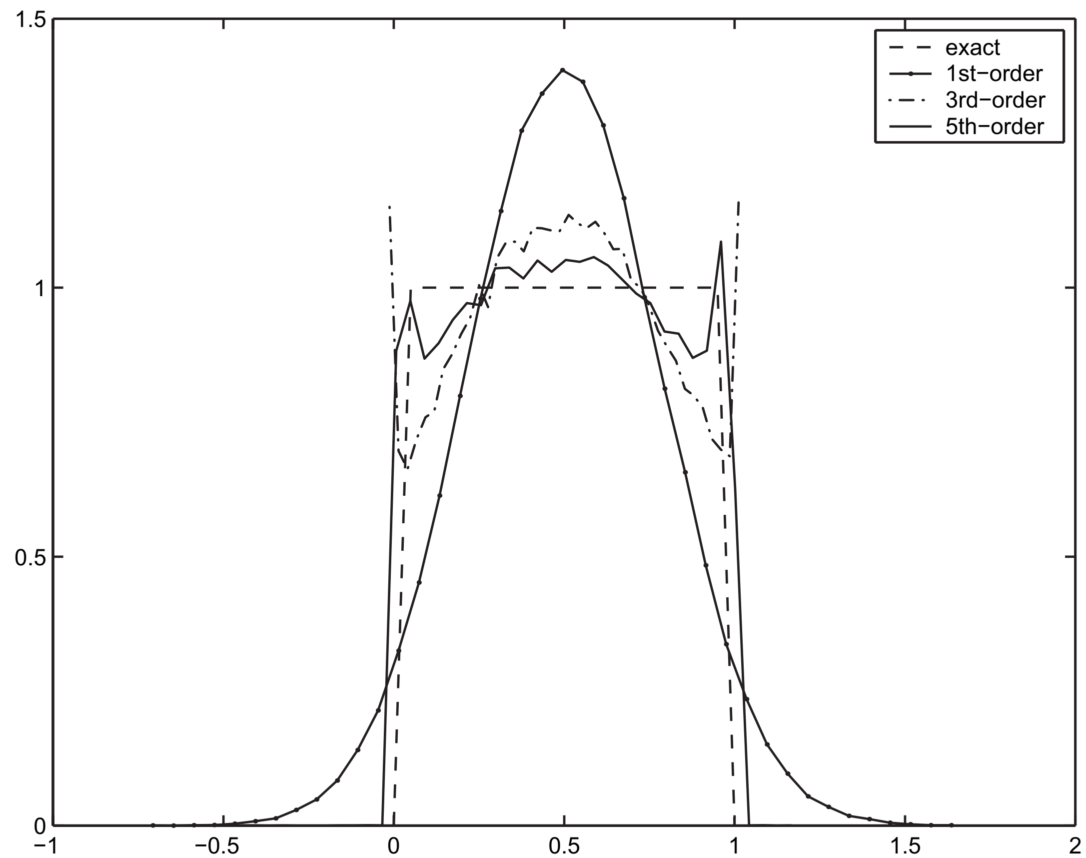
Non-Gaussian variables and gPC

Focus has been on homogeneous Chaos expansions and Hermite polynomials as originally proposed by Wiener.

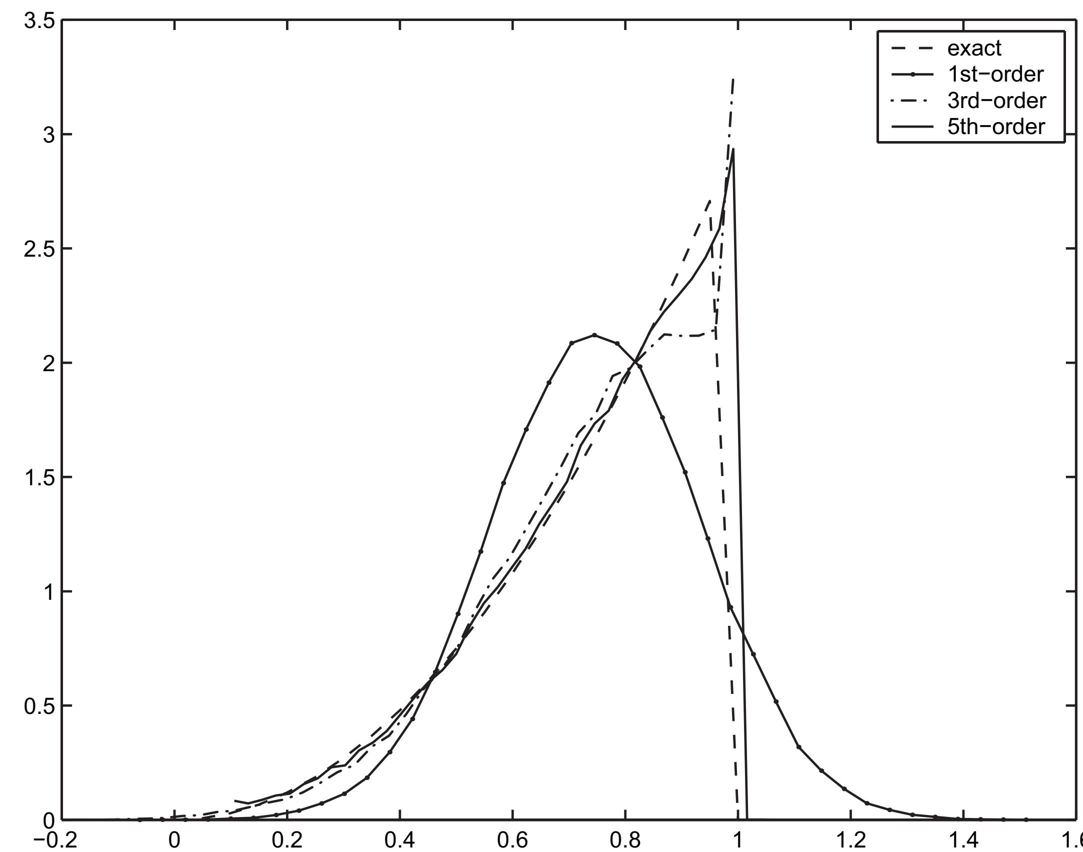
From the weak approximation results, we know this is ok

But is it a good idea ?

Xiu, 2010



Approximation of uniform density



Approximation of Beta density

Non-Gaussian variables and gPC

Recall that we introduced the polynomial chaos basis as

$$\mathsf{E}[\Phi_m(X)\Phi_n(X)] = \int \Phi_m(X(x))\Phi_n(X(x))\, dF_X(x) = \gamma_n \delta_{mn}$$

$$\gamma_n = \mathsf{E}[\Phi_n^2(X)]$$

and the Chaos expansion as

$$f(X) = \sum_{n=0}^{\infty} \hat{f}_n \Phi_n(X) \qquad \hat{f}_n = \frac{1}{\gamma_n}\mathsf{E}[f(X)\Phi_n(X)]$$

$$dF_X = \rho(x)dx$$

where the density is associated with the random variable

$$F_X = \mathsf{N}[0,1] \qquad \Rightarrow \qquad \rho(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

This suggests that the suitable basis depends on the density

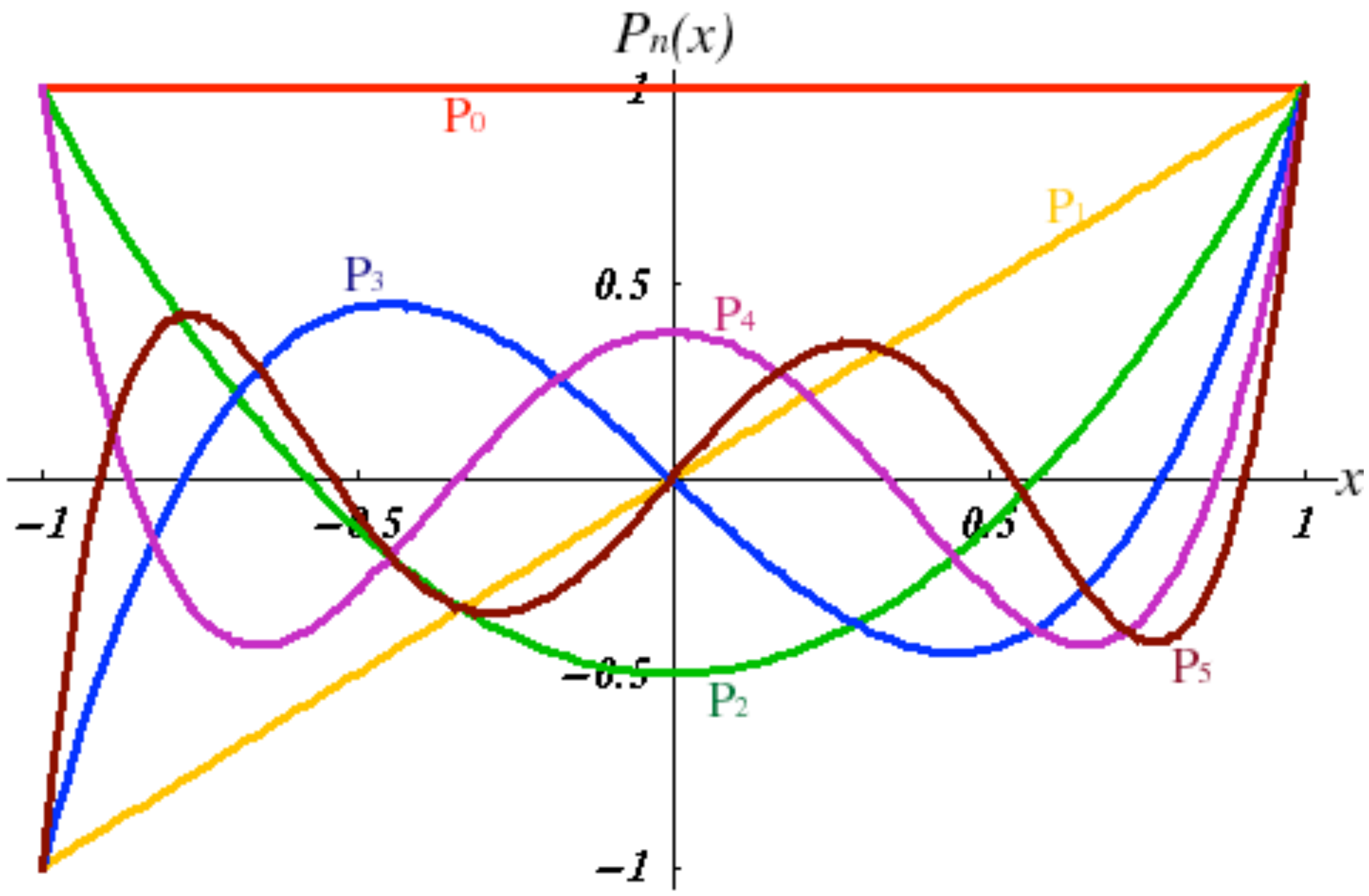
Non-Gaussian variables and gPC

Uniformly distributed variables: $U[-1,1]$

Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \dots$$

$$\rho(x) = \frac{1}{2}$$



Non-Gaussian variables and gPC

Beta-distributed variables: $B(\alpha, \beta)$

Jacobi polynomials: $P_0^{(\alpha,\beta)}(x) = 1, \;\; P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha - \beta + (\alpha + \beta + 2)x),$

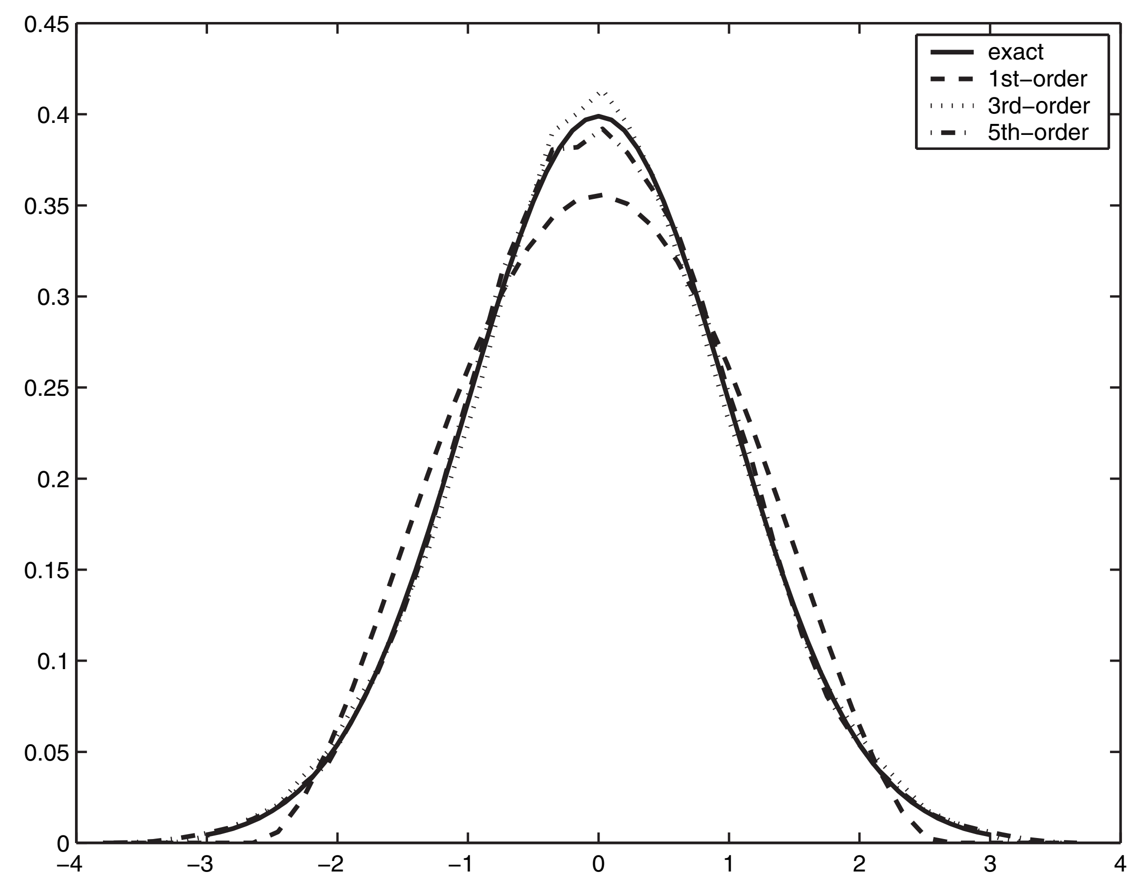
$$\rho(x) = C(\alpha, \beta)(1-x)^\alpha(1+x)^\beta, \;\; x \in [-1, 1], \;\; \alpha, \beta \geq 0$$

$$C(\alpha, \beta) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}$$

Well suited to model general densities.

Ex: Approximation of Gaussian by $P^{(10,10)}(x)$

Effective truncated Gaussian



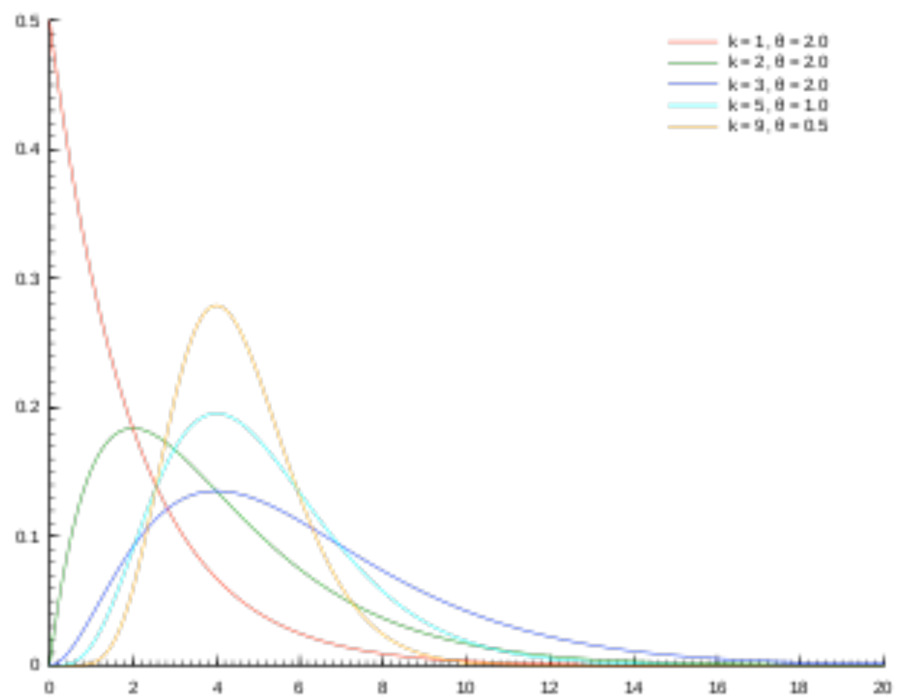
Xiu, 2010

Non-Gaussian variables and gPC

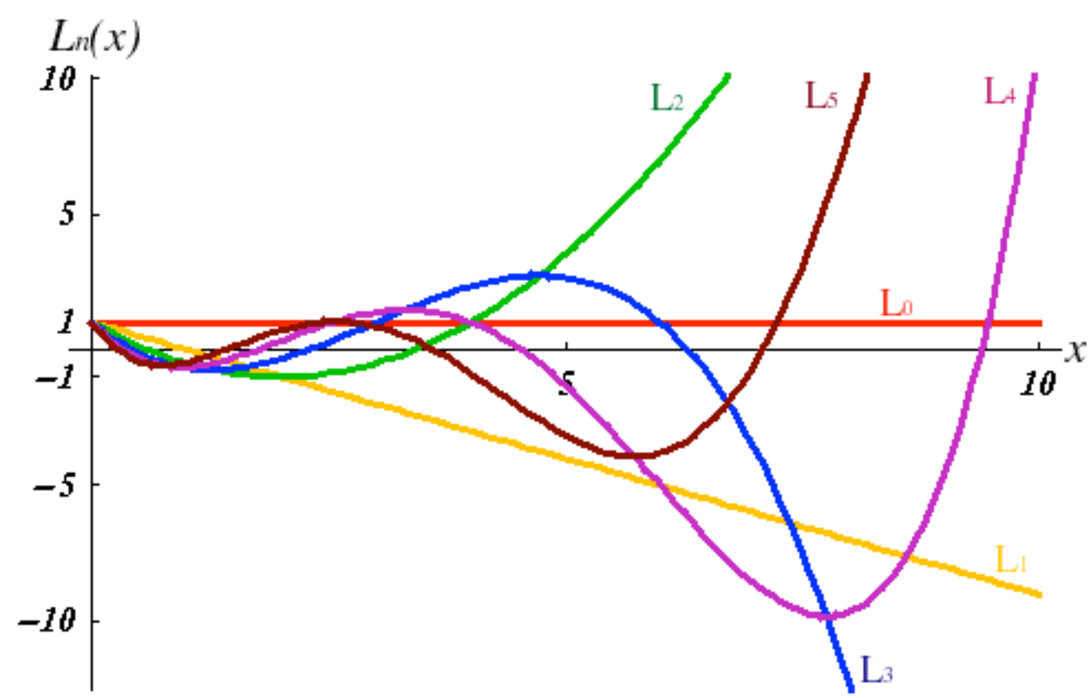
Gamma distributed variable: $\Gamma(r, c)$

Laguerre polynomials

$$L_0(x) = 1, \;\; L_1(x) = -x + 1, \;\; L_2(x) = \frac{1}{2}(x^2 - 4x + 2), \;\; \ldots$$



$$\rho(x) = x^{r-1} \frac{e^{-x/c}}{c^r \Gamma(r)}, \;\; x \geq 0, \;\; r, c > 0$$



Non-Gaussian variables and gPC

What about discrete random variables ?

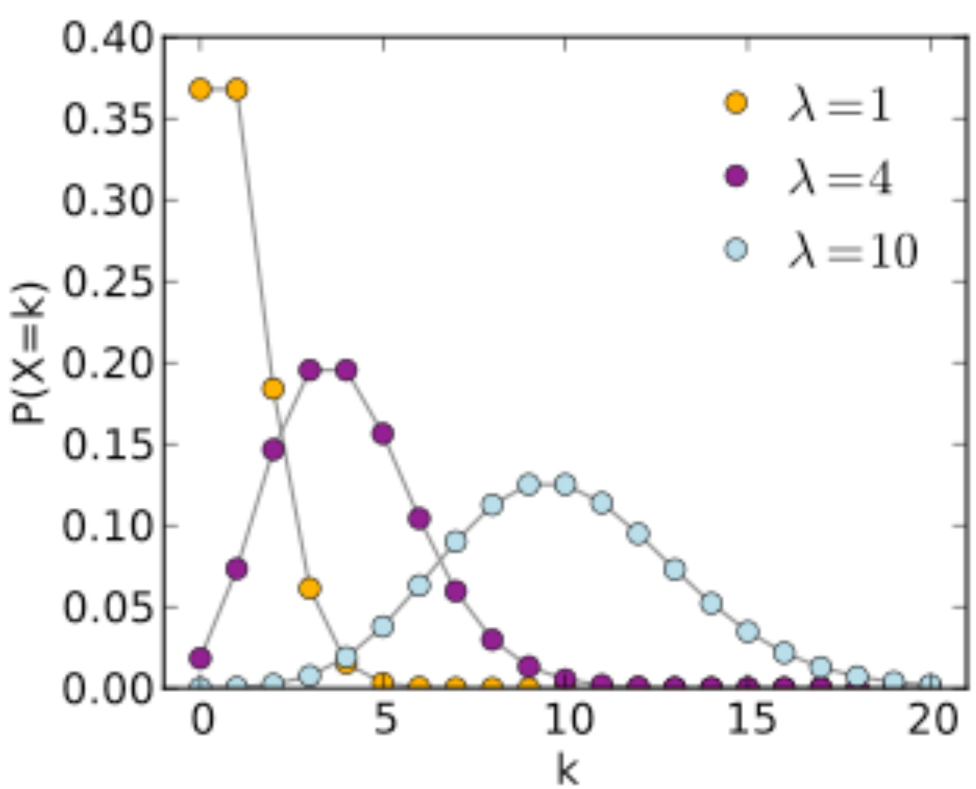
There is no essential difference

$$\mathsf{E}[\Phi_m(X)\Phi_n(X)] = \sum_i \Phi_m(x_i)\Phi_n(x_i)\rho_i = \gamma_n \delta_{nm}$$

This defines the appropriate Chaos basis

<u>Ex</u>: Poisson distribution

$$\rho(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, 3, \ldots$$



<u>Charlier polynomials</u> is the appropriate basis

Non-Gaussian variables and gPC

The much broader class of processes to consider is known as generalized Polynomial Chaos (gPC)

Continuous	Gaussian	Hermite	$(-\infty, \infty)$
	Gamma	Laguerre	$[0, \infty)$
	Beta	Jacobi	$[a, b]$
	Uniform	Legendre	$[a, b]$
Discrete	Poisson	Charlier	$\{0, 1, 2, \dots\}$
	Binomial	Krawtchouk	$\{0, 1, \dots, N\}$
	Negative binomial	Meixner	$\{0, 1, 2, \dots\}$
	Hypergeometric	Hahn	$\{0, 1, \dots, N\}$

Xiu, 2010

What about 'non-classic' cases ?

Given a weight one can always constructed a corresponding orthogonal polynomial basis

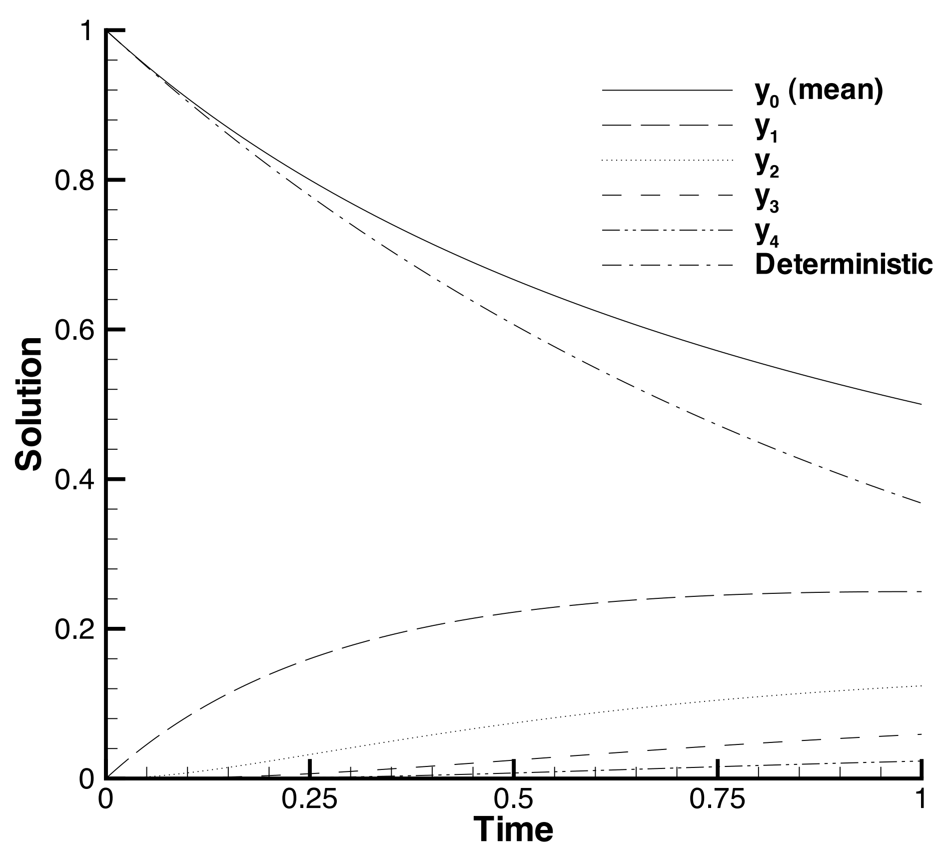
Non-Gaussian variables and gPC

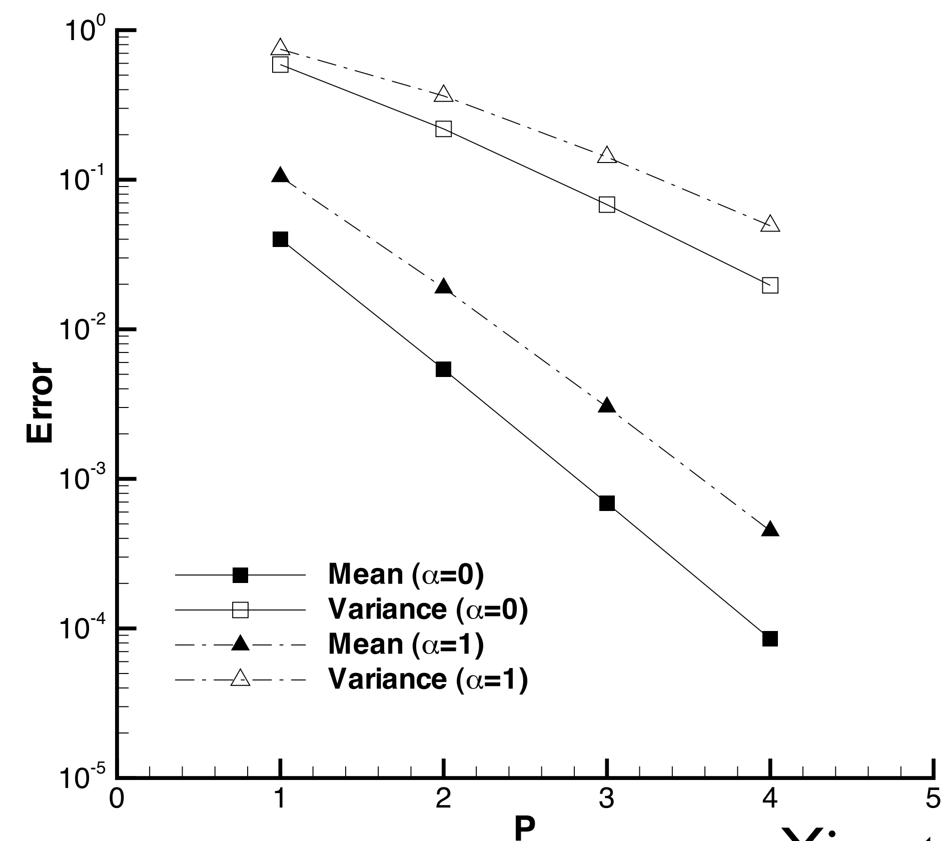
We consider again the simple ODE

$$\frac{du}{dt}(t, X) = -k(X)u, \quad u(0) = 1$$

Assume a Gamma distribution of the unknown -

$$k \sim \frac{e^{-x}x^{\alpha}}{\Gamma(\alpha+1)} \qquad \mu = \bar{k} = \alpha + 1, \;\; \sigma^2 = \alpha + 1$$





Xiu et al, 2002

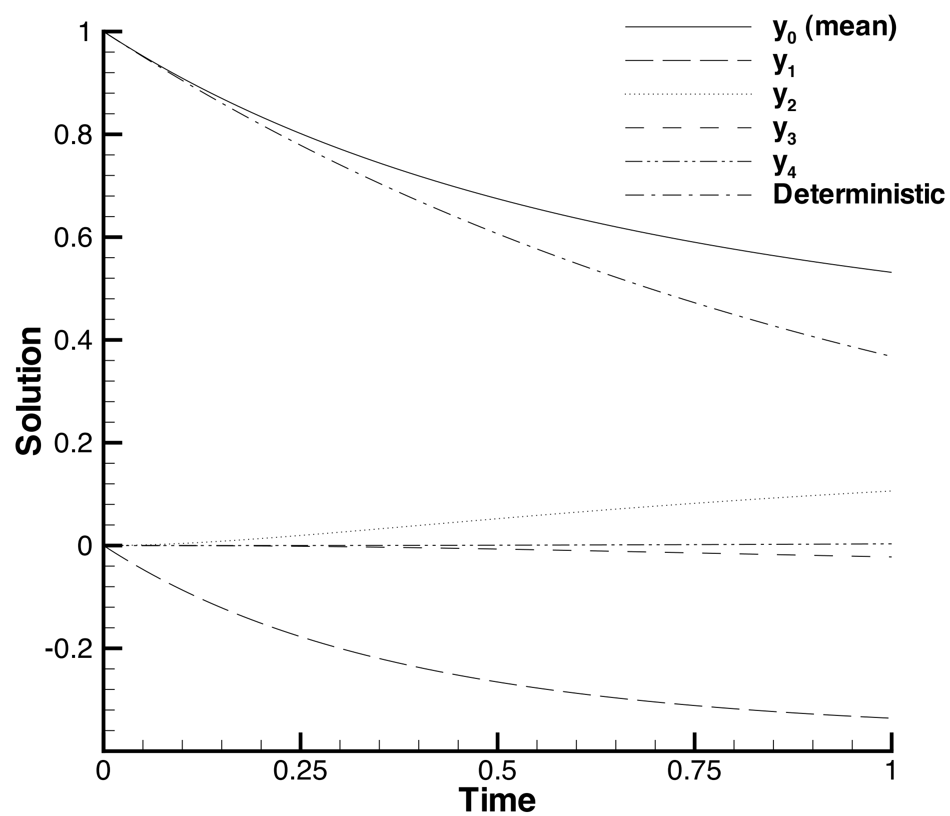
Non-Gaussian variables and gPC

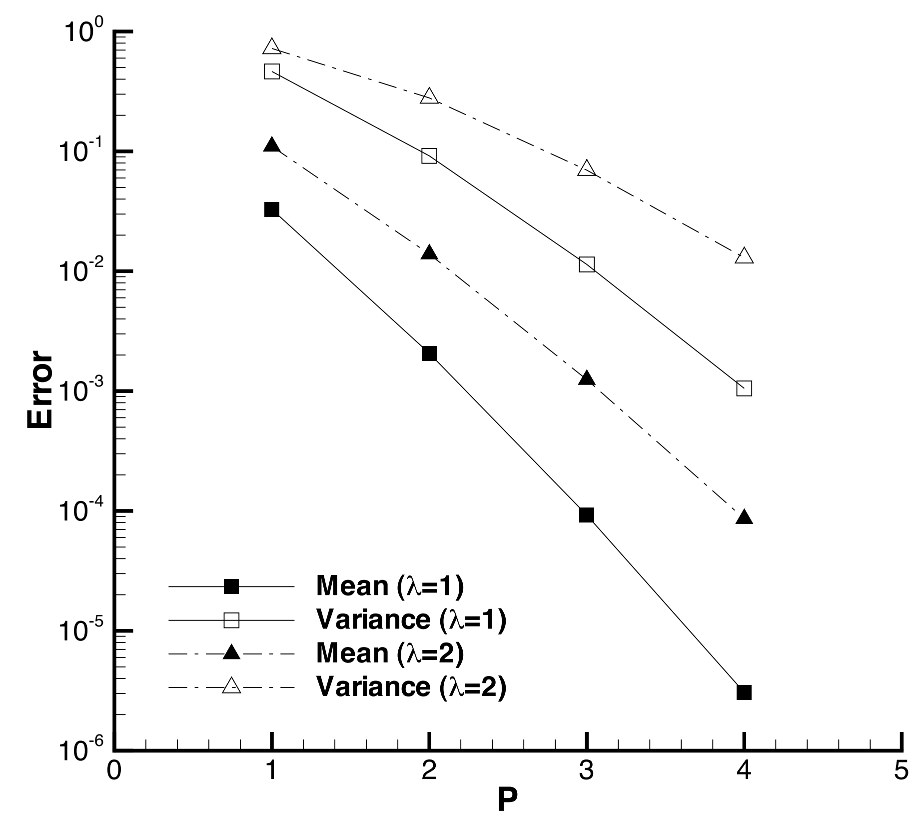
We consider again the simple ODE

$$\frac{du}{dt}(t,X) = -k(X)u, \quad u(0) = 1$$

Assume a Poisson distribution of the unknown -

$$k \sim \frac{e^{-\lambda}\lambda^x}{x!} \qquad \mu = \bar{k} = \lambda, \quad \sigma^2 = \lambda$$





Xiu, 2010

Summary

We have achieved quite a bit

- Developed and demonstrated the Stochastic Galerkin methods to quantify uncertainty in general problems

- Discussed both steady and time-dependent problems

- Introduced generalized Polynomial Chaos to most effectively deal with general random variables

Problems remain

- New solvers are required

- Computational cost