Reduced order models for parameterized problems: Lecture Three

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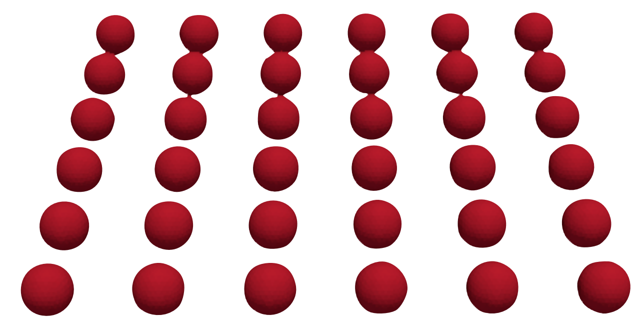
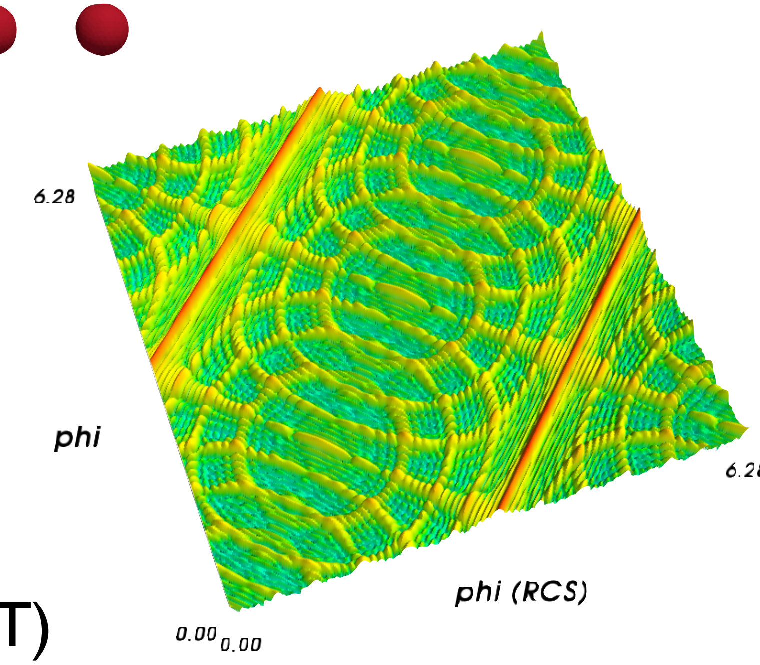




w/ assistance from B. Stamm (Aachen, D) and G. Rozza (SISSA, IT)

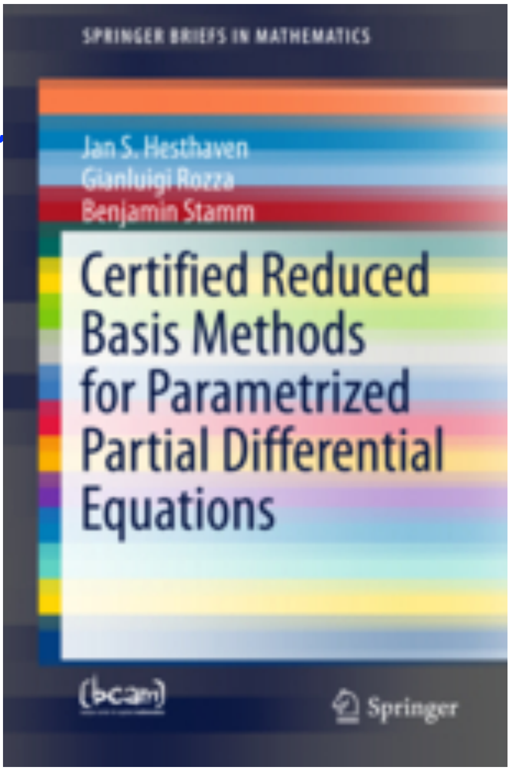
Overview of the lectures

Lecture 1: Introduction, motivation, basics

Lecture 2: Certified reduced methods

Lecture 3: The 'non's '

Hesthaven, Rozza, Stamm
Certified Reduced Basis Methods for Parametrized Partial Differential Equations
Springer Briefs in Mathematics, 2015



Free: https://infoscience.epfl.ch/record/213266?ln=en

Overall goals

Understand Reduced models

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WHAT do we mean by 'reduced models' ?

WHY should we care ?

WHEN could it work ?

HOW do we know ?

DOES it work ?

WHAT's next ?

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DOES it work ?

WHAT's next ?

The non-affine problem

The non-affine problem

The affine assumption is key to speed

Assumption:

$$a(w, v; \mu) = \sum_{q=1}^{Q_{\mathtt{a}}} \theta_{\mathtt{a}}^q(\mu)\ a_q(w, v),$$

$$f(v; \mu) = \sum_{q=1}^{Q_{\mathtt{f}}} \theta_{\mathtt{f}}^q(\mu)\ f_q(v),$$

$$\ell(v; \mu) = \sum_{q=1}^{Q_{\mathtt{l}}} \theta_{\mathtt{l}}^q(\mu)\ \ell_q(v),$$

where

$$\theta_{\mathtt{a}}^q, \theta_{\mathtt{f}}^q, \theta_{\mathtt{l}}^q : \mathbb{P} \to \mathbb{R} \qquad \mu - \text{dependent functions},$$

$$a_q : \mathbb{V} \times \mathbb{V} \to \mathbb{R} \qquad \mu - \text{independent forms},$$

$$f_q, \ell_q : \mathbb{V} \to \mathbb{R} \qquad \mu - \text{independent forms},$$

The non-affine problem

In many problems, this does not hold

- Geometric parametrization
- Material variations
- etc

In this case, we do not have the offline-online decomposition and cannot eliminate dependence on the truth problem

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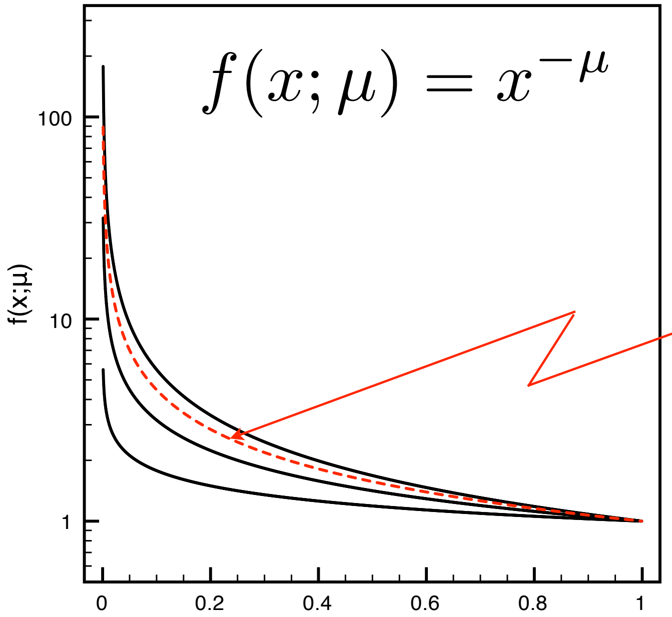
Except if we can - somehow - express non-affine terms as an affine expansion, e.g.



$$f(x;\mu) \approx \sum_{m=1}^{3} \alpha_m(\mu) x^{-\mu_m}$$

Empirical Interpolation (EIM)

We consider a general parametrized function

$$\mathcal{M} = \{ f(\cdot;\mu) \,|\, \mu \in \mathbb{P} \} \subset \mathbb{V},$$

and seek to approximate it as

$$f(x,\mu) \simeq f_N(x,\mu) = \sum_{n=0}^{N} \alpha_n(\mu)\varphi_n(x)$$

and now we seek a problem specific interpolation with

$$\varphi_n(x) = f(x;\mu_n), \qquad n = 1,\ldots,N,$$

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$$\varphi_n(x) = f(x;\mu_n), \qquad n = 1,\ldots,N,$$

How do we find the interpolation points - greedy !

Empirical Interpolation (EIM)

Given a parametrised family of functions $f(\cdot;\mu), \mu \in \mathbb{P}$, a set of $N-1$ basis functions $\varphi_1, \dots, \varphi_{N-1}$ and $N-1$ interpolation points $x_1, \dots, x_{N-1}$ let us define

$$\mu_N = \arg\max_{\mu \in \mathbb{P}} \| f(\cdot;\mu) - \mathrm{I}_{N-1} f(\cdot;\mu) \|_{L^\infty(\Omega)}.$$

$\Rightarrow$ the worst approximation results if taking μ_N.

Thus the basis should be enriched by $f(\cdot;\mu_N)$.

Set

$$x_N = \arg\max_{x \in \Omega} |f(x;\mu_N) - \mathrm{I}_{N-1} f(x;\mu_N)|,$$

and

$$\varphi_N(x) = \frac{f(x;\mu_N) - \mathrm{I}_{N-1} f(x;\mu_N)}{f(x_N;\mu_N) - \mathrm{I}_{N-1} f(x_N;\mu_N)}, \qquad \forall x \in \Omega.$$

Empirical Interpolation (EIM)

Step N:

`Given:` $\{\varphi_1, \ldots, \varphi_{N-1}\}$, $\{x_1, \ldots, x_{N-1}\}$.

- Solve the interpolation problem: Find $\{\alpha_n(\mu)\}_{n=1}^{N-1}$ s.t.

$$\sum_{n=1}^{N-1} \varphi_n(x_i)\alpha_n(\mu) = f(x_i;\mu), \qquad i = 1, \ldots, N-1.$$

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- Compute the interpolating function

$$\mathrm{I}_{N-1} f(\cdot;\mu) = \sum_{n=1}^{N-1} \alpha_n(\mu)\varphi_n$$

$$\big(\mathrm{I}_{N-1} f(x_i;\mu) = f(x_i;\mu),\ i = 1, \ldots, N-1\big).$$

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- Define

$$\mu_N = \arg\max_{\mu \in \mathbb{P}} \|f(\cdot;\mu) - \mathrm{I}_{N-1} f(\cdot;\mu)\|_{L^\infty(\Omega)},$$

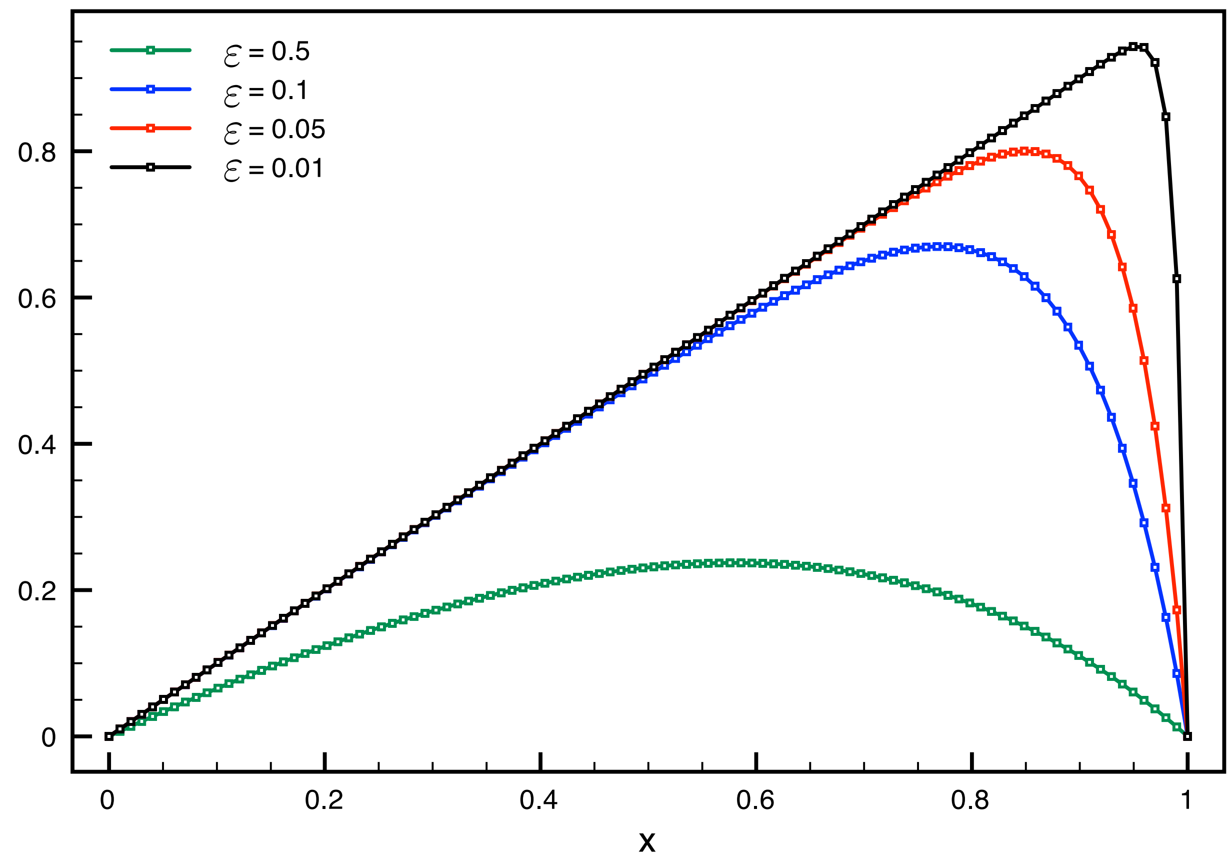
$$x_N = \arg\max_{x \in \Omega} |f(x;\mu_N) - \mathrm{I}_{N-1} f(x;\mu_N)|,$$

$$q_N = \frac{f(x;\mu_N) - \mathrm{I}_{N-1} f(x;\mu_N)}{f(x_N;\mu_N) - \mathrm{I}_{N-1} f(x_N;\mu_N)}$$

EIM - example

Consider the parametrized family of functions:

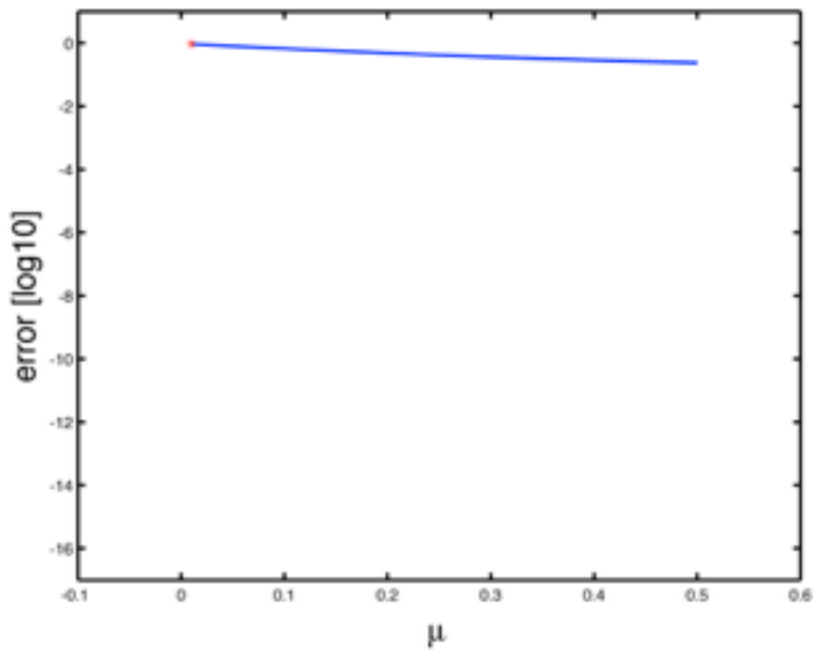
$$u(x;\mu) = x - \frac{e^{\frac{x}{\mu}} - 1}{e^{\frac{1}{\mu}} - 1}, \qquad \text{for } x \in (0,1), \mu \in [0.01, 0.5].$$



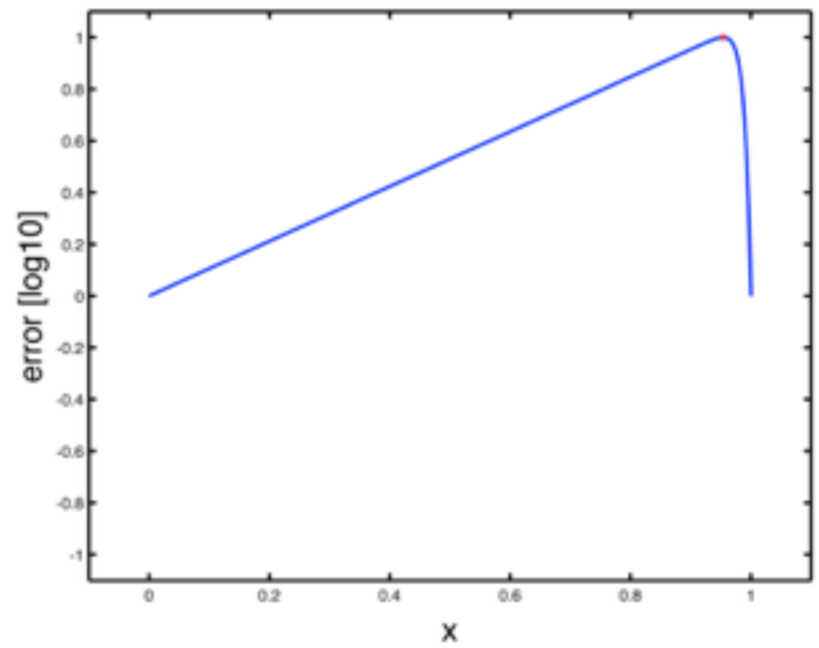
EIM - example

$n = 1$

$\varepsilon(\mu)$



$\varphi_n(x)$



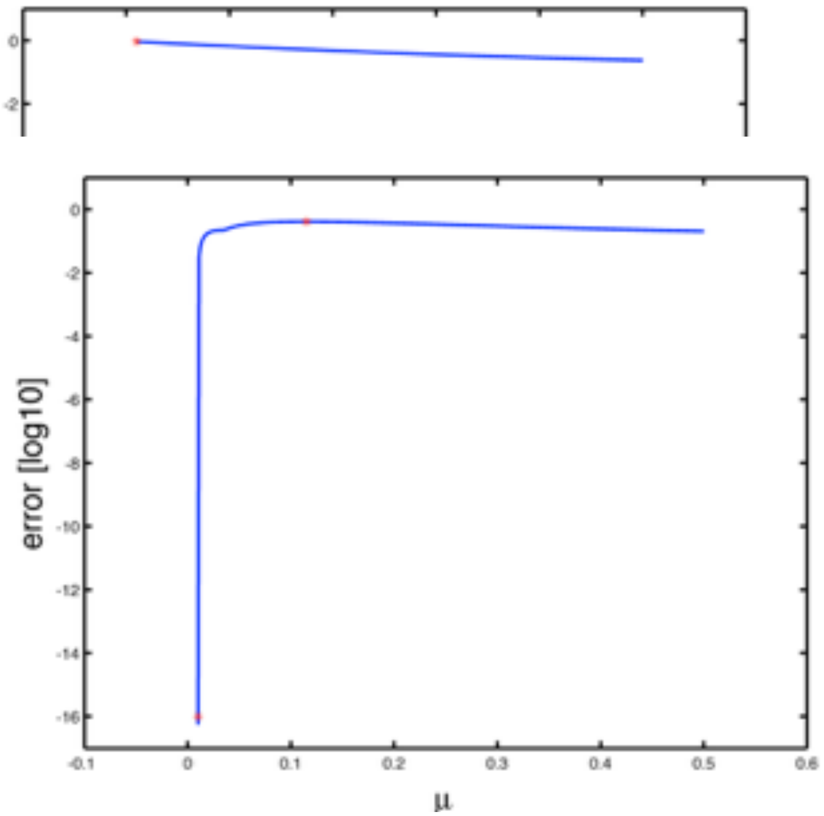
EIM - example

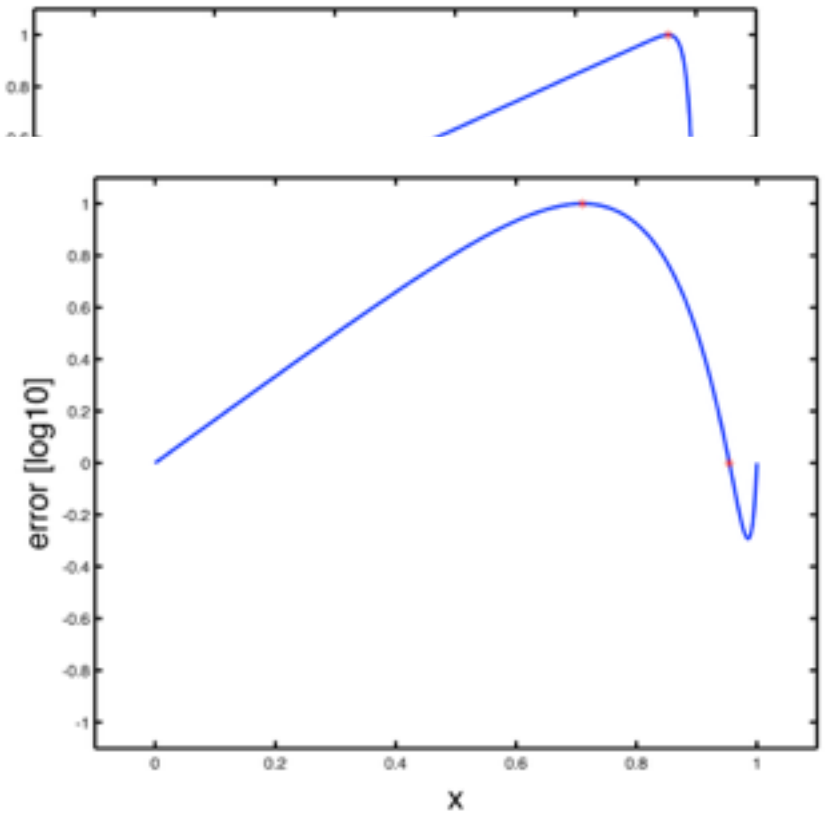
$\varepsilon(\mu)$

$\varphi_n(x)$

r

$n = 2$





EIM - example

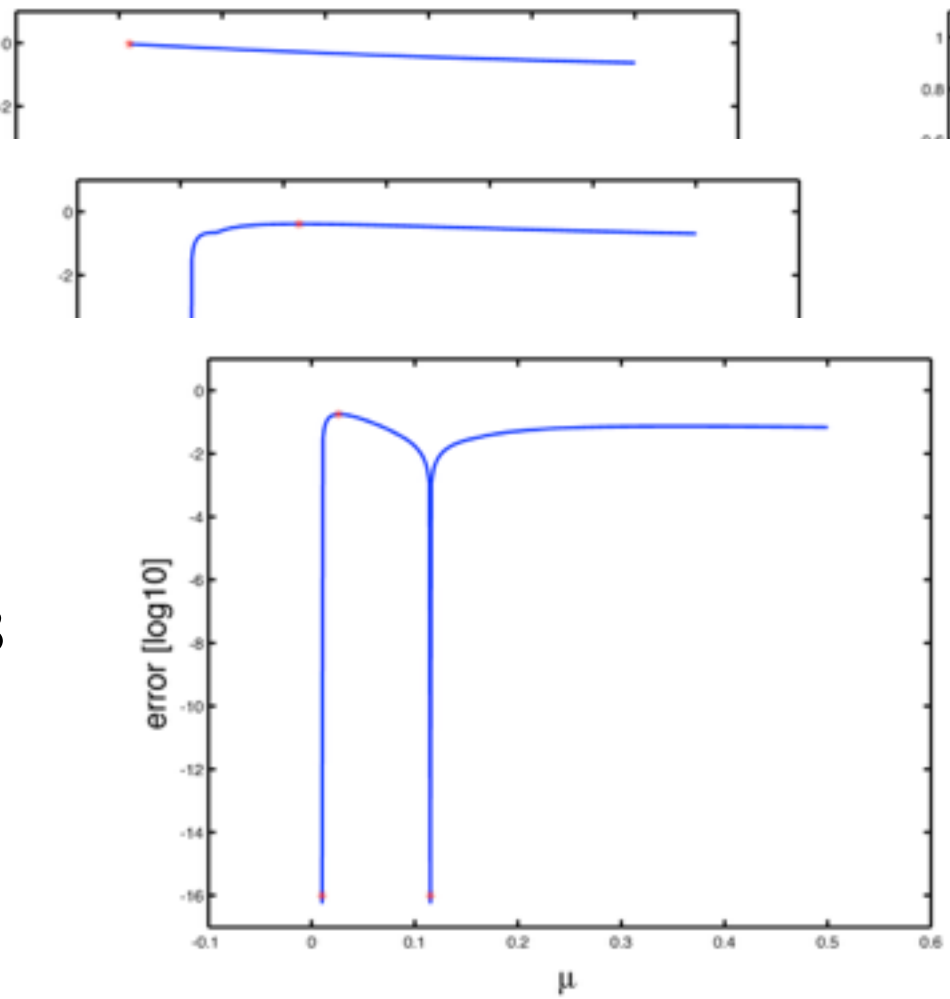
$\varepsilon(\mu)$

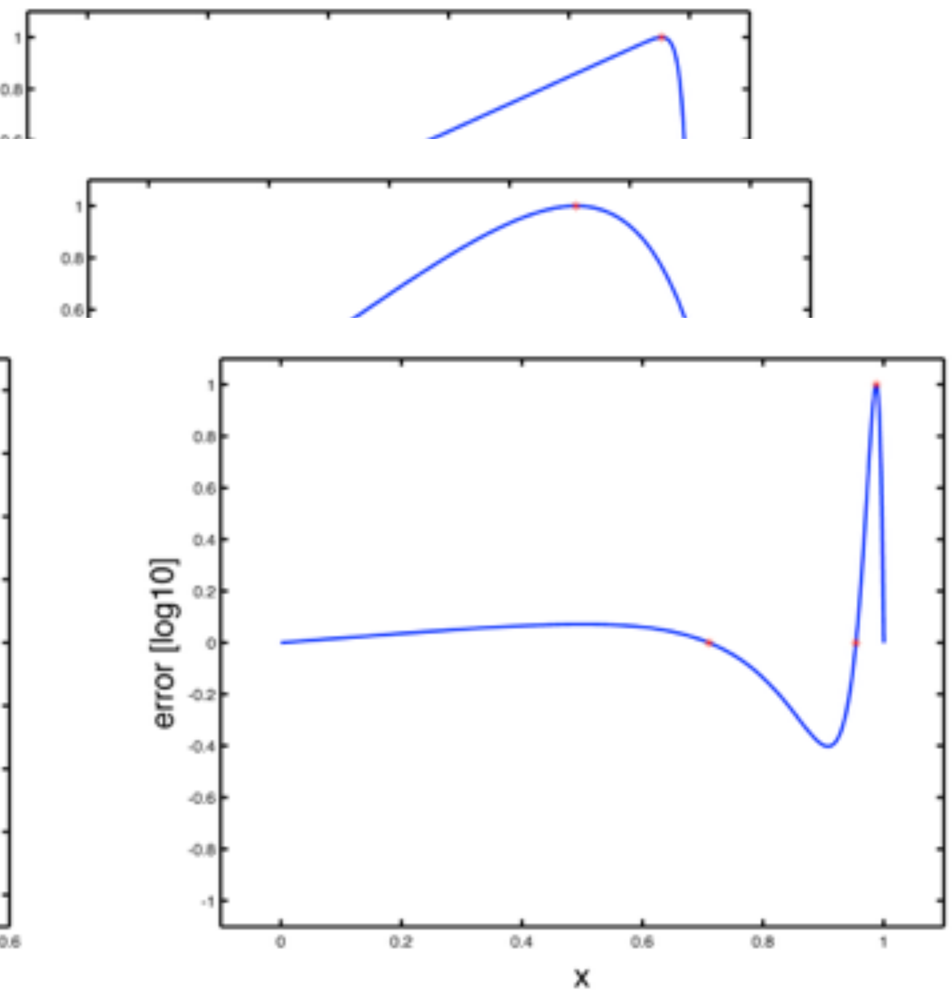
$\varphi_n(x)$

r

$n =$

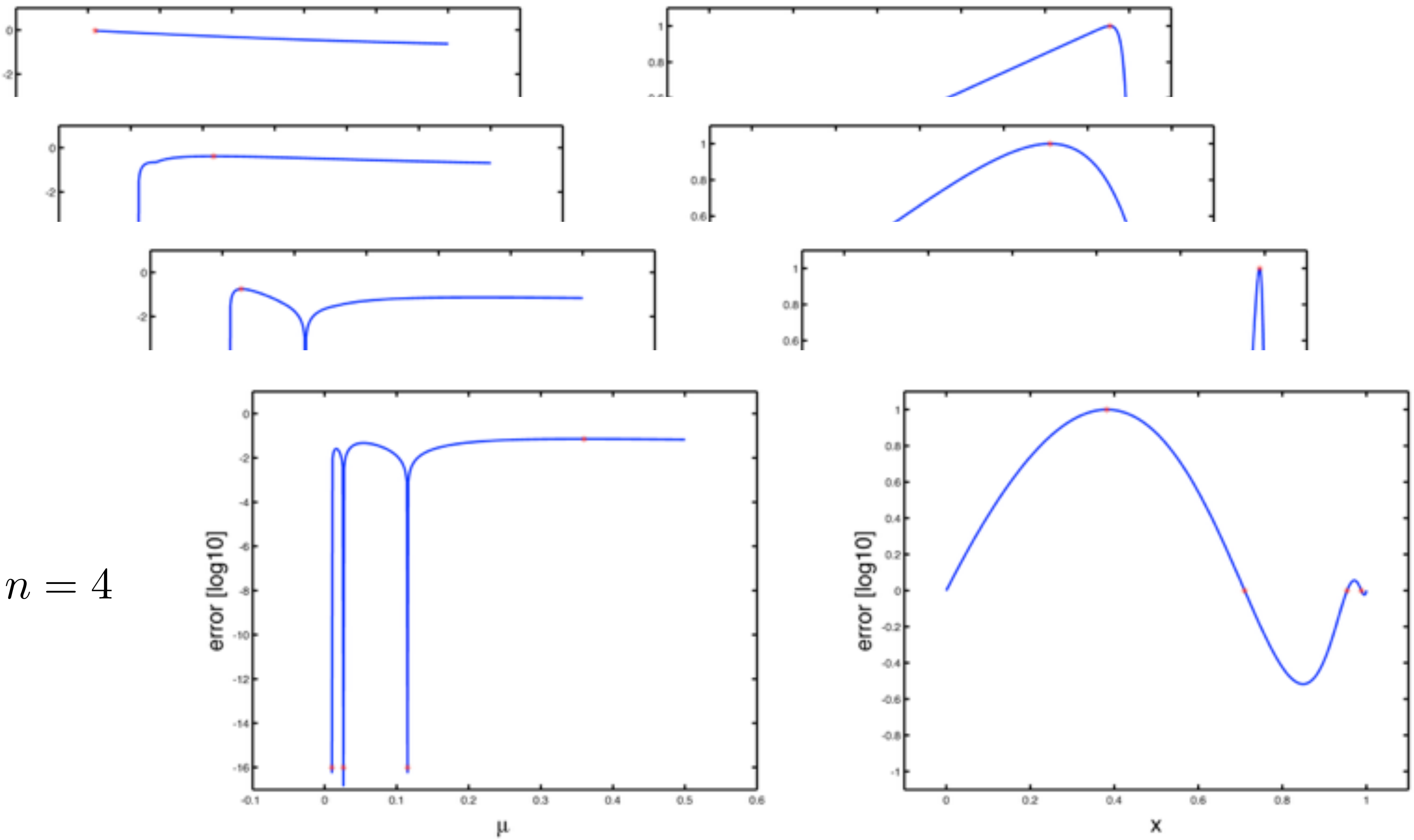
$n = 3$





EIM - example

$\varepsilon(\mu)$ $\varphi_n(x)$



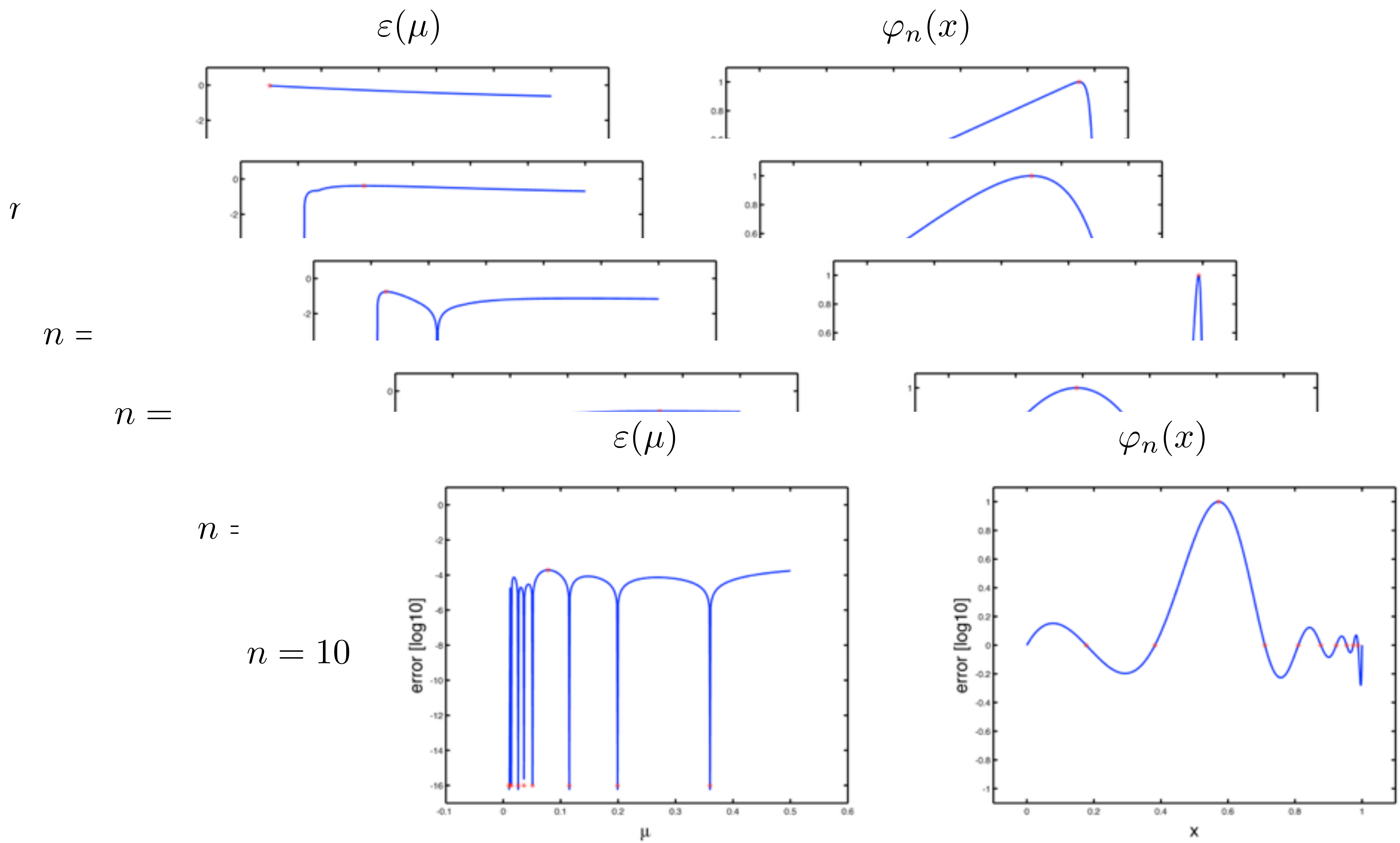
r

$n =$

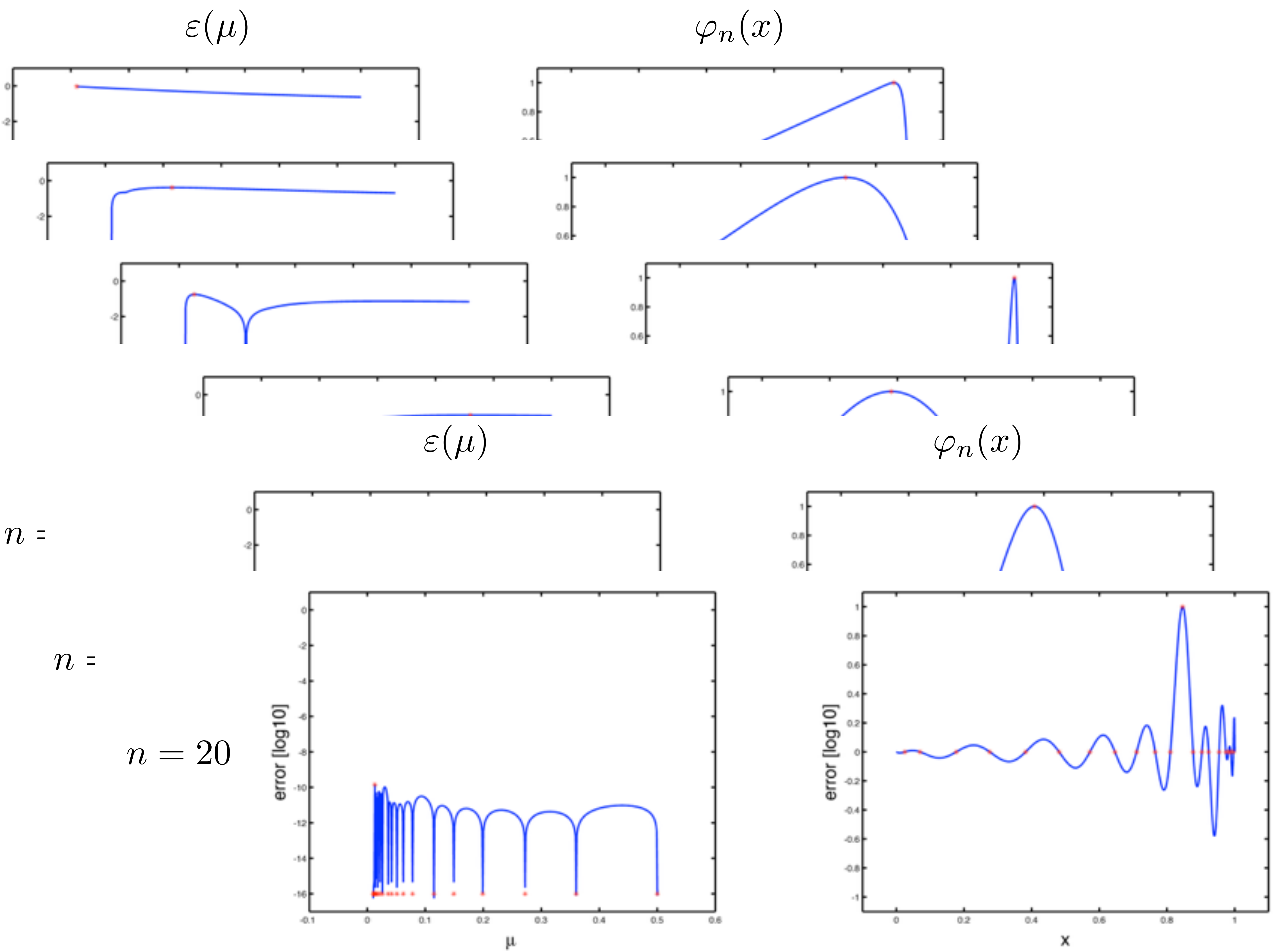
$n =$

$n = 4$

EIM - example



EIM - example

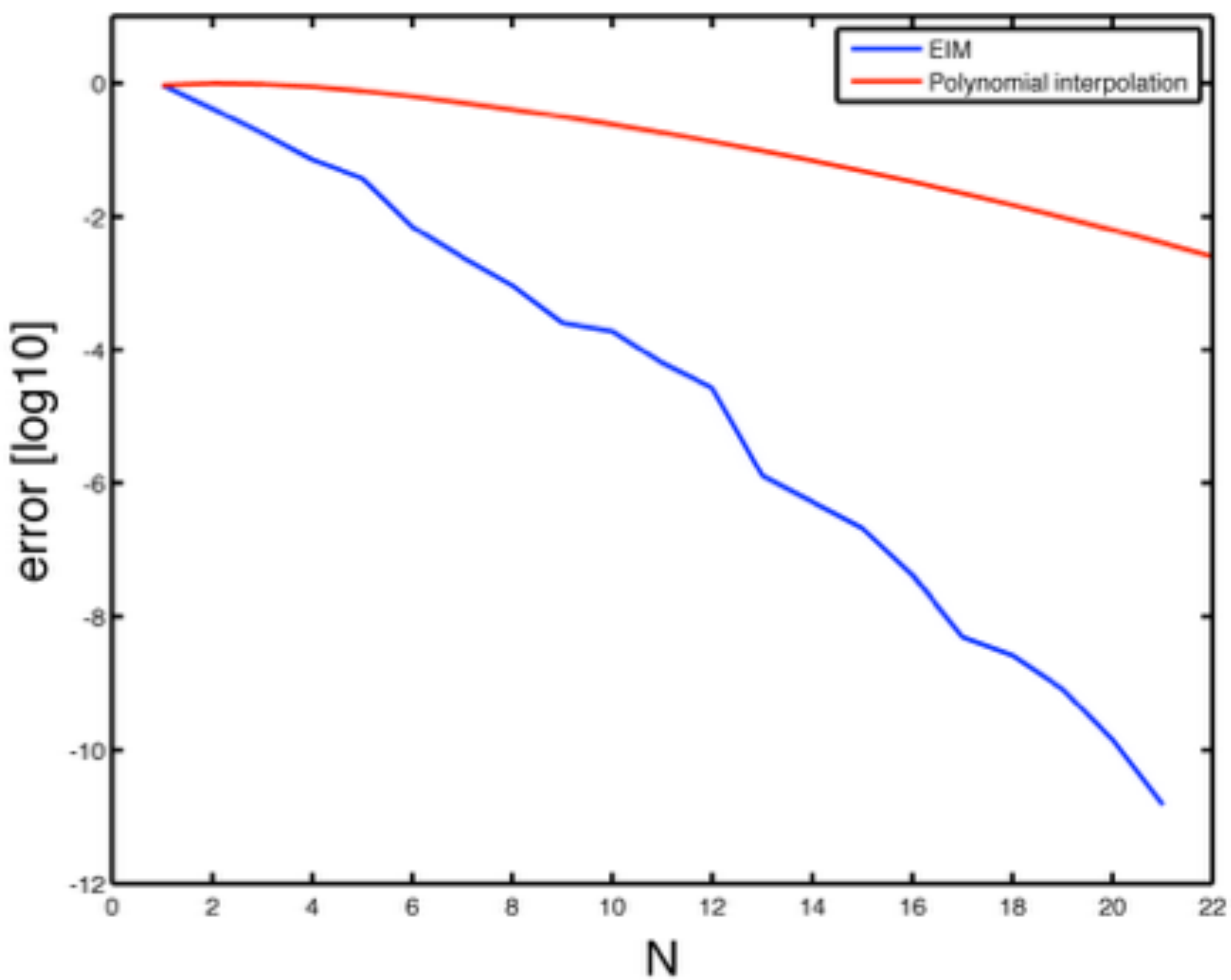


$n = 20$

EIM - example

Consider the parametrized family of functions:

$$u(x;\mu) = x - \frac{e^{\frac{x}{\mu}} - 1}{e^{\frac{1}{\mu}} - 1}, \qquad \text{for } x \in (0,1), \mu \in [0.01, 0.5].$$



EIM - example

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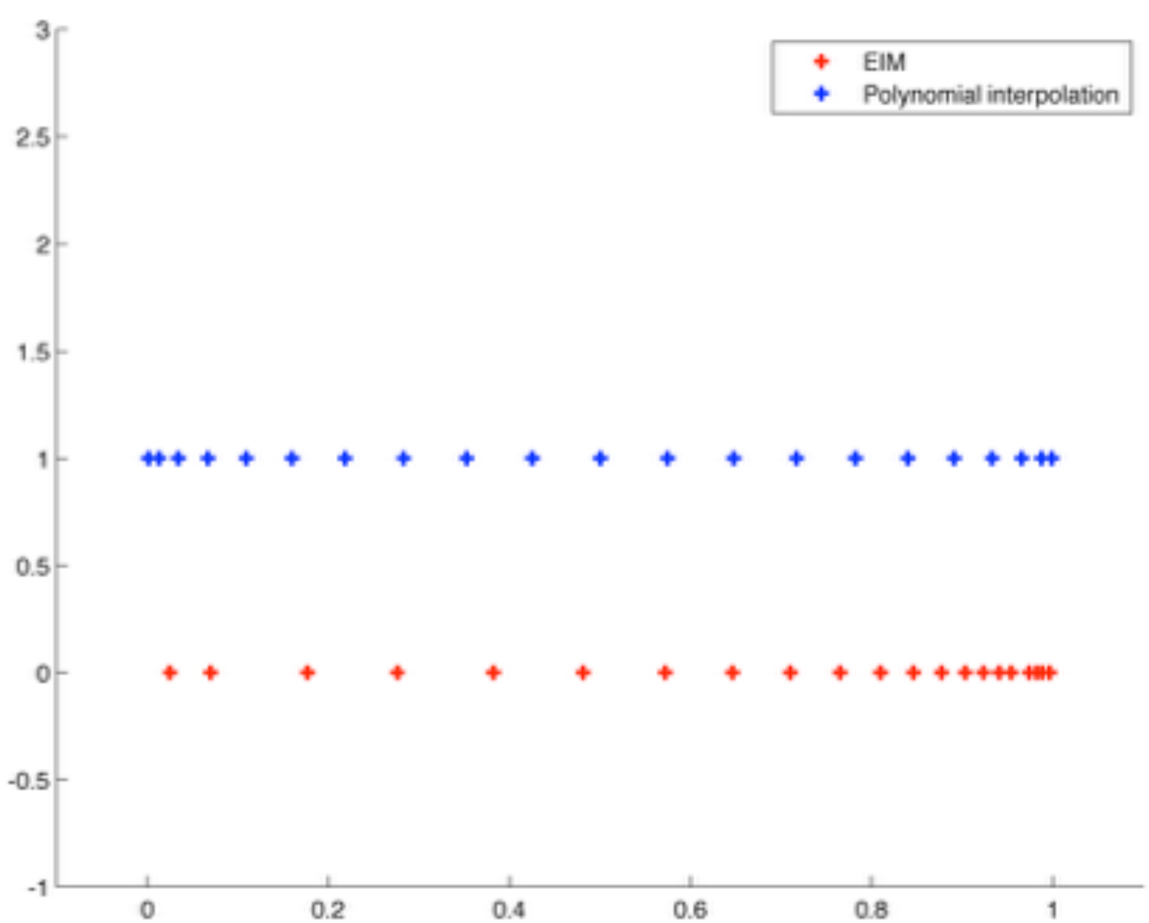


Interpolation: Bad accuracy until there are enough interpolation points in the boundary layer.

EIM - errors

The error analysis of the interpolation procedure classically involves the Lebesgue constant $\Lambda_N = \sup_{x\in\Omega} \sum_{i=1}^N |h_i^N(x)|$, where the h_i^N is the associated Lagrange basis.

A (in practice very pessimistic) upper-bound for the Lebesque constant is $2^N - 1$.

Lemma:

For any $f \in \mathcal{M}$, the interpolation error satisfies

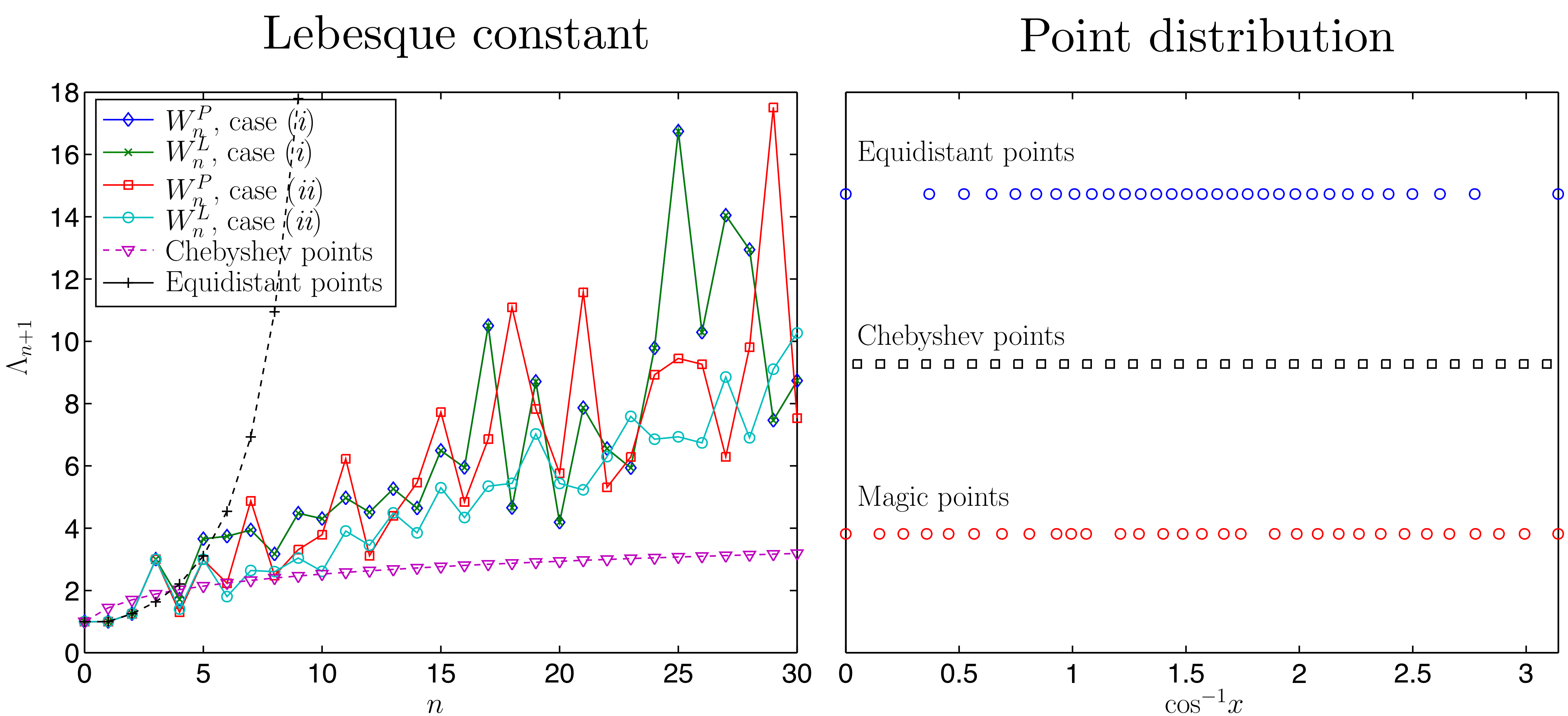
$$\|f - \mathrm{I_N} f\|_{L^\infty(\Omega)} \leq (1 + \Lambda_N) \inf_{v_N \in \mathbb{V}_N} \|f - v_N\|_{L^\infty(\Omega)}.$$

Comparison with polynomial interpolation:

Equidistant points: $\Lambda_N \sim \frac{2^{N+1}}{eN \log N}$

Chebychev points: $\Lambda_N < \frac{2}{\pi} \log(N+1) + 1$

EIM - errors

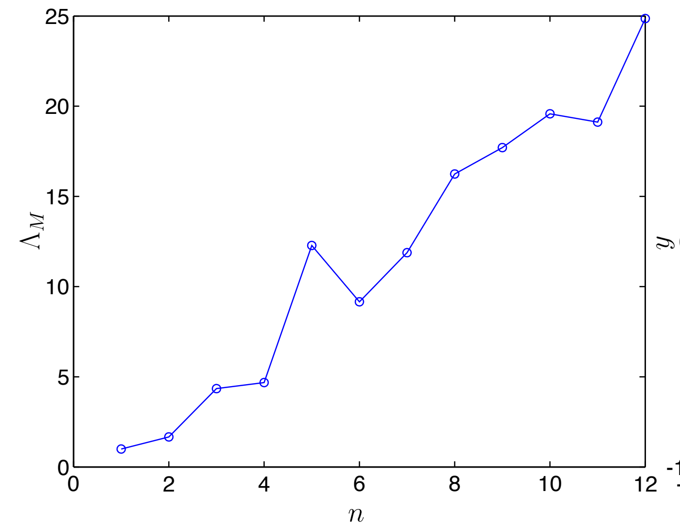


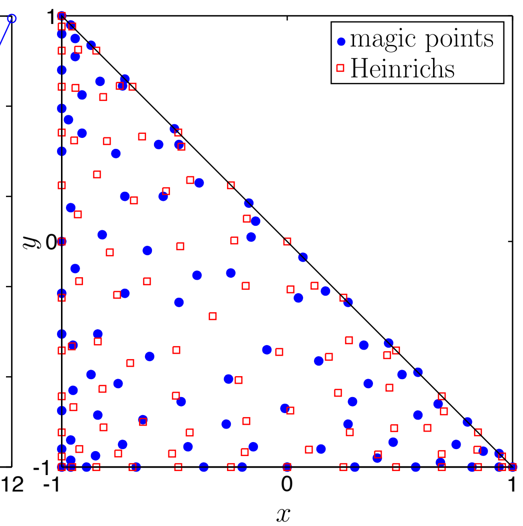
Magic points:

- Hierarchical set of points.
- Application to any domain Ω as we will see in the next slides.

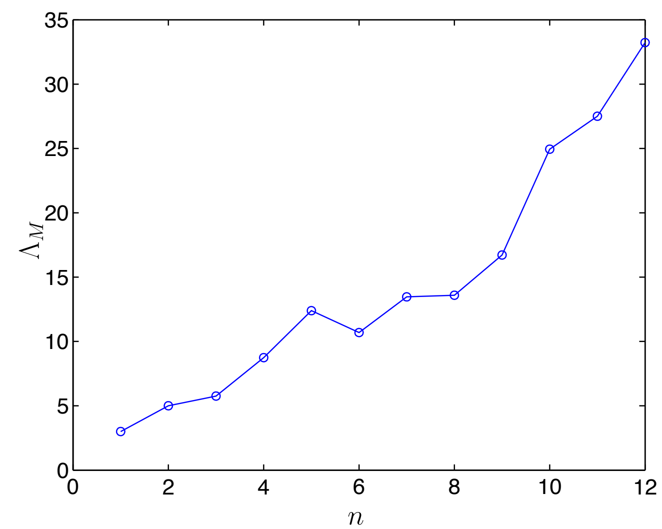
EIM - extensions

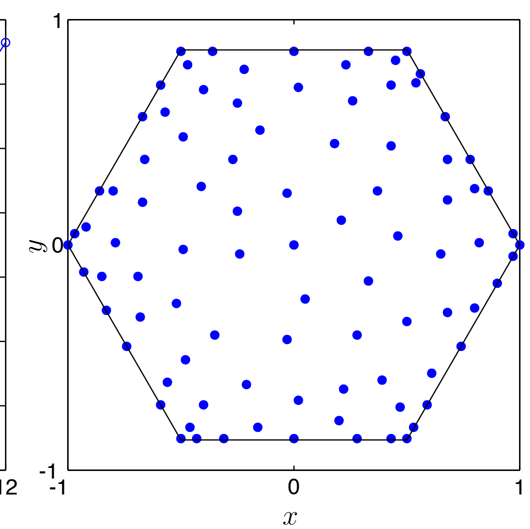
On the triangle:



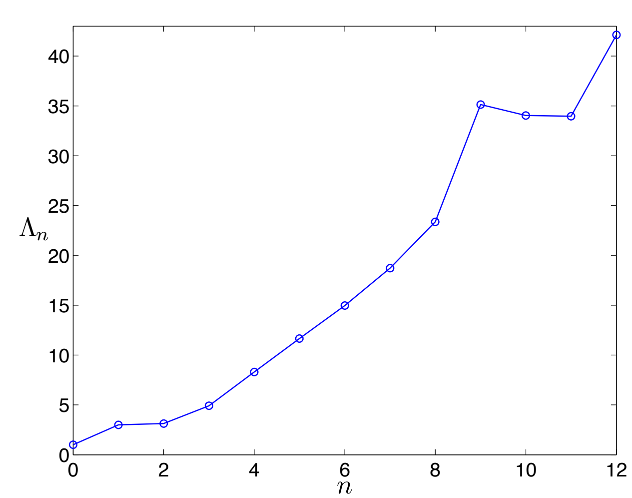


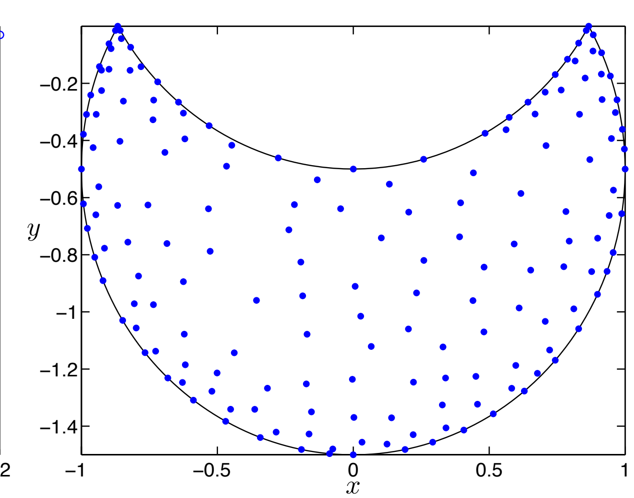
On a hexagon:





On a half-moon:





A non-affine example

Let us consider problems described by integral equations

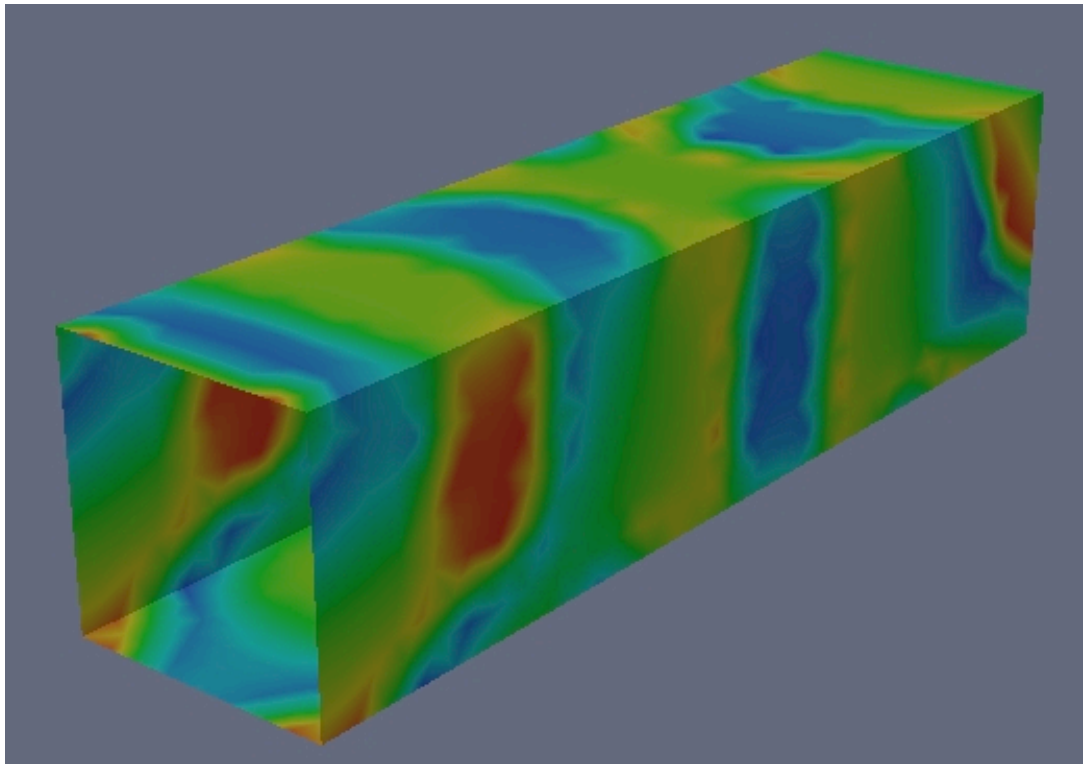
Electric field integral equation (EFIE)

$$ik \int_{\Gamma \times \Gamma} G_k(\boldsymbol{x}, \boldsymbol{y}) \left[\boldsymbol{j}(\boldsymbol{x}) \cdot \boldsymbol{j}^t(\boldsymbol{y}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{j}(\boldsymbol{x}) \mathrm{div}_\Gamma \boldsymbol{j}^t(\boldsymbol{y}) \right] d\boldsymbol{x} d\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{j}^t)$$

$$G_k(\boldsymbol{x}, \boldsymbol{y}) := \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}.$$

Truth approximation is a standard MoM solver.

CERFACS



Integral equations

After discretization we again have

$$a(\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}) = f(\boldsymbol{v}_h; \boldsymbol{\mu}) \qquad \forall \boldsymbol{v}_h \in \mathbb{V}_h.$$

Output functional is

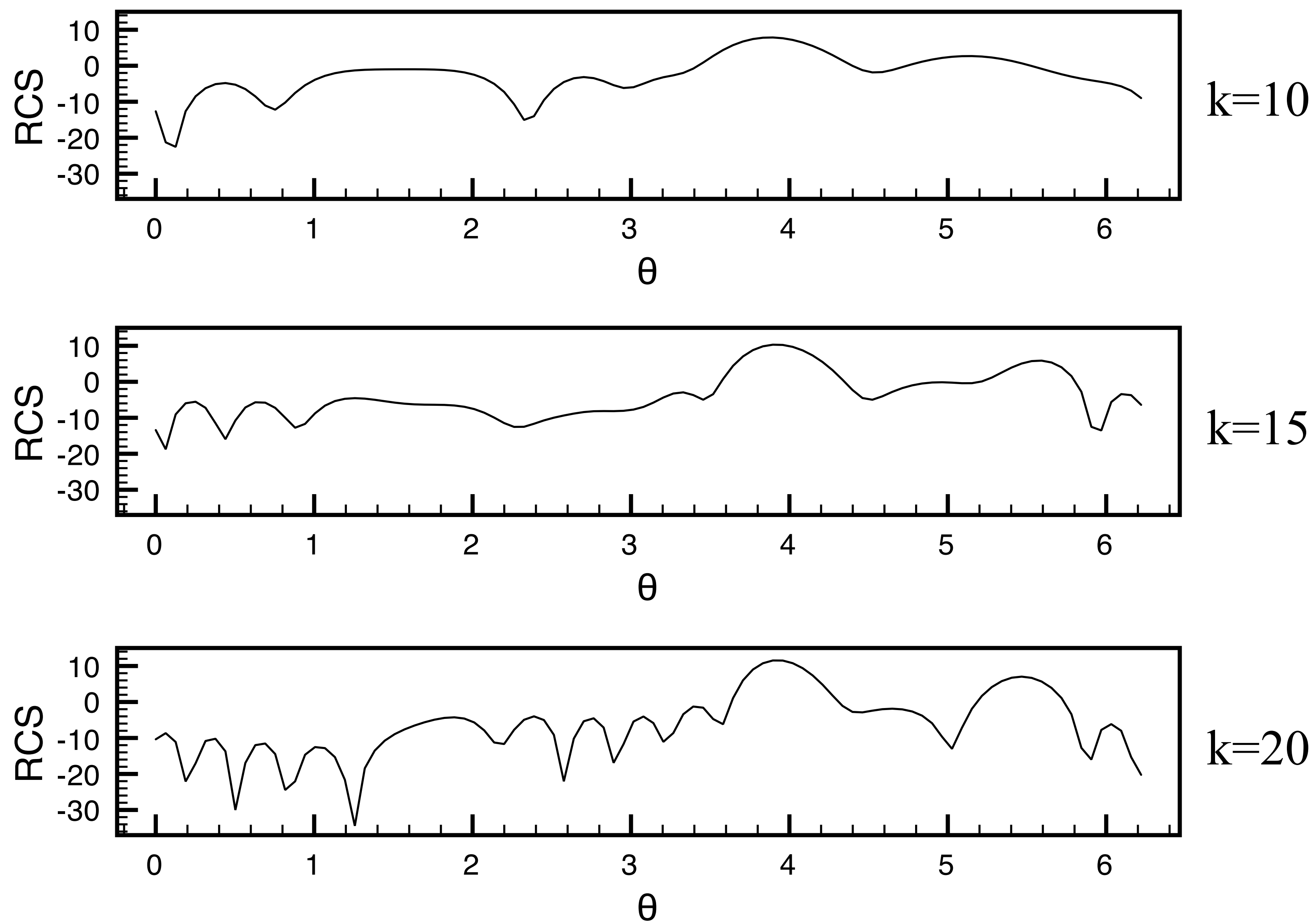
$$A_\infty(\boldsymbol{u}, \hat{\boldsymbol{d}}) = \frac{ikZ}{4\pi} \int_\Gamma \hat{\boldsymbol{d}} \times (\boldsymbol{u}(\boldsymbol{x}) \times \hat{\boldsymbol{d}}) e^{-ik\boldsymbol{x}\cdot\hat{\boldsymbol{d}}} d\boldsymbol{x}$$

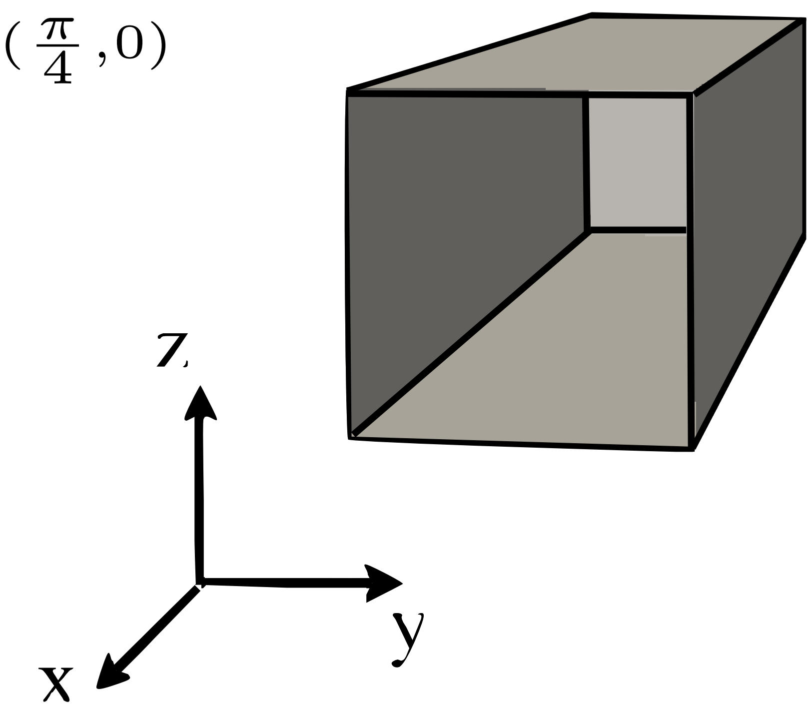
$$\mathrm{RCS}(\boldsymbol{u}, \hat{\boldsymbol{d}}) = 10 \log_{10} \left(\frac{|A_\infty(\boldsymbol{u}, \hat{\boldsymbol{d}})|^2}{|A_\infty(\boldsymbol{u}, \hat{\boldsymbol{d}}_0)|^2} \right)$$

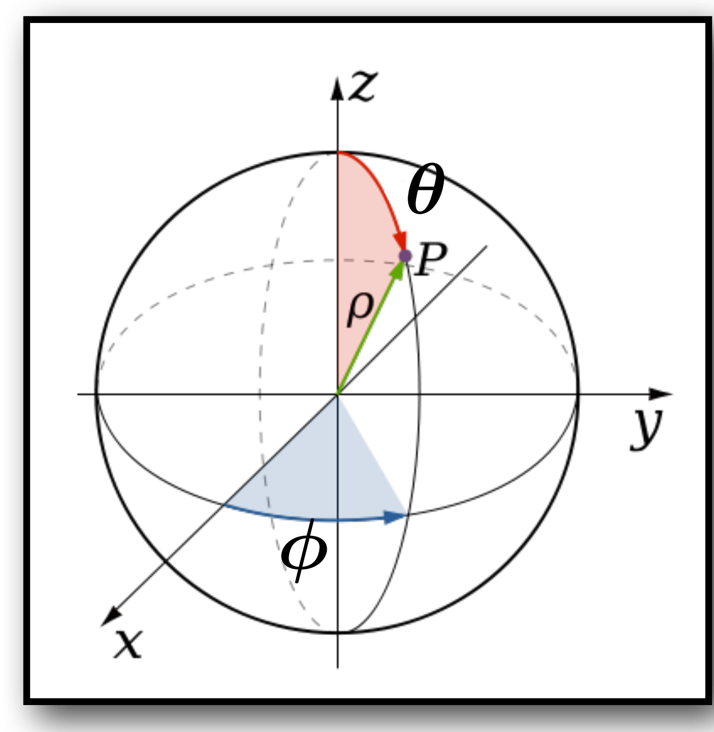
$\boldsymbol{u}$: current on surface
$\hat{\boldsymbol{d}}$: given directional unit vector
$\hat{\boldsymbol{d}}_0$: reference unit direction

Integral equations

Incident field $\boldsymbol{E}^i(\boldsymbol{x};k) = -\boldsymbol{p}\, e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{s}}_{(\frac{\pi}{4},0)}}$







Integral equations

One problem - the affine assumption fails

Caution: This is not feasible in the framework of the EFIE!

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = ikZ \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|} \left\{ \boldsymbol{u}_h(\boldsymbol{x}) \cdot \overline{\boldsymbol{v}_h(\boldsymbol{y})} - \frac{1}{k^2} \mathrm{div}_{\Gamma,\boldsymbol{x}} \boldsymbol{u}_h(\boldsymbol{x}) \cdot \overline{\mathrm{div}_{\Gamma,\boldsymbol{y}} \boldsymbol{v}_h(\boldsymbol{y})} \right\} d\boldsymbol{x}\, d\boldsymbol{y}$$

$$f(\boldsymbol{v}_h; \boldsymbol{\mu}) = \boldsymbol{n} \times (\boldsymbol{p} \times \boldsymbol{n}) \int_\Gamma e^{ik\boldsymbol{x} \cdot \hat{\boldsymbol{s}}_{(\theta,\phi)}} \cdot \overline{\boldsymbol{v}_h(\boldsymbol{x})}\, d\boldsymbol{x}$$

Integral equations

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Solution - empirical interpolation method (EIM)

Seek $\{\boldsymbol{\mu}_m\}_{m=1}^M$ such that

$$\mathcal{I}_M(f)(\boldsymbol{x}; \boldsymbol{\mu}) = \sum_{m=1}^{M} \alpha_m(\boldsymbol{\mu}) f(\boldsymbol{x}; \boldsymbol{\mu}_m)$$

RBM for Integral Equations

For the EFIE formulation this results in

$$
\begin{aligned}
a(\boldsymbol{w},\boldsymbol{v};k) \approx 1 \int_{\Gamma\times\Gamma} \frac{\boldsymbol{w}(\boldsymbol{x})\cdot\overline{\boldsymbol{v}(\boldsymbol{y})}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}d\boldsymbol{x}\,d\boldsymbol{y} \\
- \frac{1}{k^2}\int_{\Gamma\times\Gamma} \frac{\mathrm{div}_\Gamma\boldsymbol{w}(\boldsymbol{x})\,\overline{\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{y})}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}d\boldsymbol{x}\,d\boldsymbol{y} \\
+ \sum_{m=1}^{M}\alpha_m(k)\int_{\Gamma\times\Gamma} G^{ns}_{k_m}(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{w}(\boldsymbol{x})\cdot\overline{\boldsymbol{v}(\boldsymbol{y})}d\boldsymbol{x}\,d\boldsymbol{y} \\
- \sum_{m=1}^{M}\frac{\alpha_m(k)}{k^2}\int_{\Gamma\times\Gamma} G^{ns}_{k_m}(|\boldsymbol{x}-\boldsymbol{y}|)\mathrm{div}_\Gamma\boldsymbol{w}(\boldsymbol{x})\,\overline{\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{y})}d\boldsymbol{x}\,d\boldsymbol{y}
\end{aligned}
$$

blue: parameter independent
red: parameter dependent

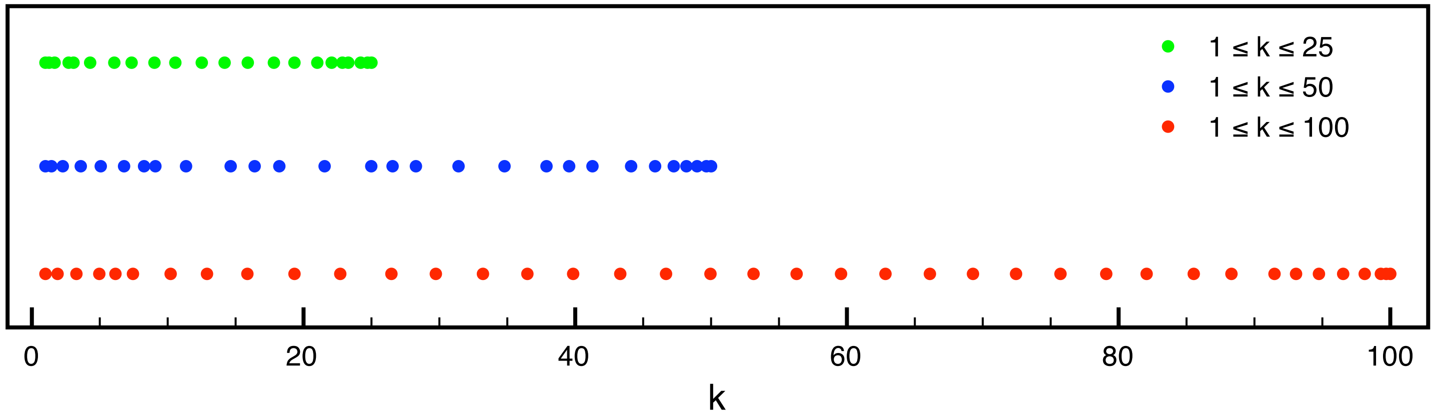
and for the source

$$
F(\boldsymbol{v};\boldsymbol{\mu}) \approx \sum_{m=1}^{M_f}\alpha_f(\boldsymbol{\mu})\int_\Gamma \gamma_t\mathbf{E}^i(\boldsymbol{y};\boldsymbol{\mu}_m)\cdot\overline{\boldsymbol{v}(\boldsymbol{y})}d\boldsymbol{y}
$$

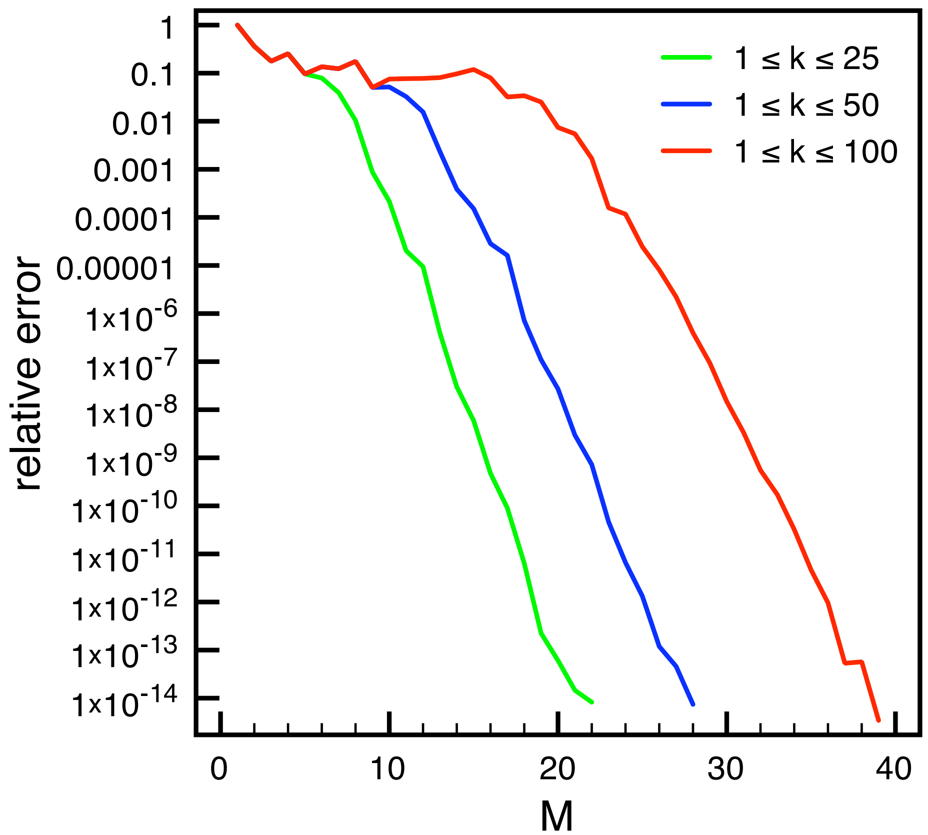
RBM for Integral Equations

Results for EIM

$$f(x;k) = \frac{e^{ikx} - 1}{x}, \quad x \in (0, R_{\max}], k \in [1, k_{\max}]$$



Picked parameters k_m in the parameter domain

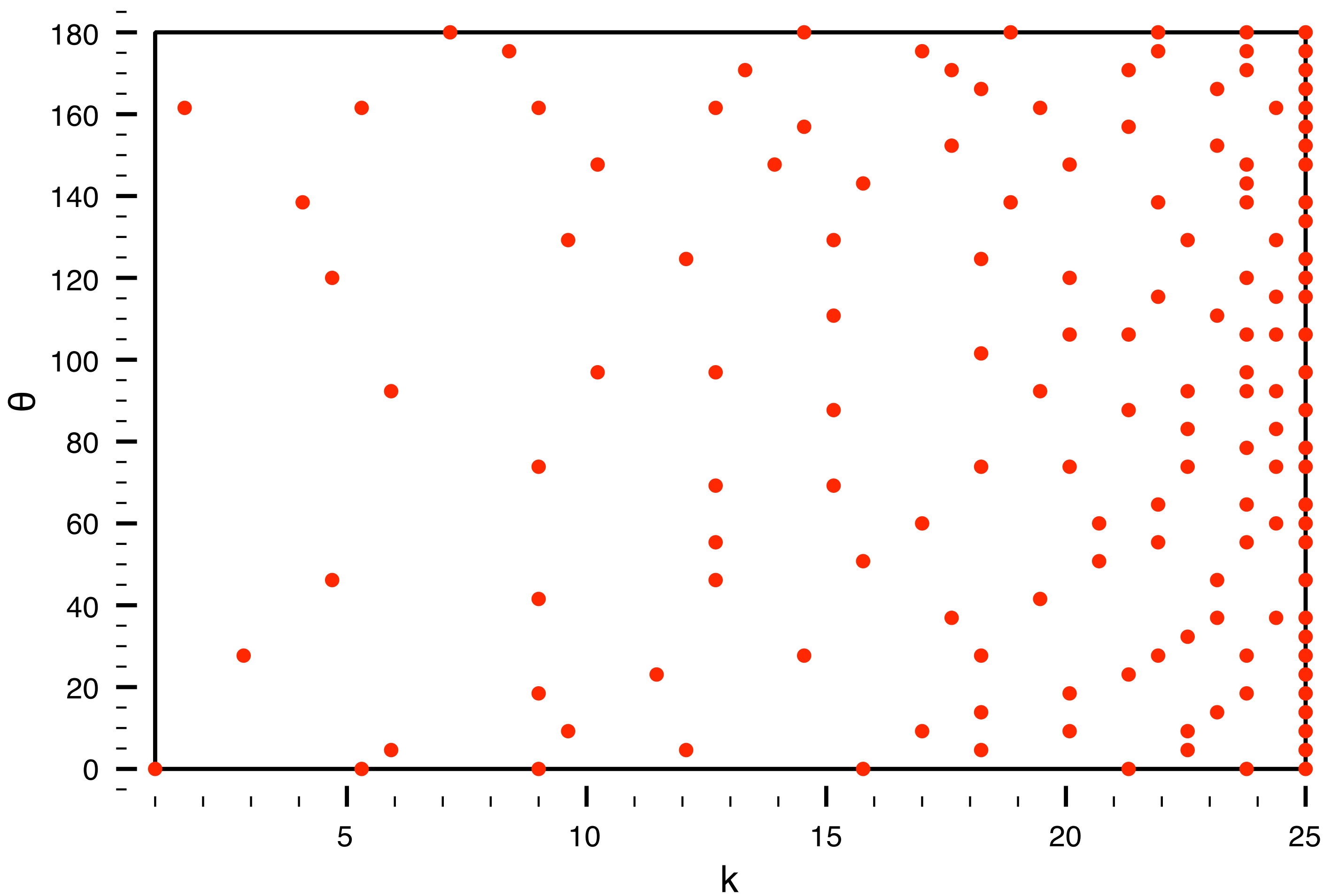


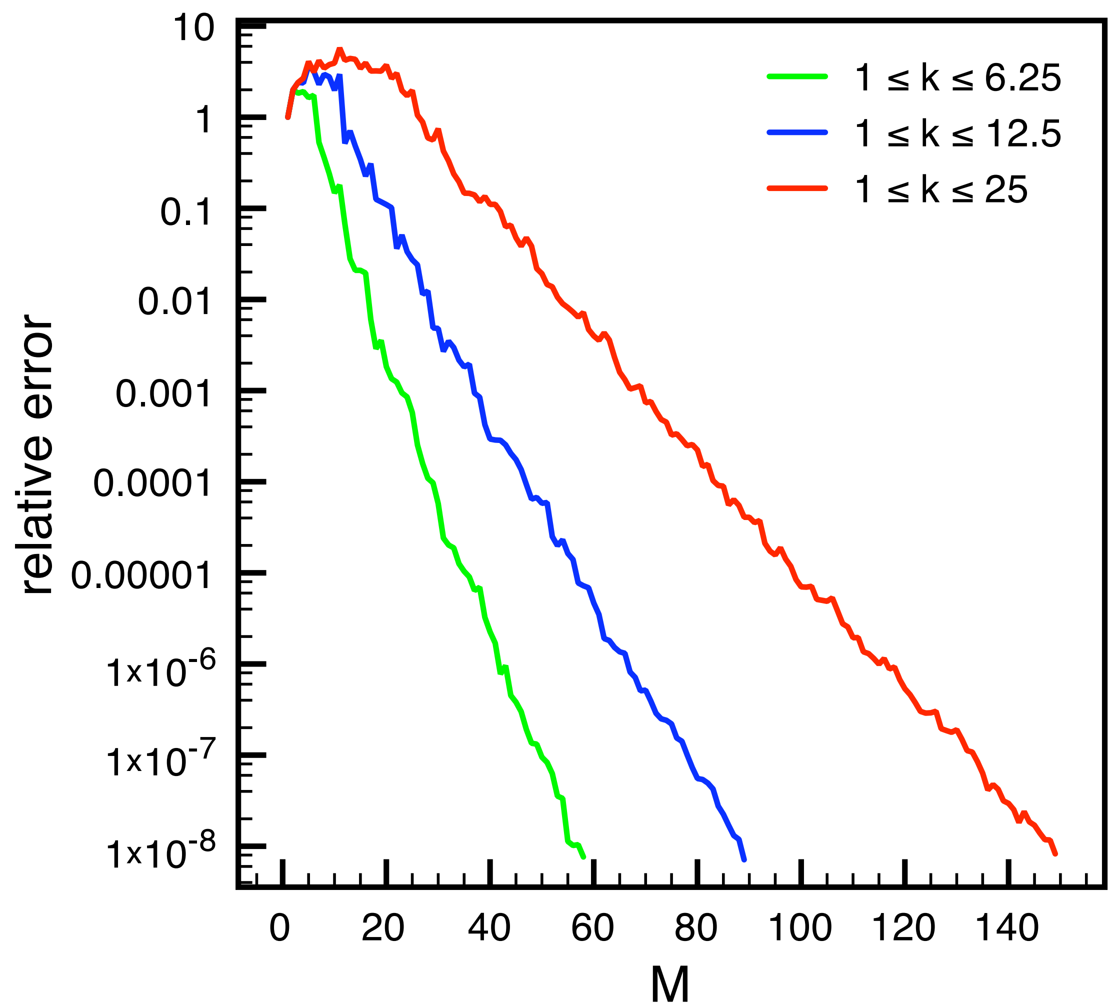
Interpolation error depending on the length of the expansion

RBM for Integral Equations

Results for EIM

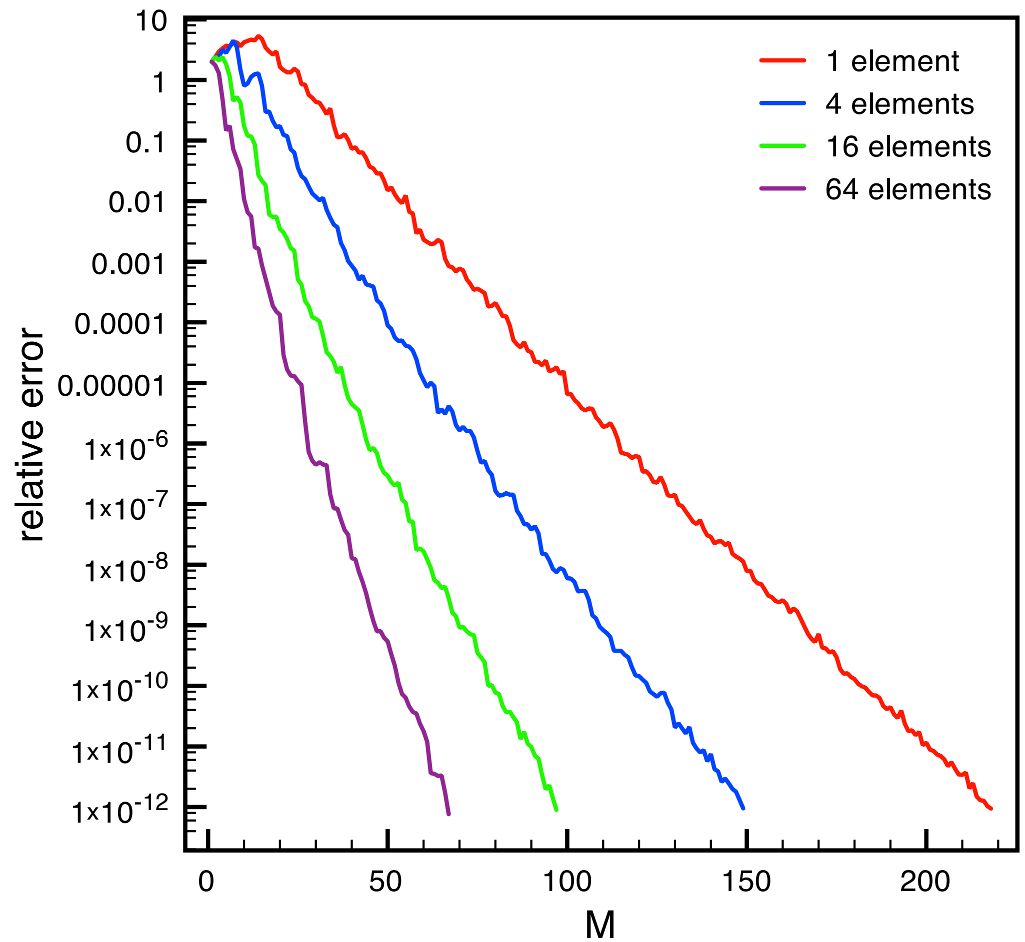
$$f(\boldsymbol{x};\boldsymbol{\mu}) = e^{ik\hat{\boldsymbol{s}}(\theta,\phi)\cdot\boldsymbol{x}}, \quad \boldsymbol{x}\in\Gamma, \boldsymbol{\mu}\in\mathcal{D},$$
$$\boldsymbol{\mu} = (k,\theta), \quad \phi \text{ fixed},$$
$$\mathcal{D} = [1, k_{\max}] \times [0,\pi]$$

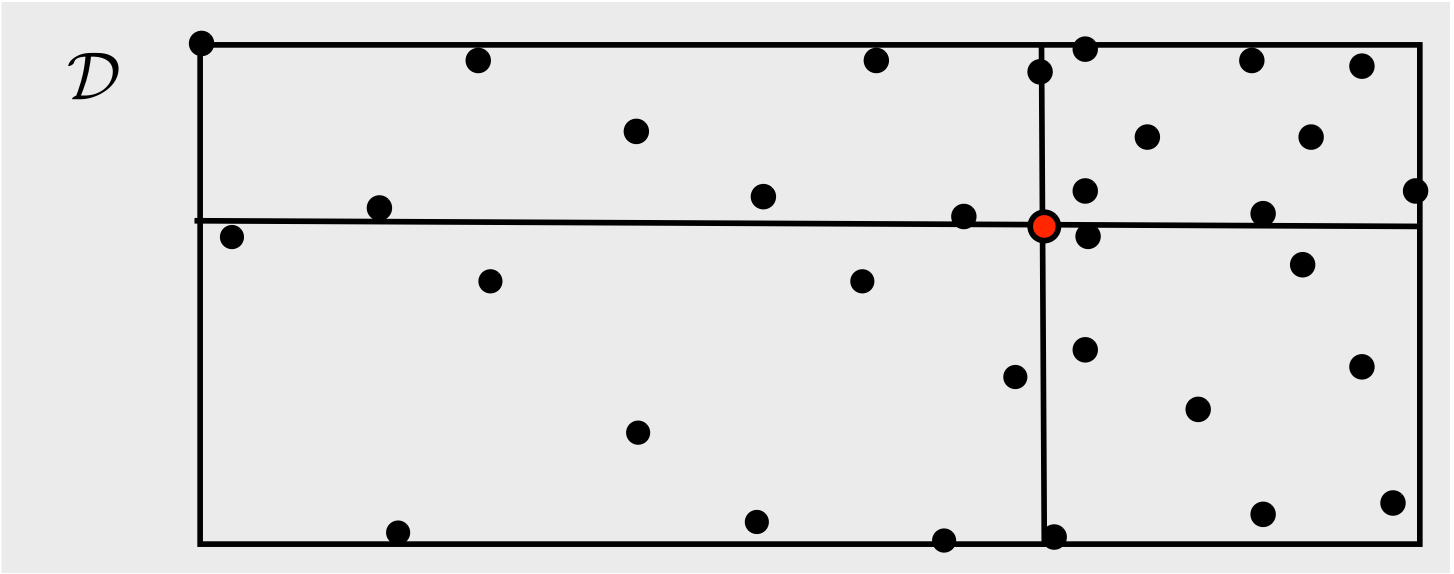


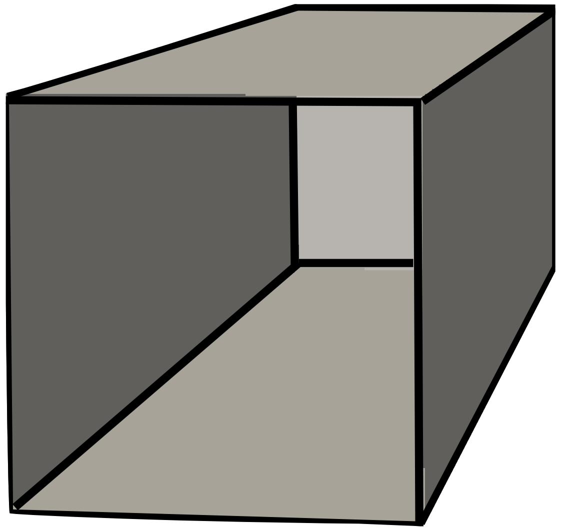


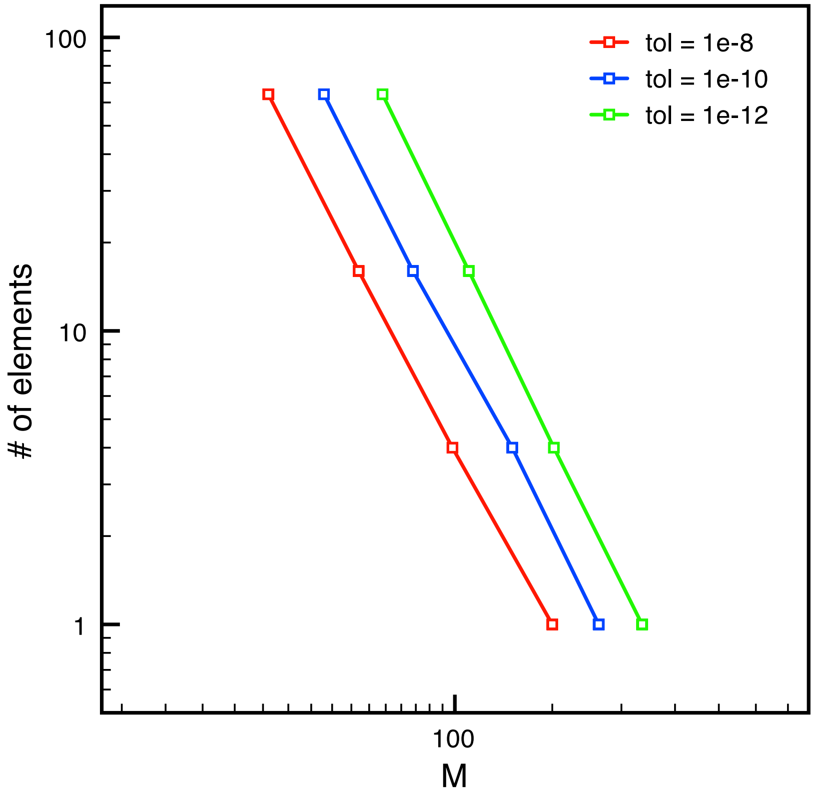
RBM for Integral Equations

Extension to an element based EIM





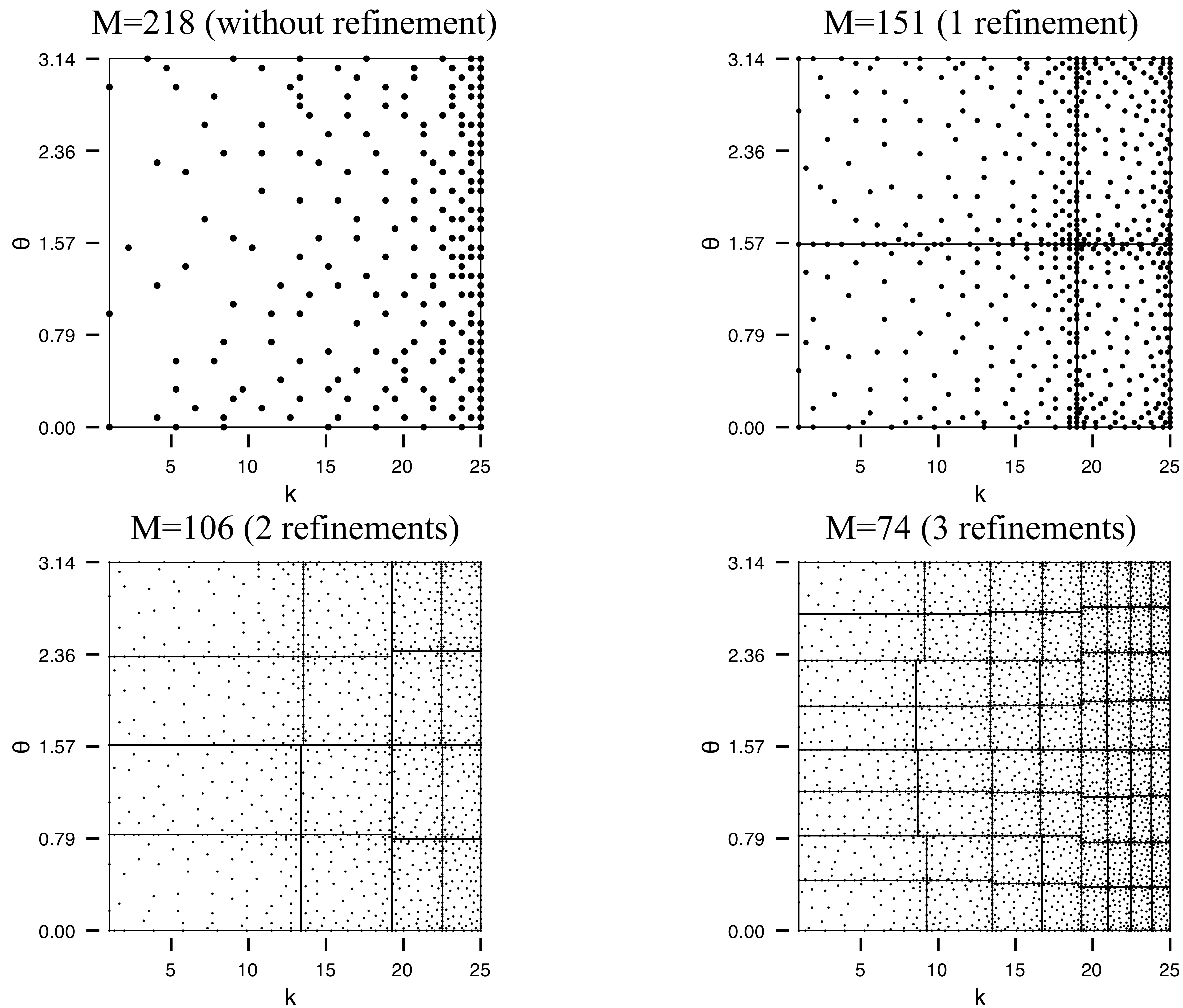




Objective is to reduce online cost

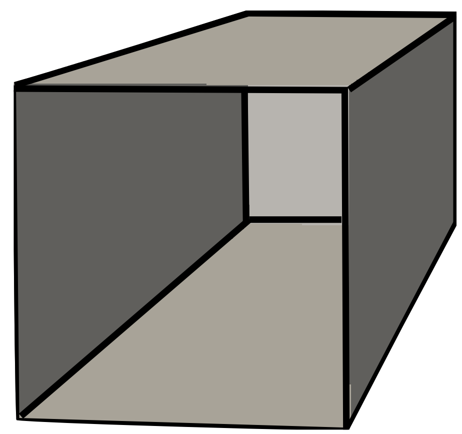
RBM for Integral Equations

Picked parameter values and EIM elements
(tol=1e-12):

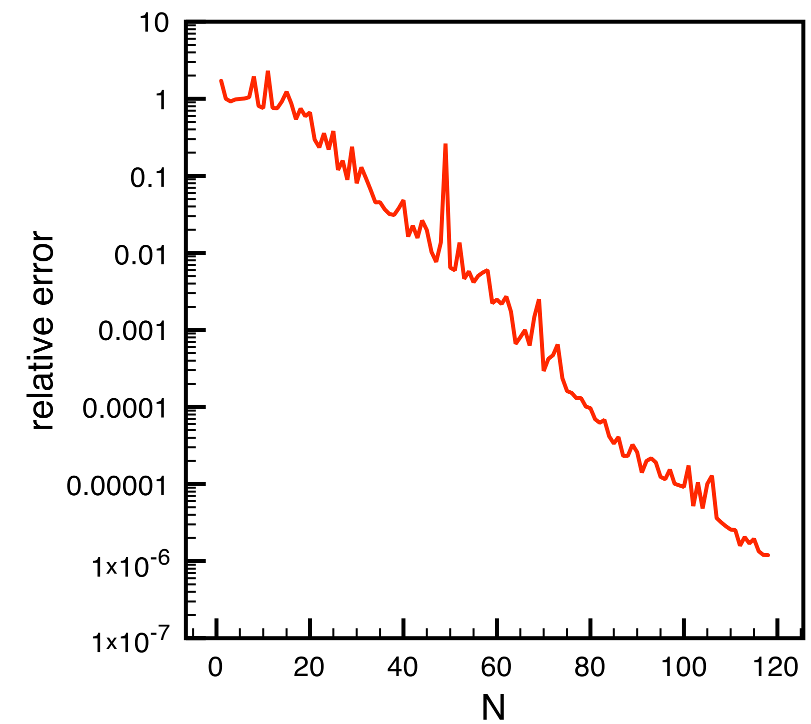


More complex examples

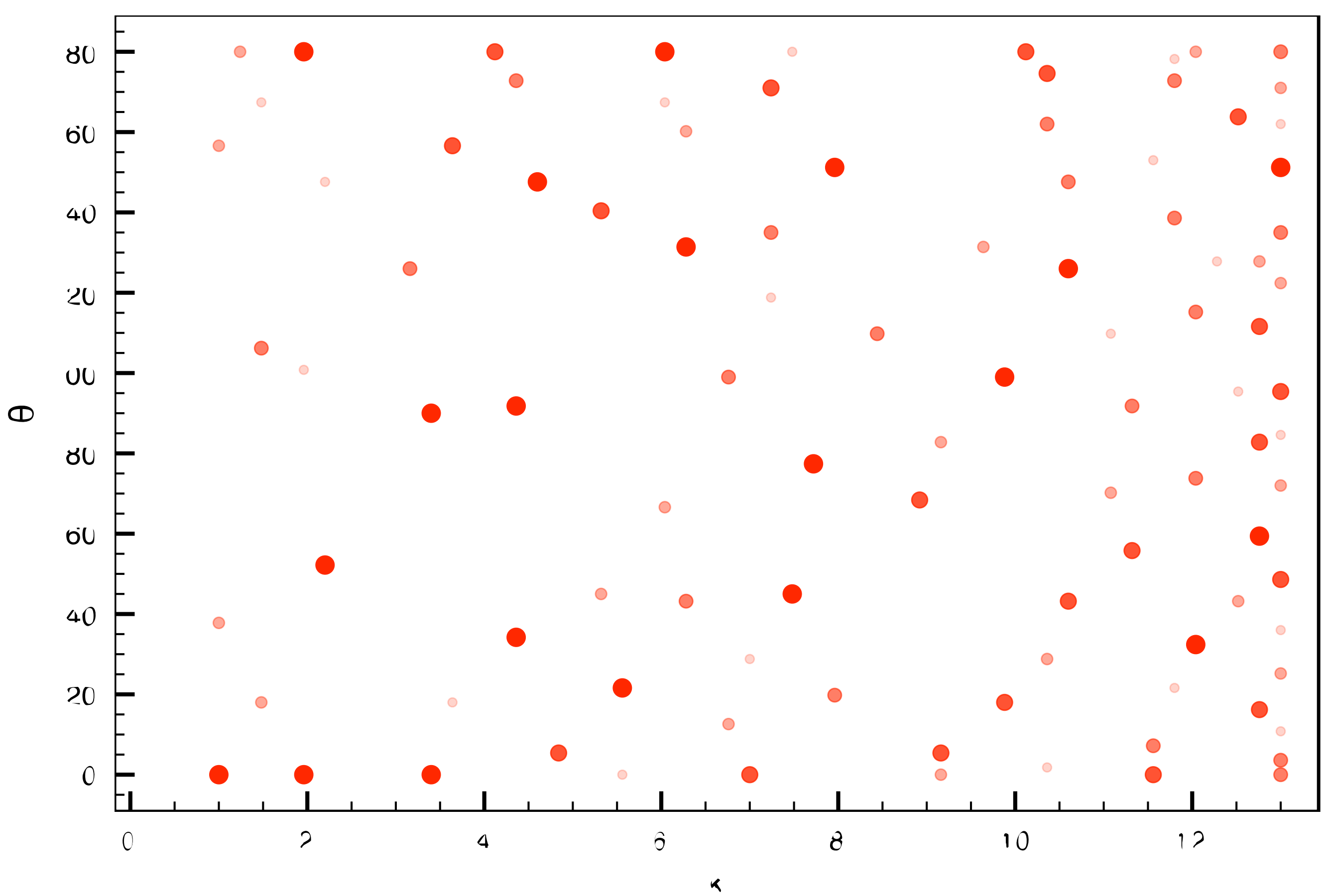
2 parameters, $\boldsymbol{\mu} = (k, \theta)$ with $\mathcal{D} = [1, 13] \times [0, \pi]$
$\phi = 0$ fixed



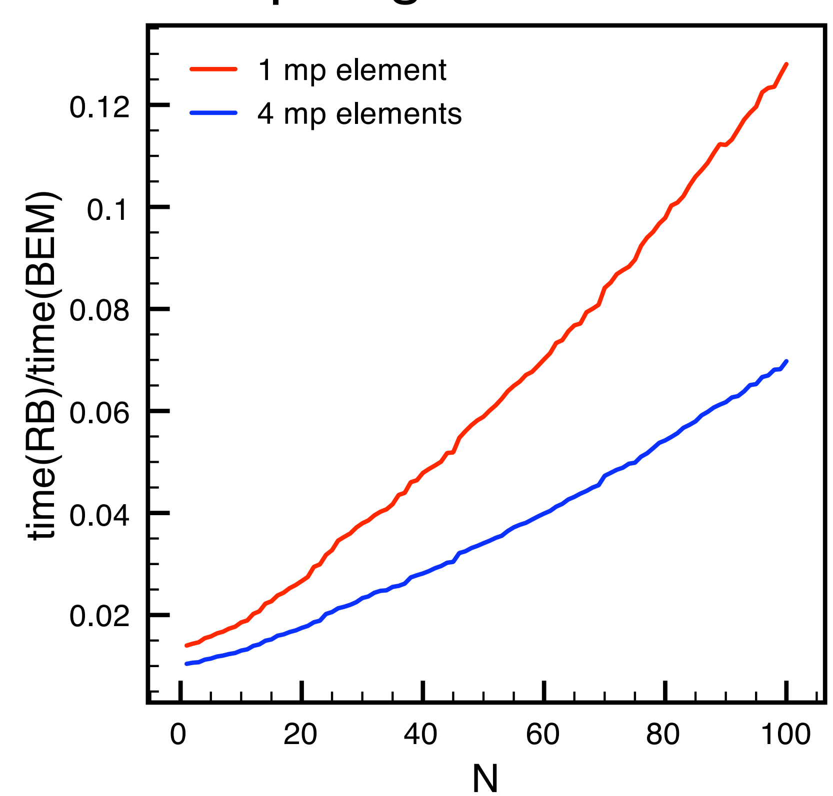
Convergence:



Picked parameters:

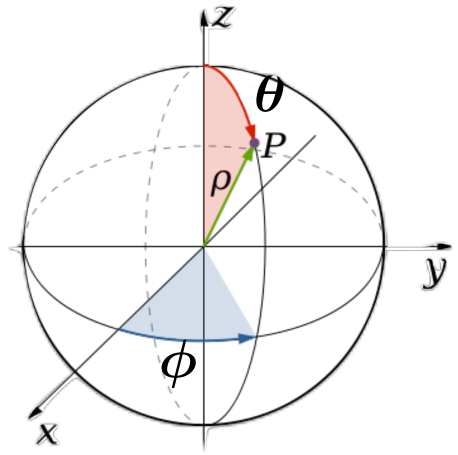


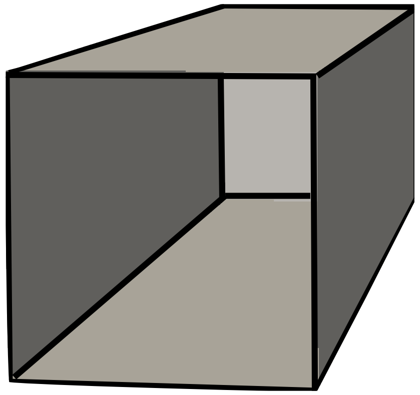
Computing time:



Scattering example

2 parameters, $\boldsymbol{\mu} = (k, \theta)$ with $\mathcal{D} = [1, 25] \times [0, \pi]$
$\phi = 0$ fixed





Scattering example

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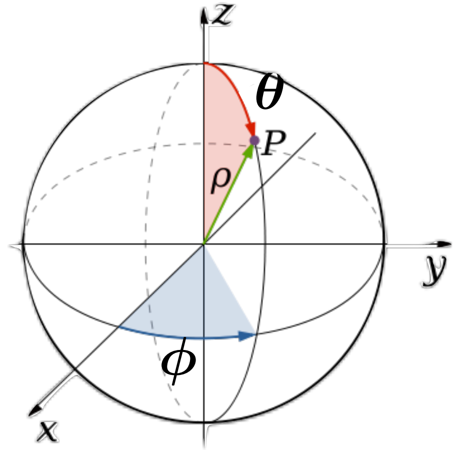
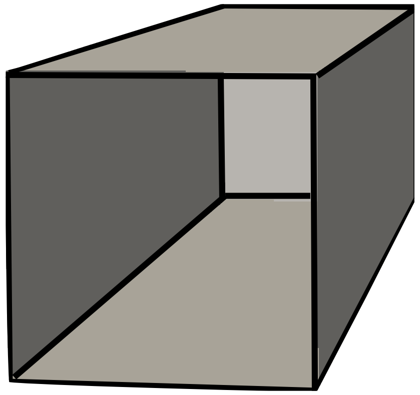
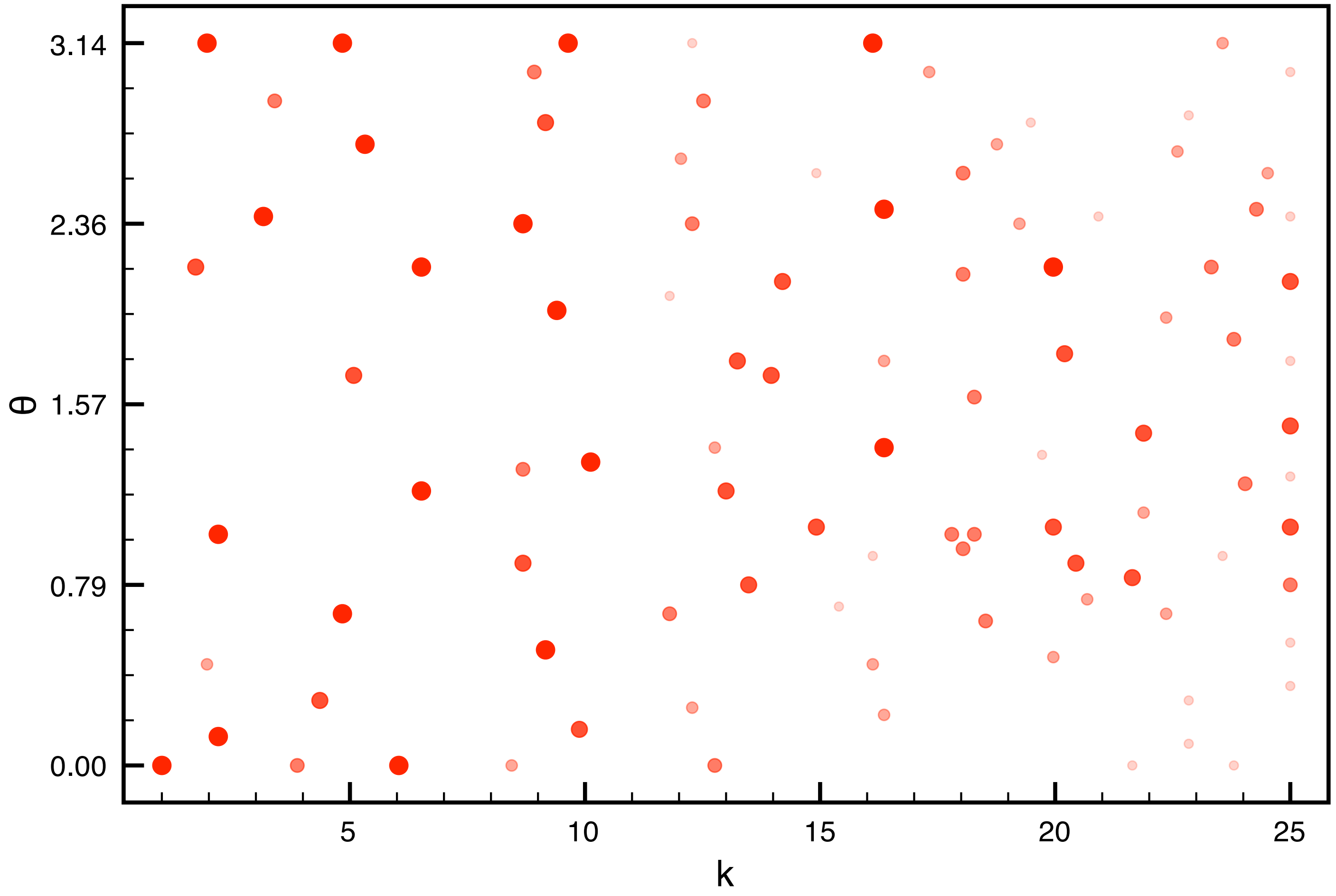




Picked parameters:



Scattering example

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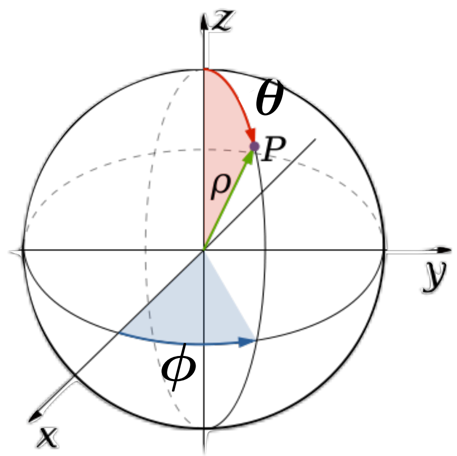
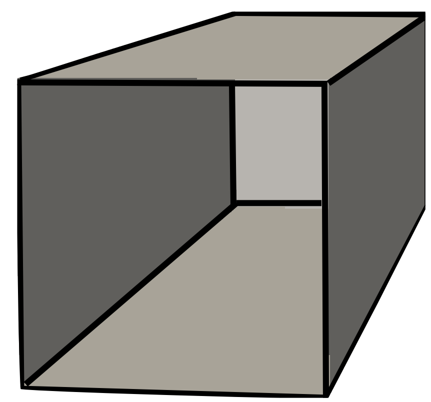
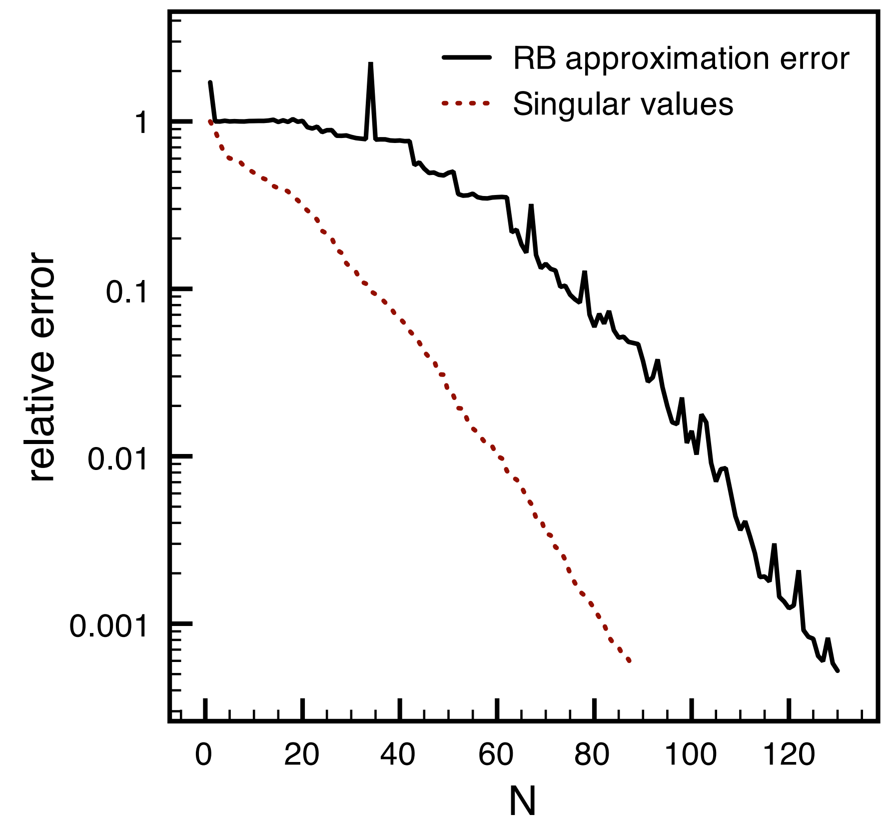
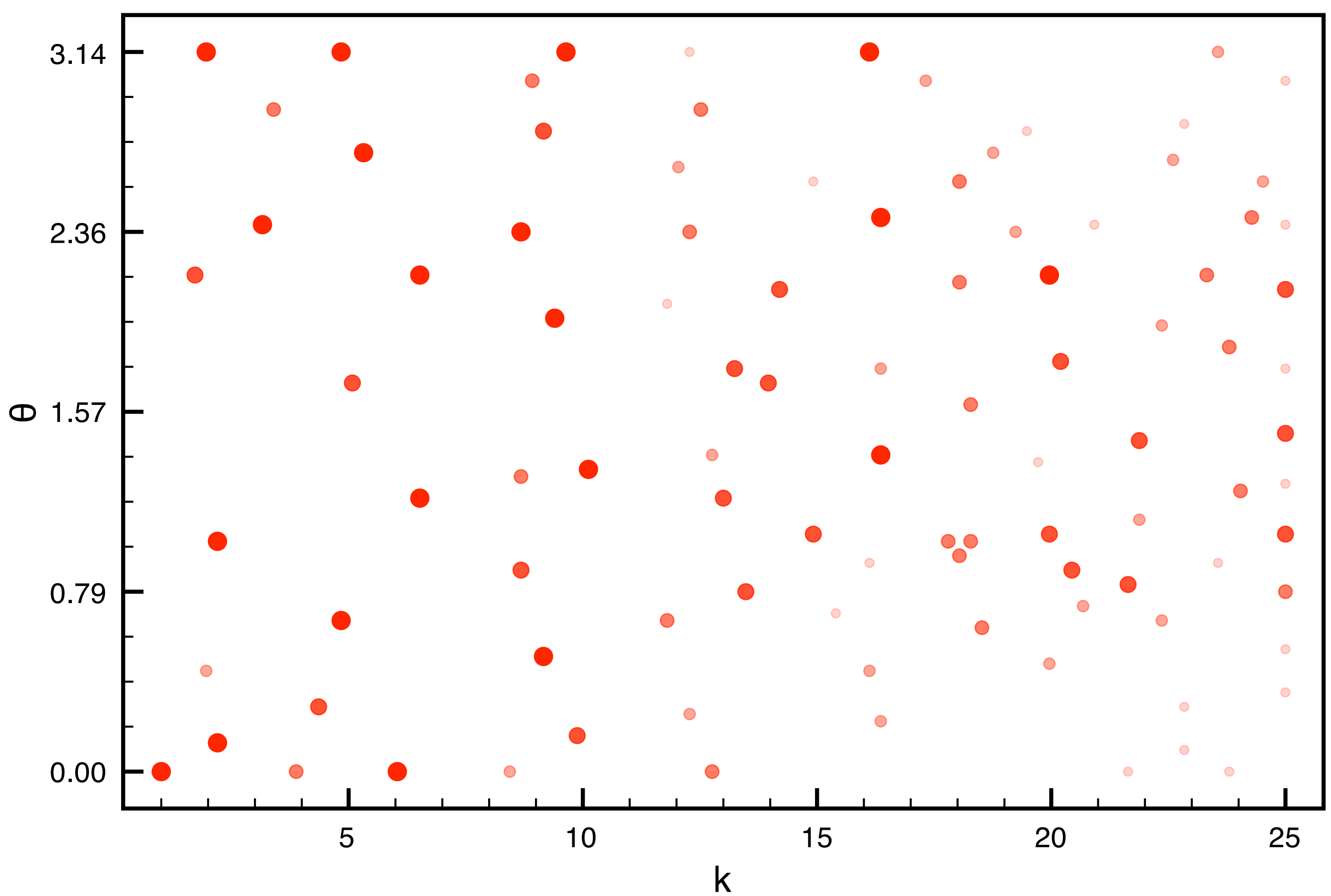
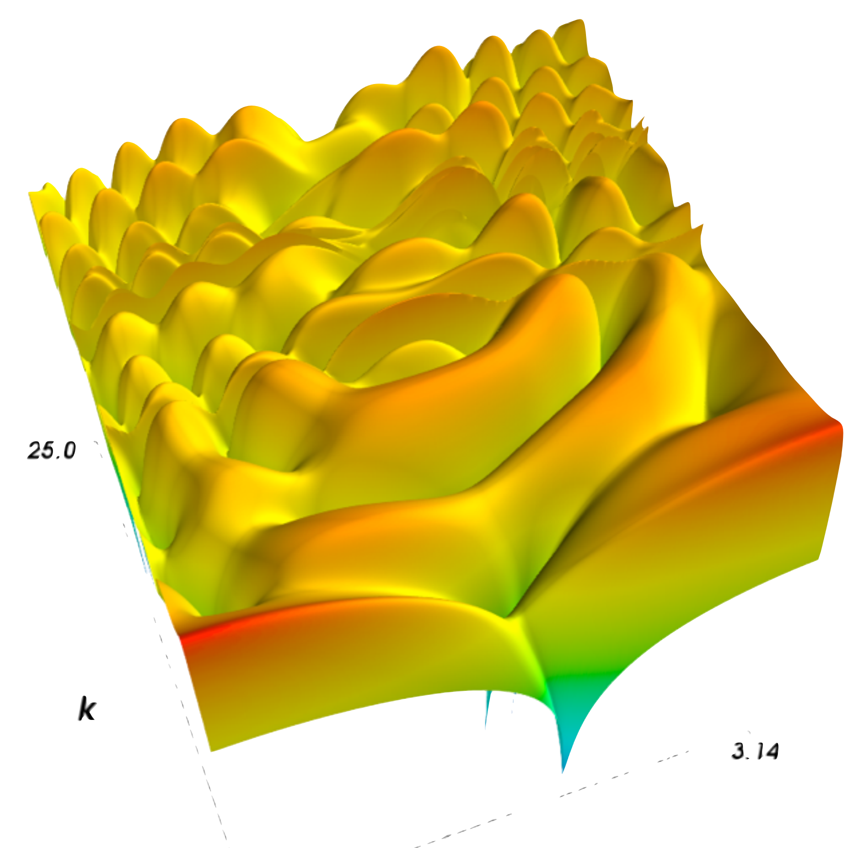






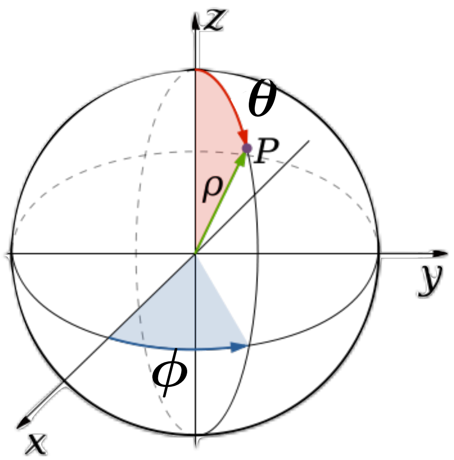
Picked parameters:

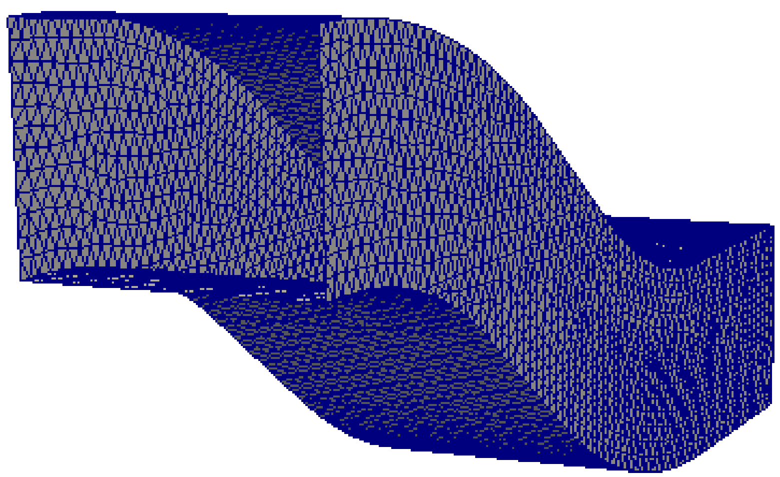




Scattering example

1 parameter, $\mu = k$ with $\mathcal{D} = [1, 25.5]$
$(\theta, \phi) = (\frac{\pi}{6}, 0)$ fixed





Scattering example

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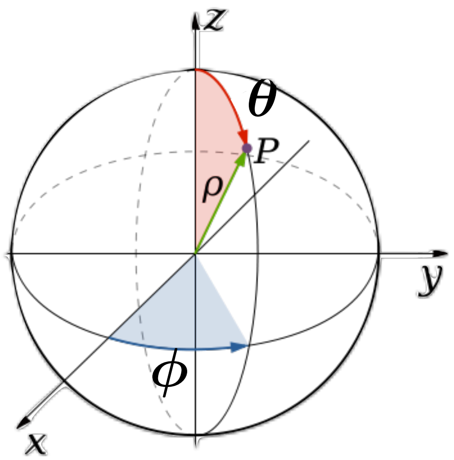
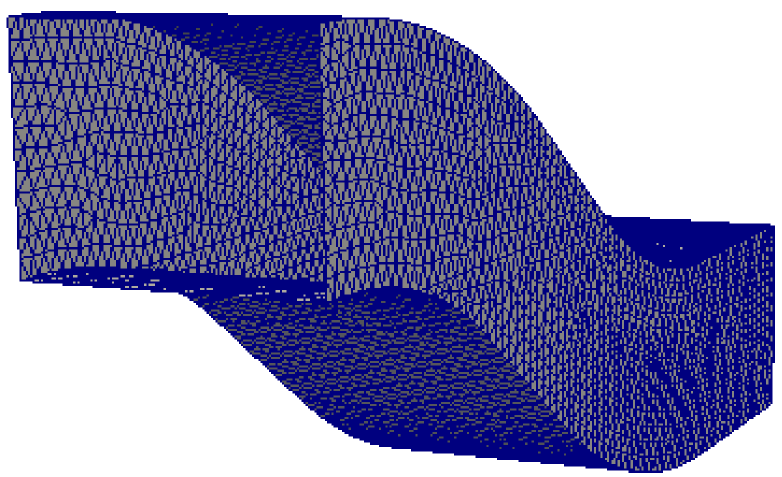
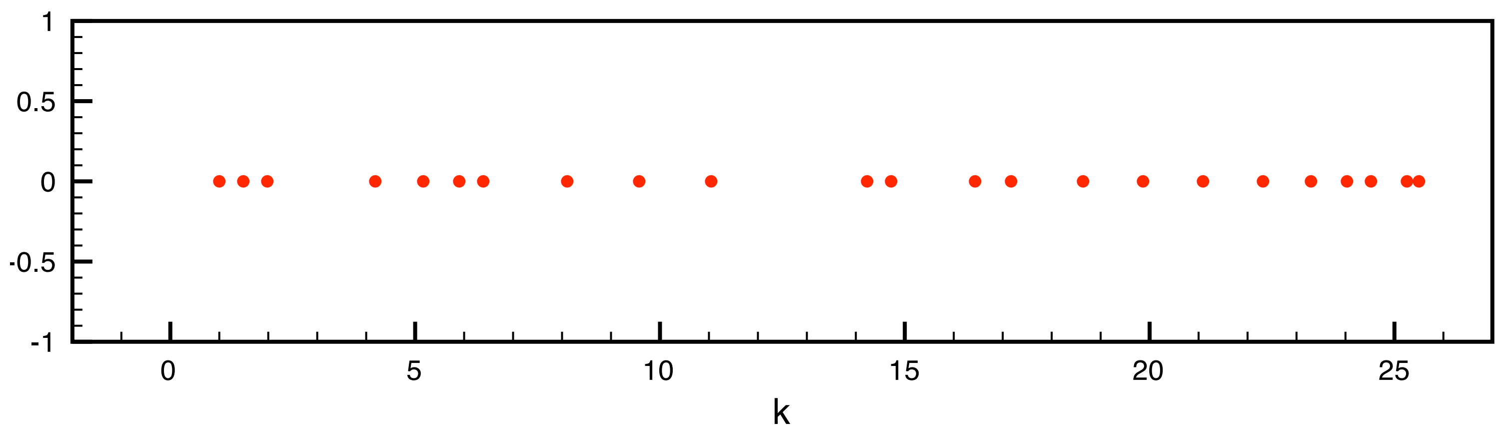




Repartition of 23 first picked parameters:



Scattering example

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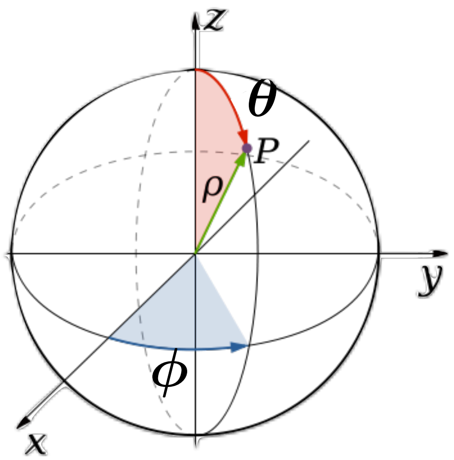
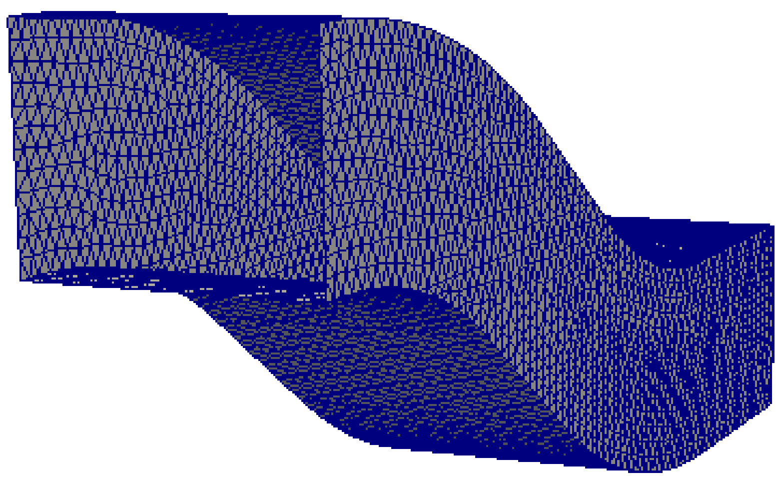
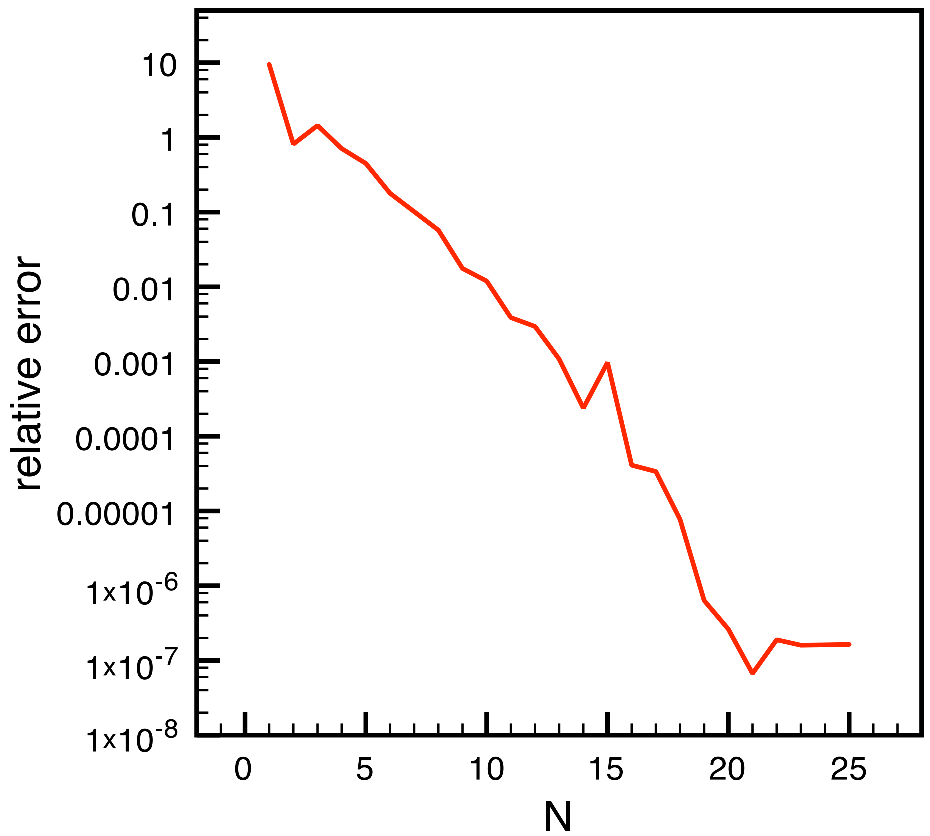
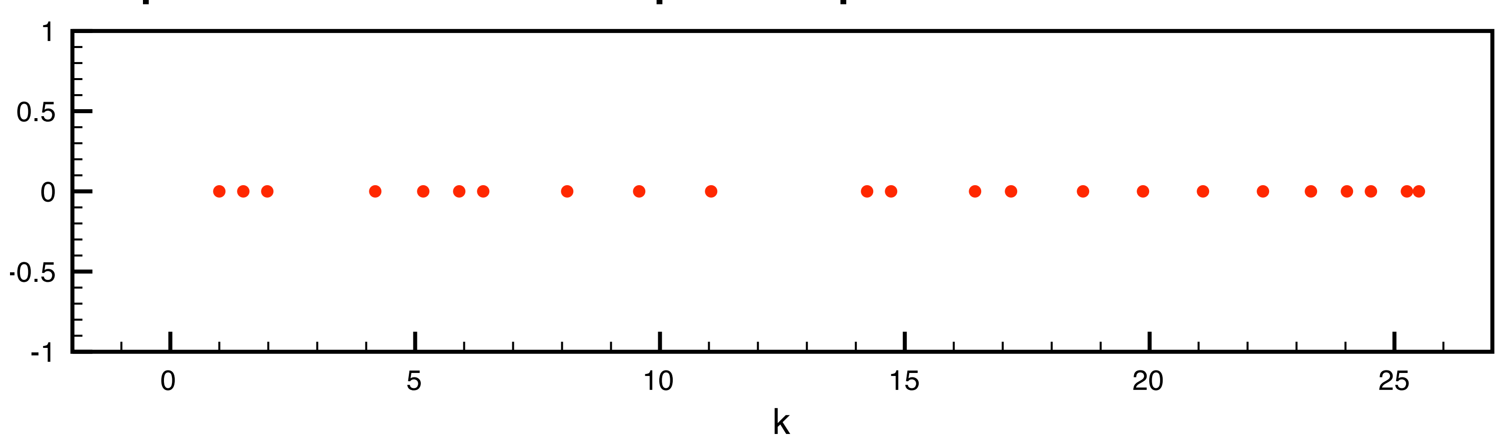






Repartition of 23 first picked parameters:



Scattering example

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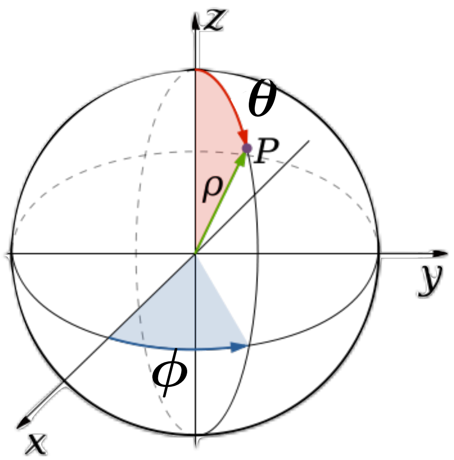
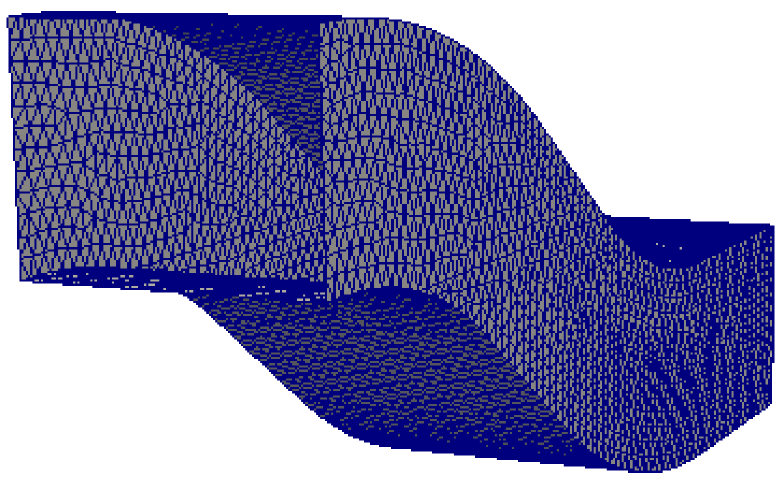
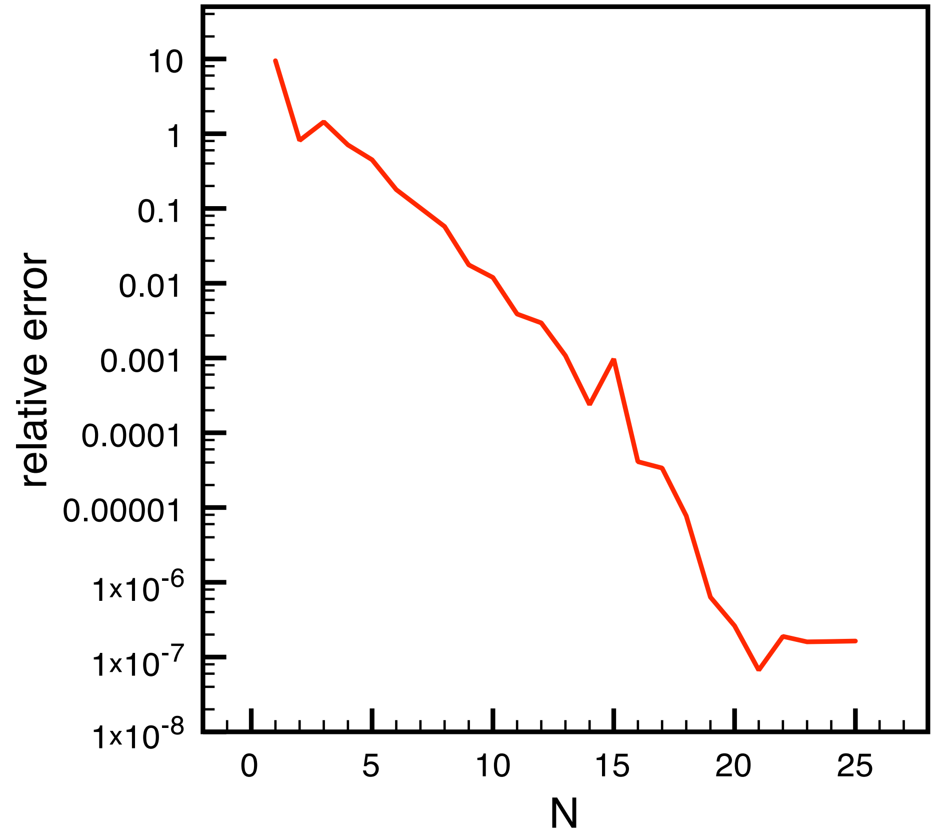
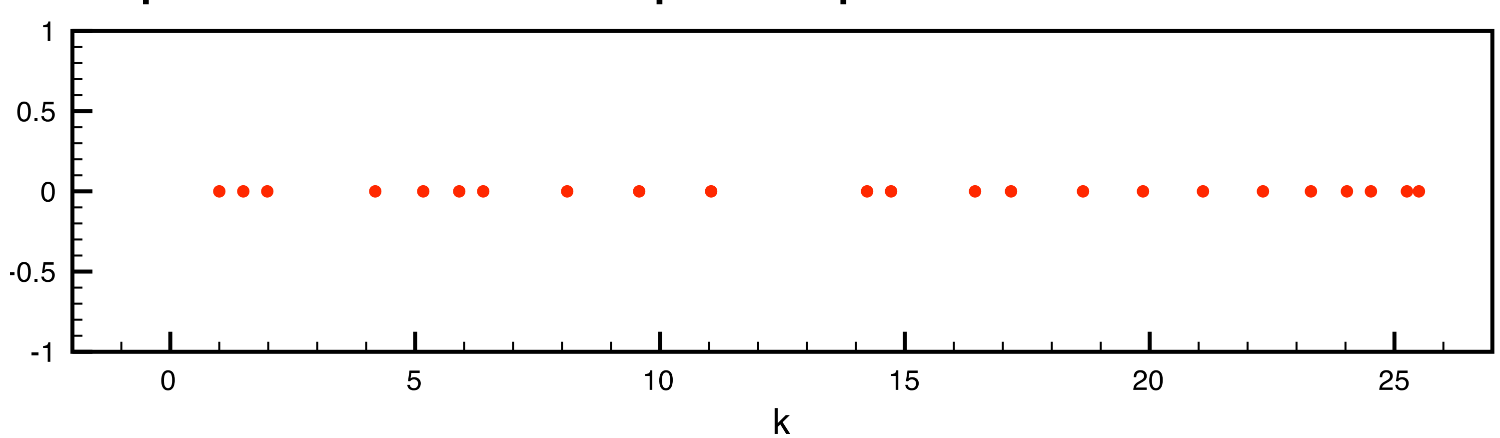
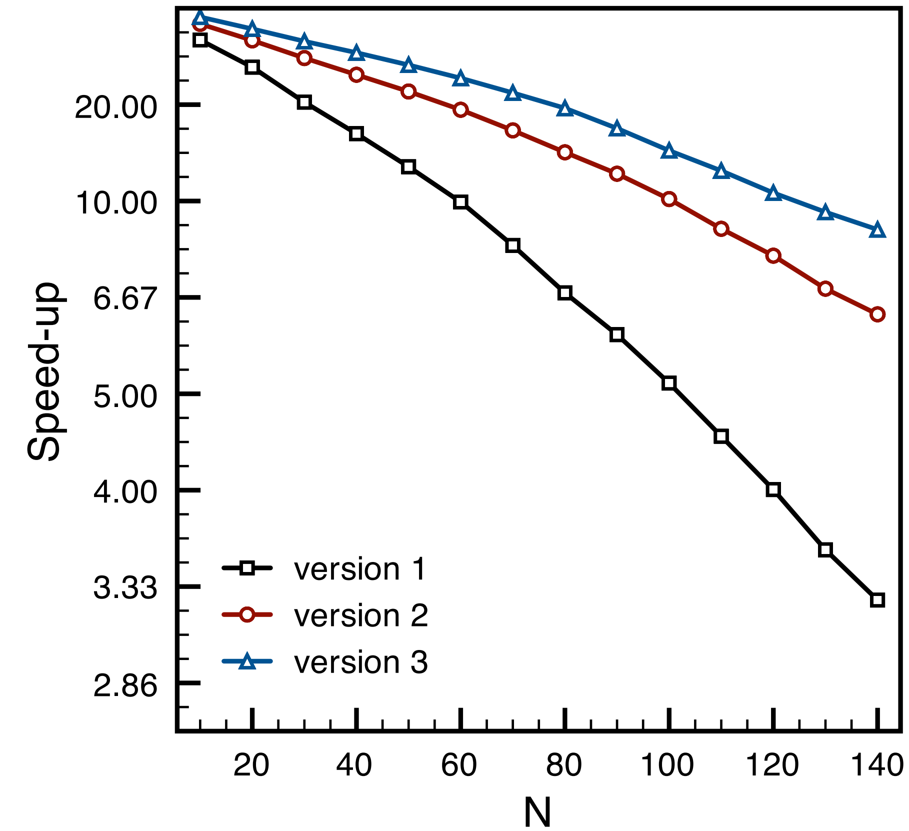






Repartition of 23 first picked parameters:





The non-compliant problem

Non-compliant case

A case is called non-compliant if

Compliant output: if $s(\mu) = \ell(u(\mu); \mu) = f(u(\mu); \mu)$.

Here: if $s(\mu) = \ell(u(\mu); \mu) \neq f(u(\mu); \mu)$.

In that case we must also solve the dual problem

Exact solution: for some $\mu \in \mathbb{P}$, find $\psi(\mu) \in \mathbb{V}$ such that

$$a(v, \psi(\mu); \mu) = -\ell(v; \mu), \qquad \forall v \in \mathbb{V}.$$

Truth solution: for some $\mu \in \mathbb{P}$, find $\psi_\delta(\mu) \in \mathbb{V}_\delta$ such that

$$a(v_\delta, \psi_\delta(\mu); \mu) = -\ell(v_\delta; \mu), \qquad \forall v_\delta \in \mathbb{V}_\delta.$$

Non-compliant case

The resulting RB approximation $u_{\mathtt{rb}} \in \mathbb{V}_{\mathtt{pr}}, \psi_{\mathtt{rb}} \in \mathbb{V}_{\mathtt{du}}$ solve

$$
\begin{aligned}
a(u_{\mathtt{rb}}(\mu), v_{\mathtt{rb}}; \mu) &= f(v_{\mathtt{rb}}), \quad \forall v_{\mathtt{rb}} \in \mathbb{V}_{\mathtt{pr}}, \\
a(v_{\mathtt{rb}}, \psi_{\mathtt{rb}}(\mu); \mu) &= -\ell(v_{\mathtt{rb}}), \quad \forall v_{\mathtt{rb}} \in \mathbb{V}_{\mathtt{du}}.
\end{aligned}
$$

Then, the RB output can be evaluated as

$$
s_{\mathtt{rb}}(\mu) = \ell(u_{\mathtt{rb}}) - r_{\mathtt{pr}}(\psi_{\mathtt{rb}}; \mu)
$$

where

$$
\begin{aligned}
r_{\mathtt{pr}}(v; \mu) &= f(v) - a(u_{\mathtt{rb}}, v; \mu), \\
r_{\mathtt{du}}(v; \mu) &= -\ell(v) - a(v, \psi_{\mathtt{rb}}; \mu)
\end{aligned}
$$

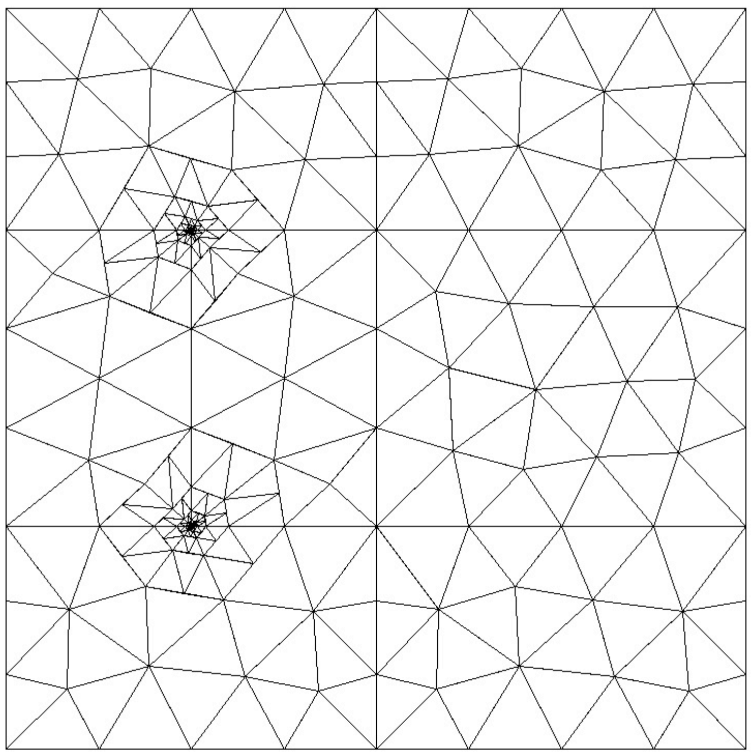
are the primal and the dual residuals.

The output error bound takes the form

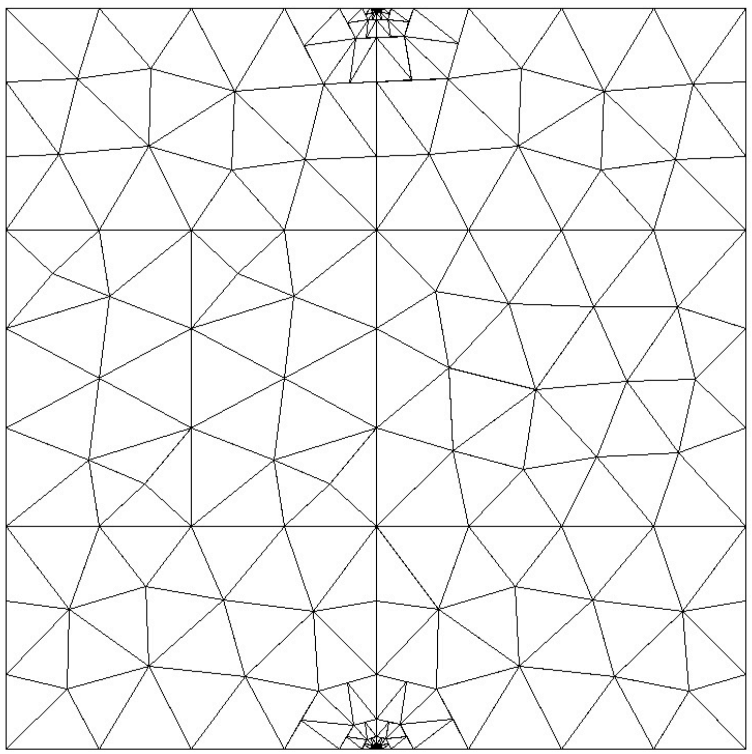
$$
\eta_{\mathrm{s}}(\mu) \equiv \frac{\|r_{\mathtt{pr}}(\,\cdot\,; \mu)\|_{\mathbb{V}'}}{(\alpha_{\mathrm{LB}}(\mu))^{1/2}} \, \frac{\|r_{\mathtt{du}}(\,\cdot\,; \mu)\|_{\mathbb{V}'}}{(\alpha_{\mathrm{LB}}(\mu))^{1/2}}.
$$

Including the adjoint

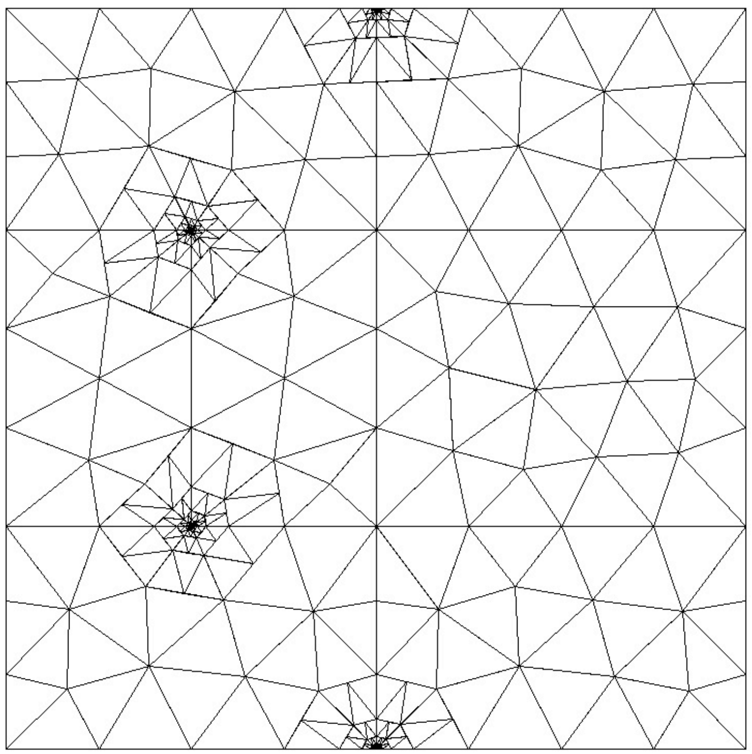
Note: we can allow different approximation spaces based on different parts of the problem



Primal

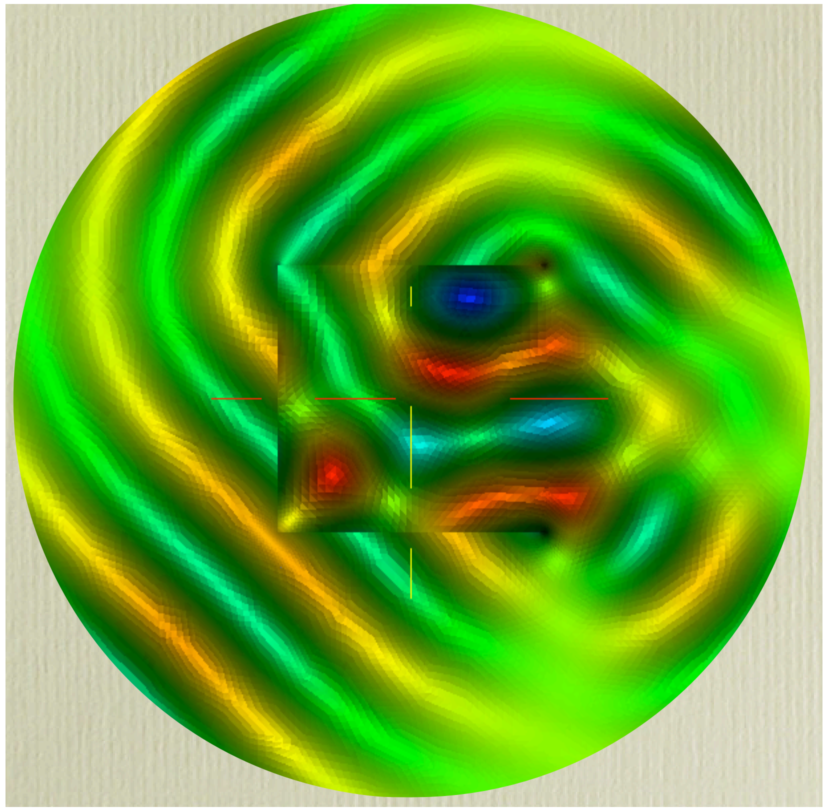


Dual



Primal-Dual

Let's consider an example



2D EM scattering off an open PEC cavity

Parameters are angles and frequency

We consider an output of interest known as the RCS

$$F(\omega,\theta,\psi) = \frac{\omega}{\sqrt{8\pi\omega}} \int_S \left[n_x H_y - n_y H_x + (\cos\psi n_x E_z + \sin\psi n_y E_z) \right] e^{-i\omega(x\cos\psi + y\sin\psi)}\, ds$$

$$s(\omega,\theta,\psi) = 10\log_{10}\left[2\pi \frac{|F(\omega,\theta,\psi)|^2}{|E_z^{inc}(\omega,\theta)|^2} \right]$$

Treated by empirical interpolation

2D EM problems

We consider the 2D Maxwell problem

$$\begin{cases} -\epsilon\omega^2 E_x + \dfrac{1}{\mu}\dfrac{\partial}{\partial y}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) = i\omega J_x \\ -\epsilon\omega^2 E_y - \dfrac{1}{\mu}\dfrac{\partial}{\partial x}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) = i\omega J_y \end{cases}$$

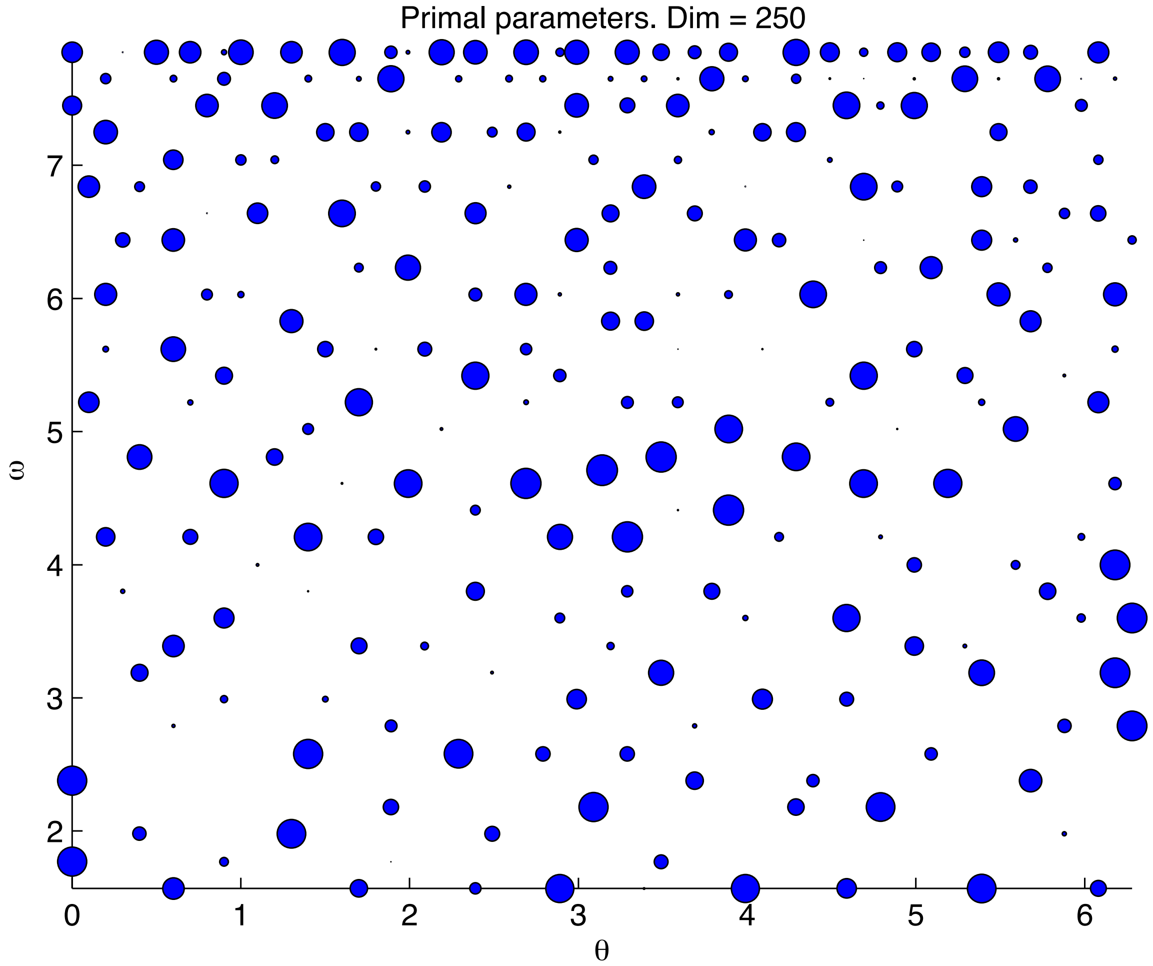
Parameters can be in the

- Materials
- Sources
- Frequencies
- Geometries

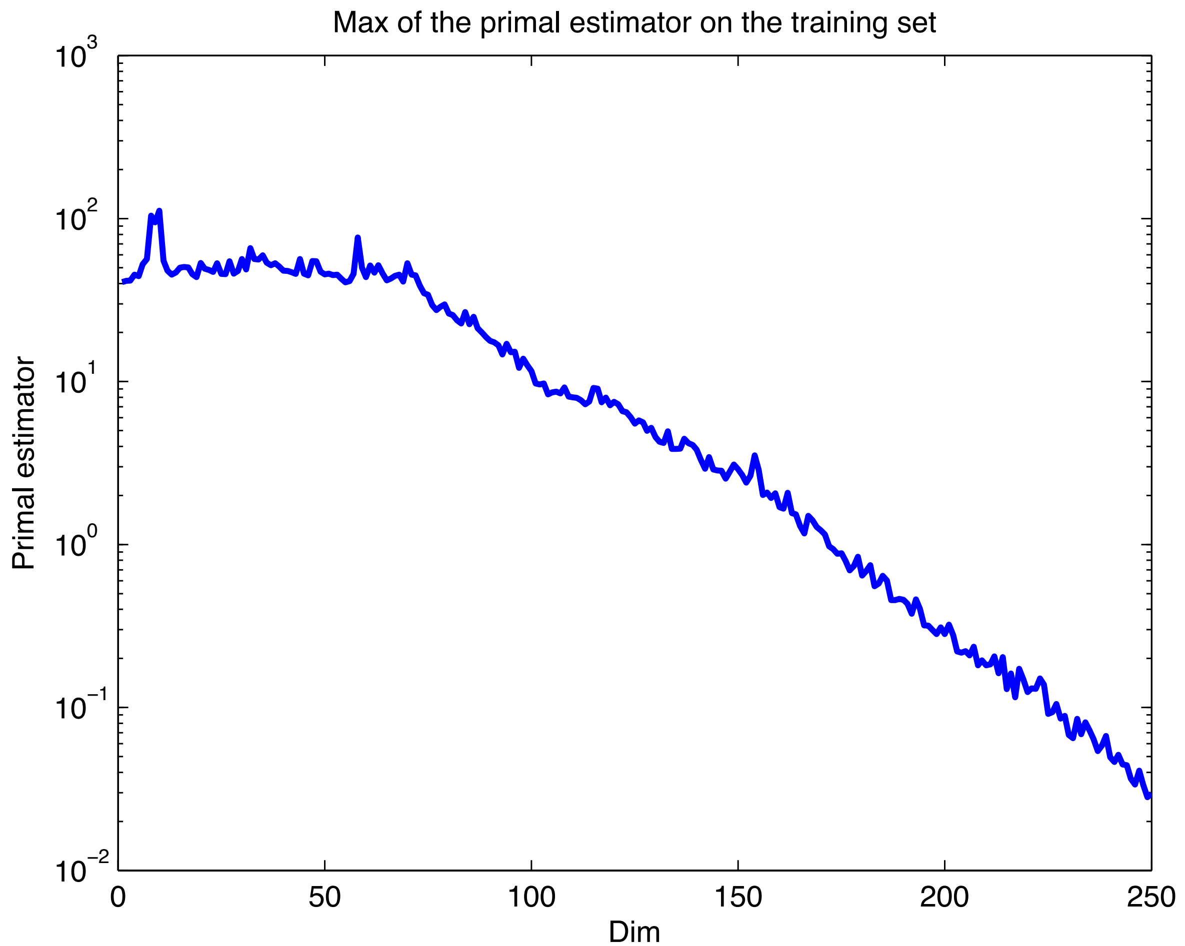
Problem is non-coercive and internal problems can have resonances

2D Scattering example

- ✓ Problem is affine in the frequency
- ✓ Non-affine in the angle(s) and output
- ✓ Both primal and dual problem are solved



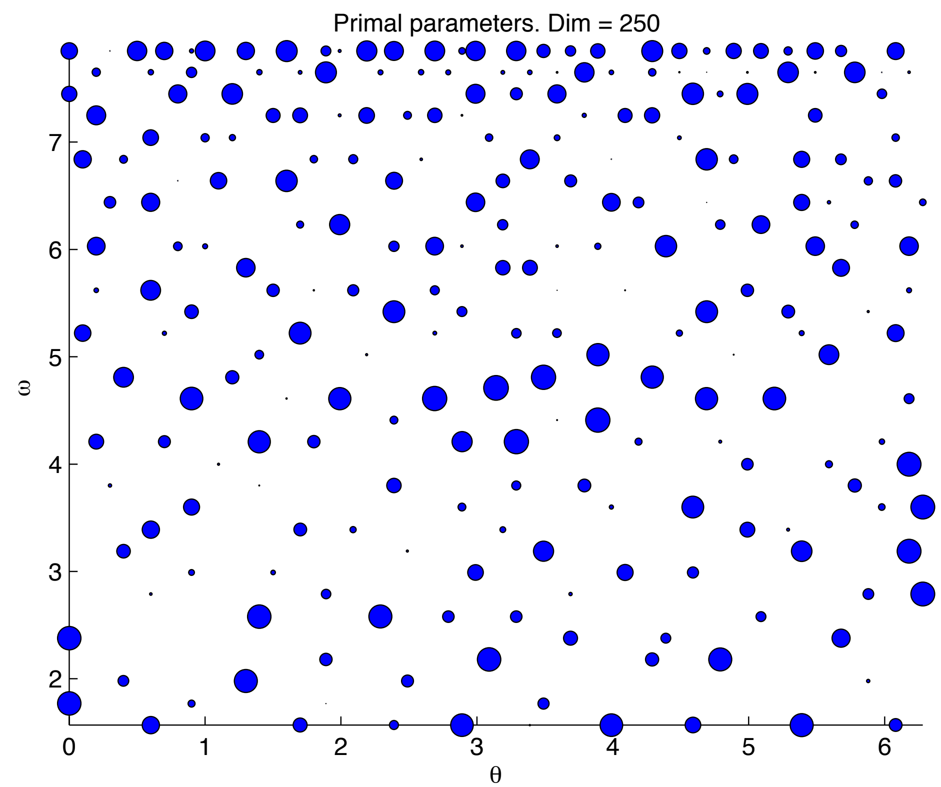
Greedy selection of samples

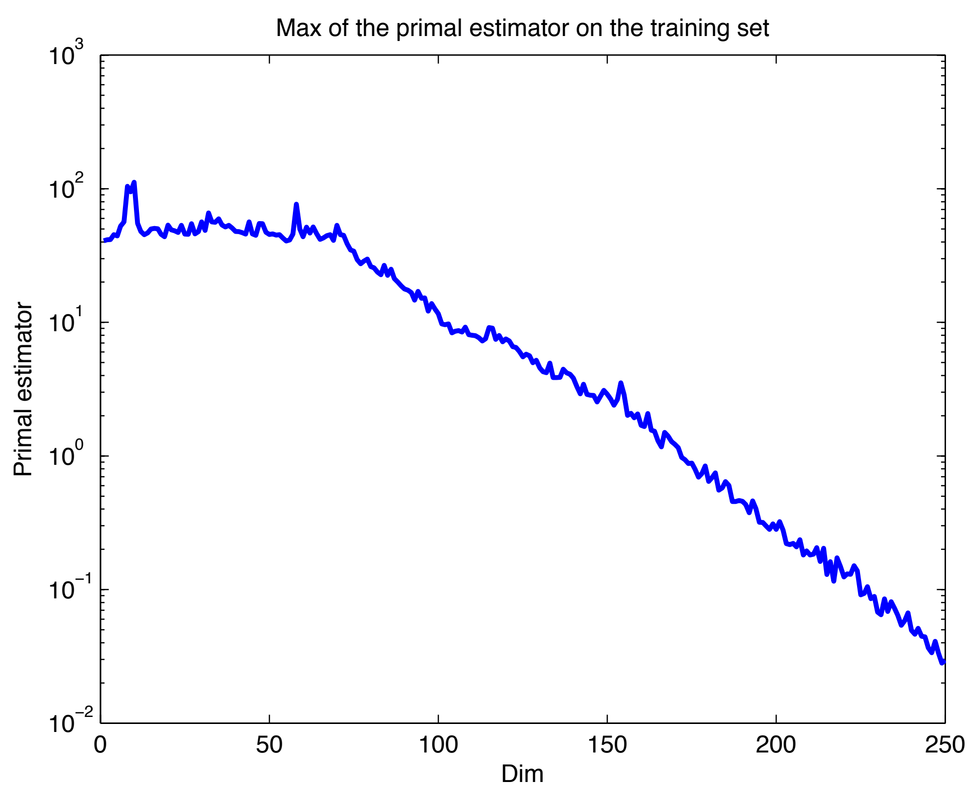


Max error in primal error estimator

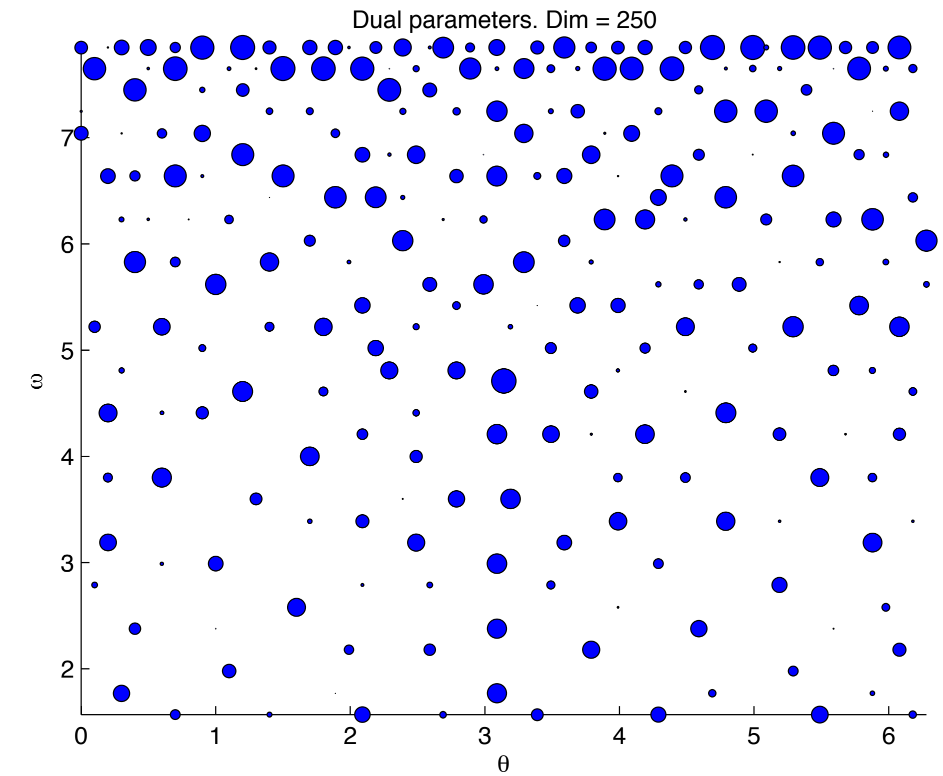
Primal vs dual solution

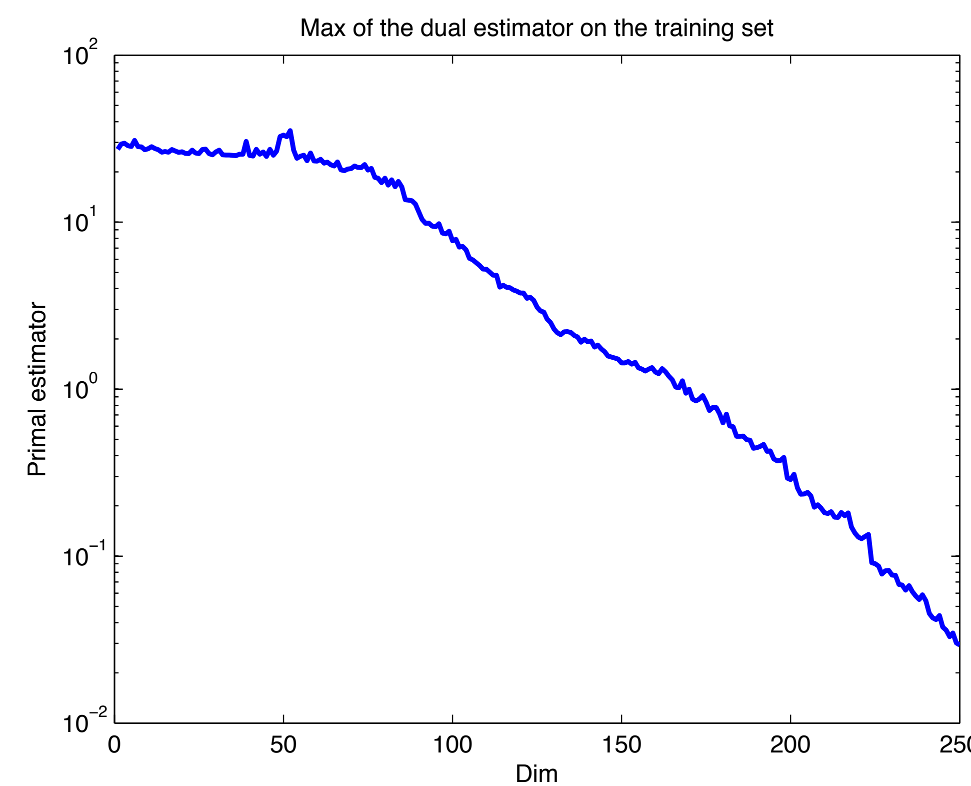
Primal problem



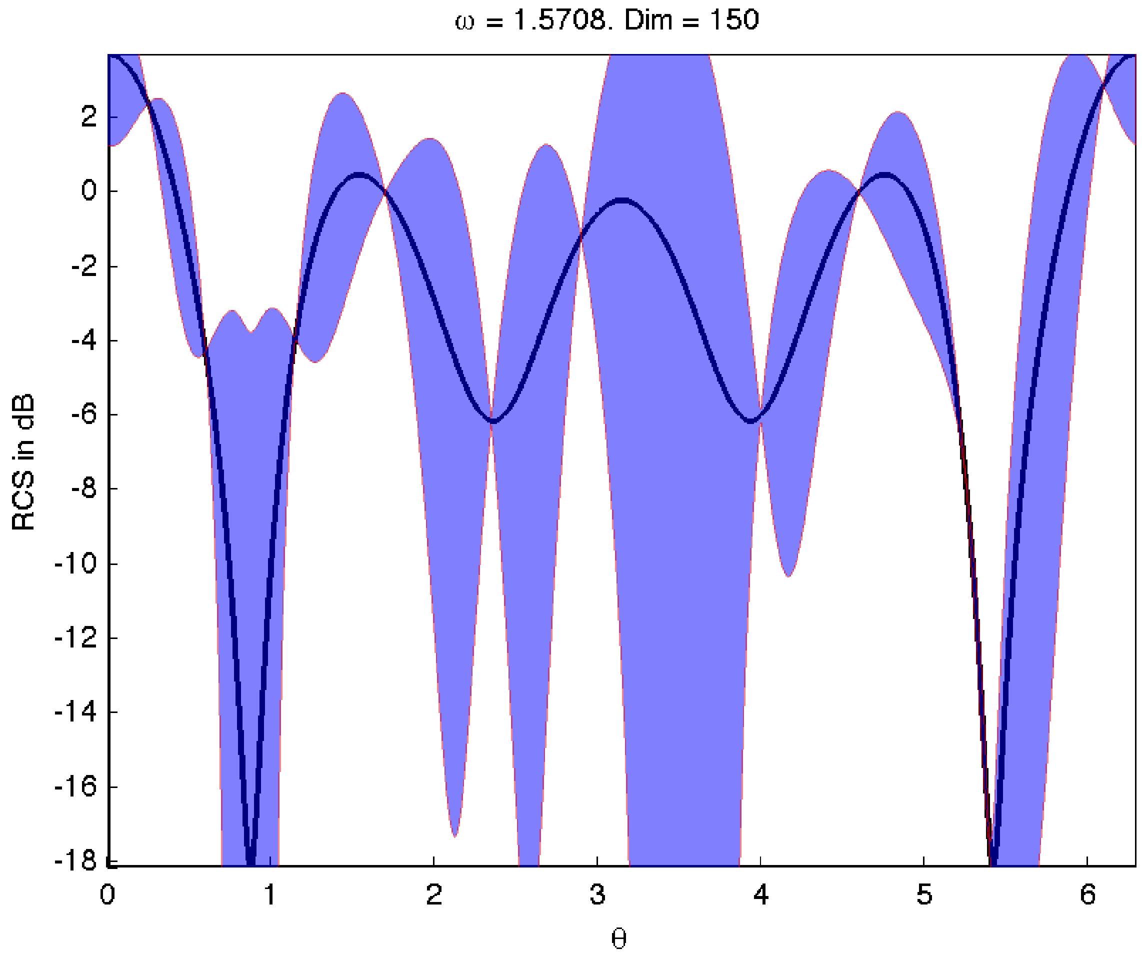


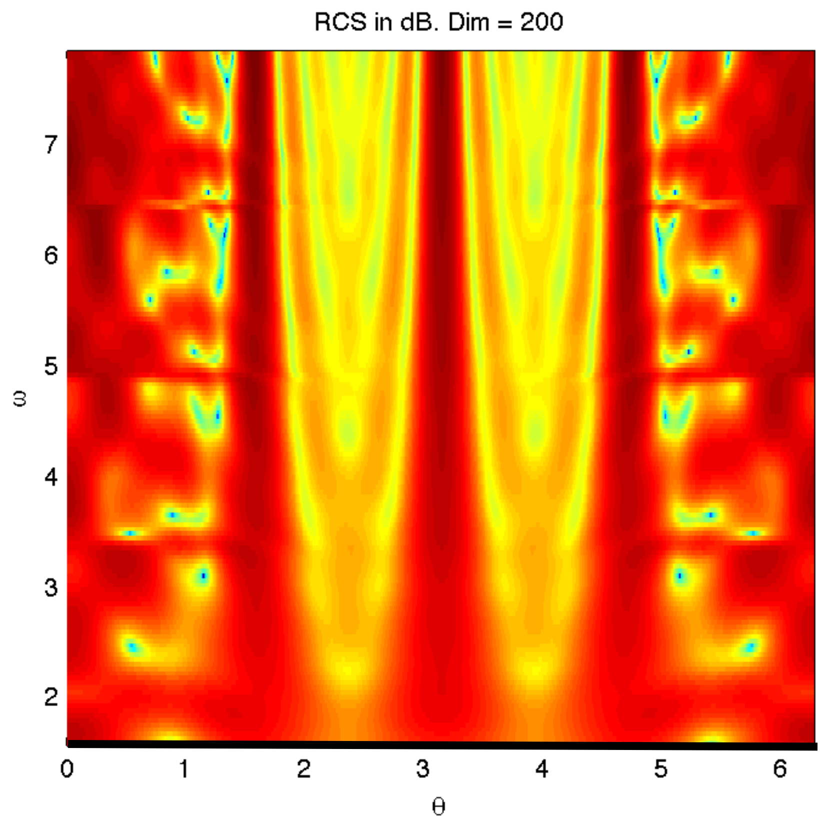
Dual problem



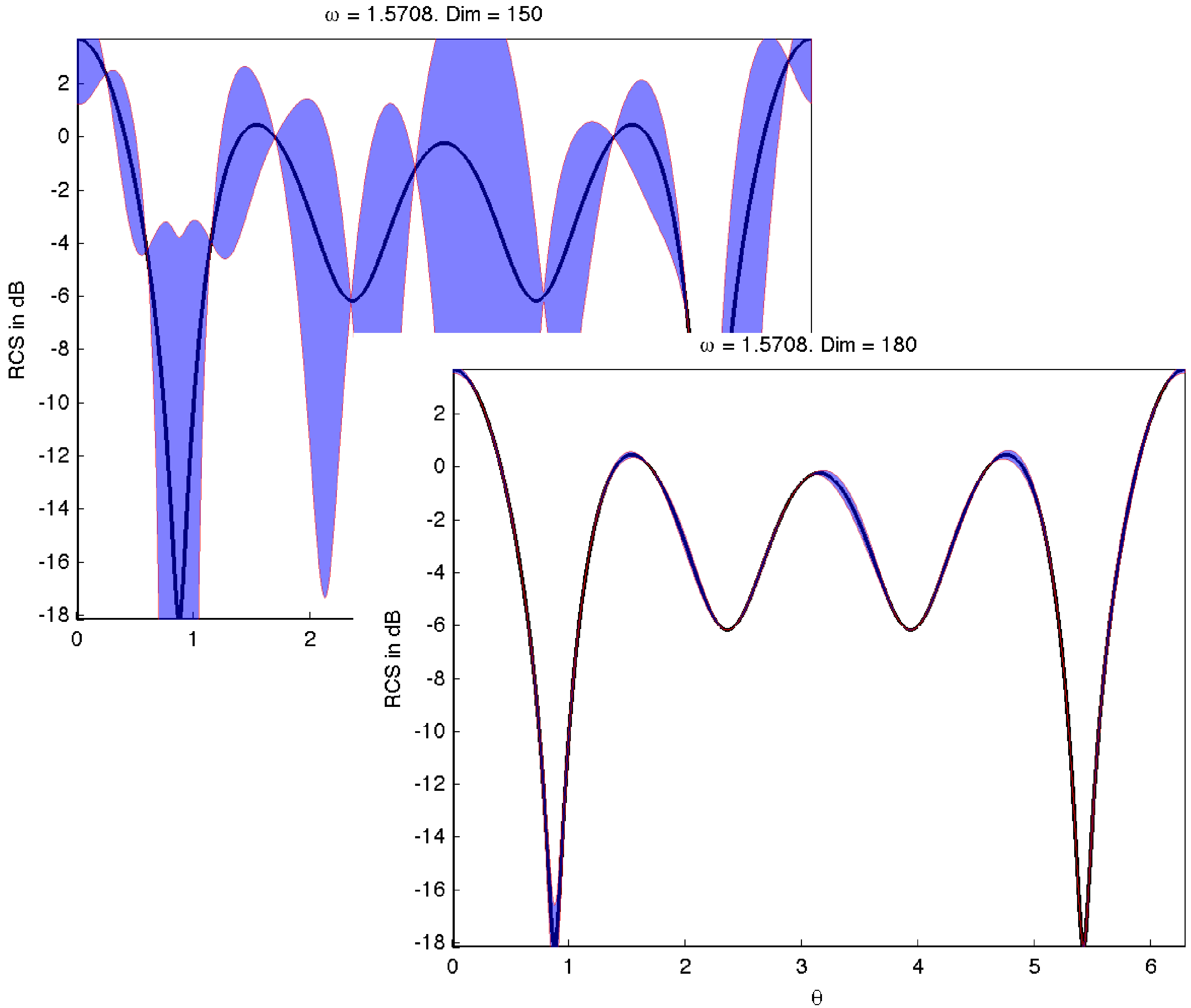


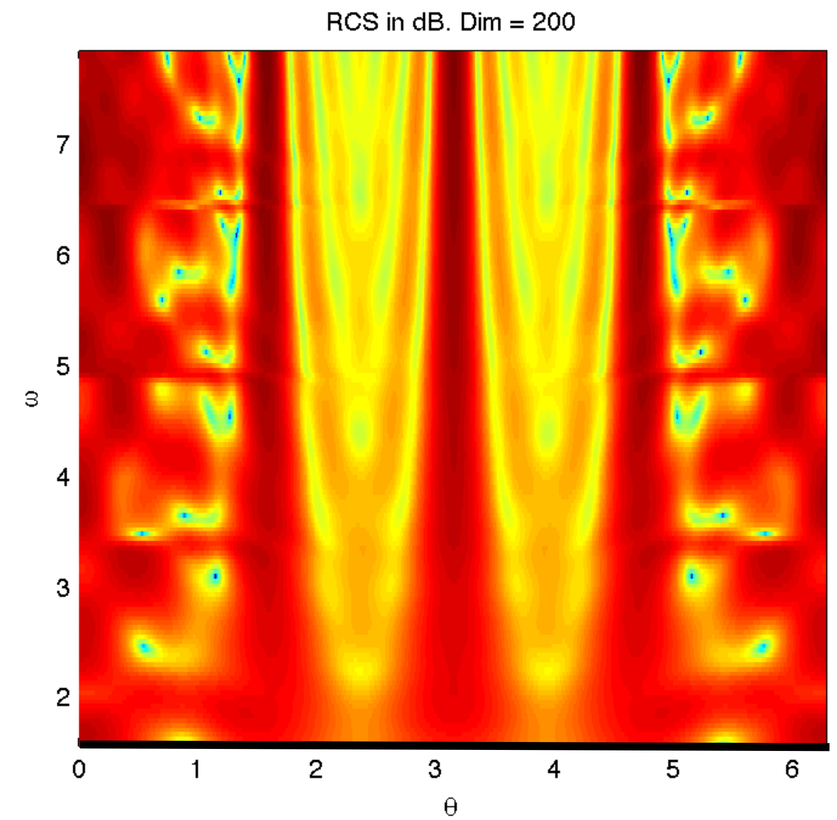
2D Scattering example



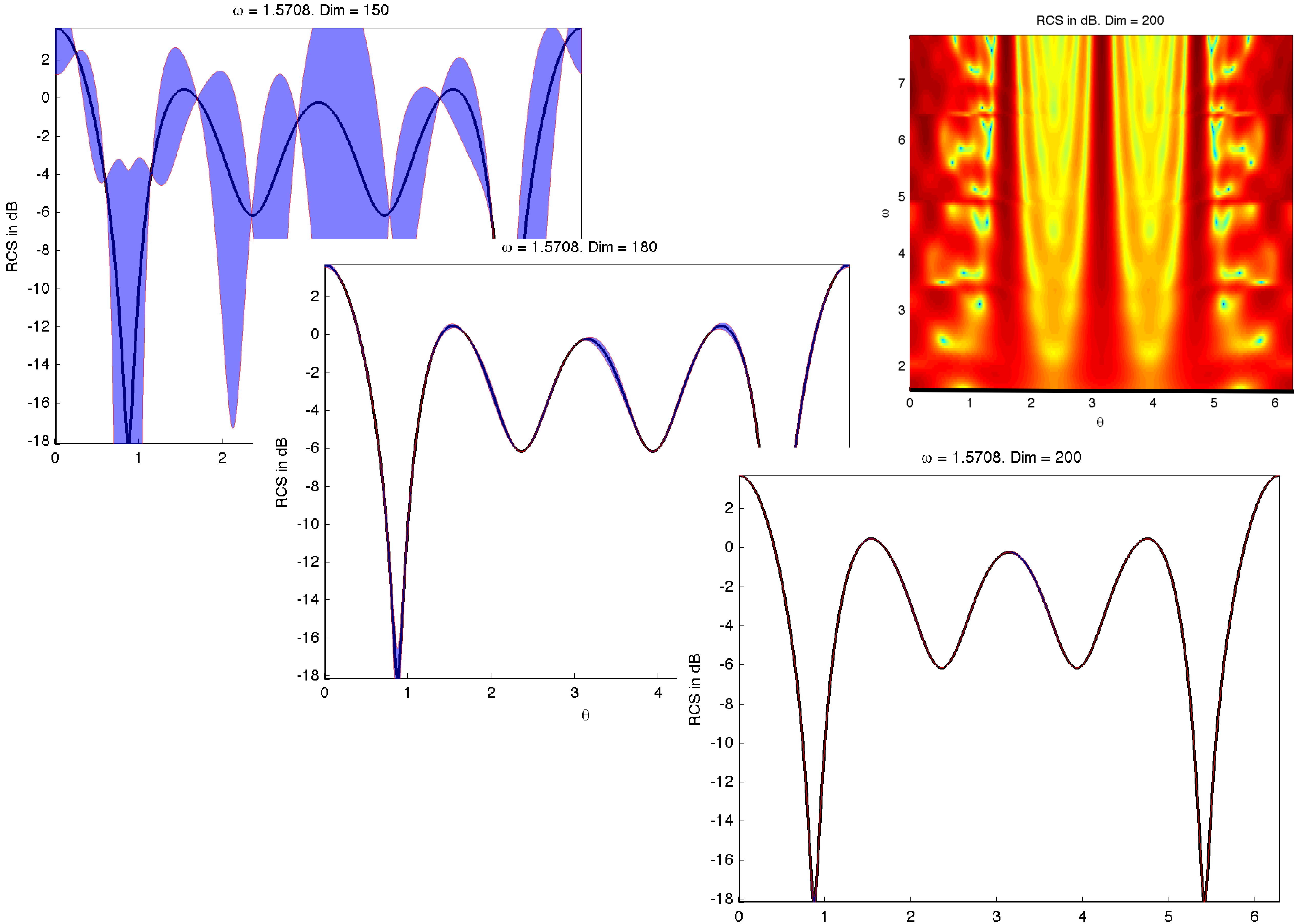


2D Scattering example

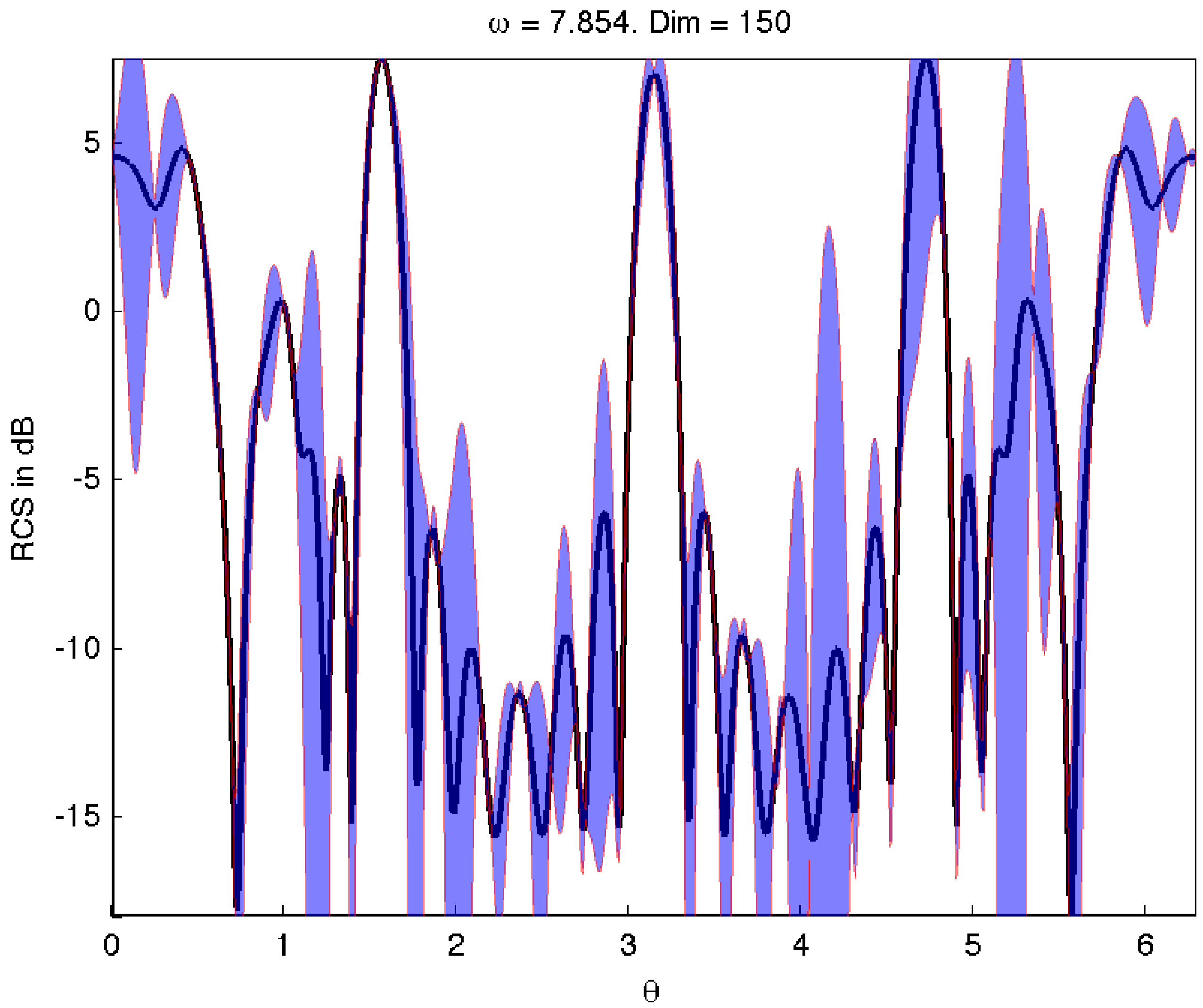


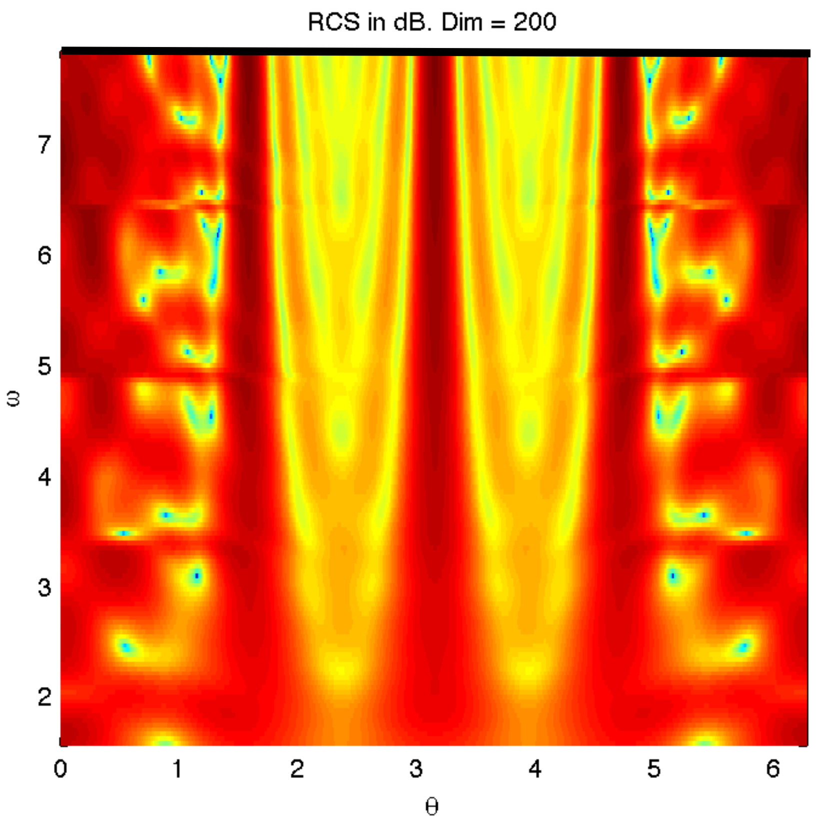


2D Scattering example

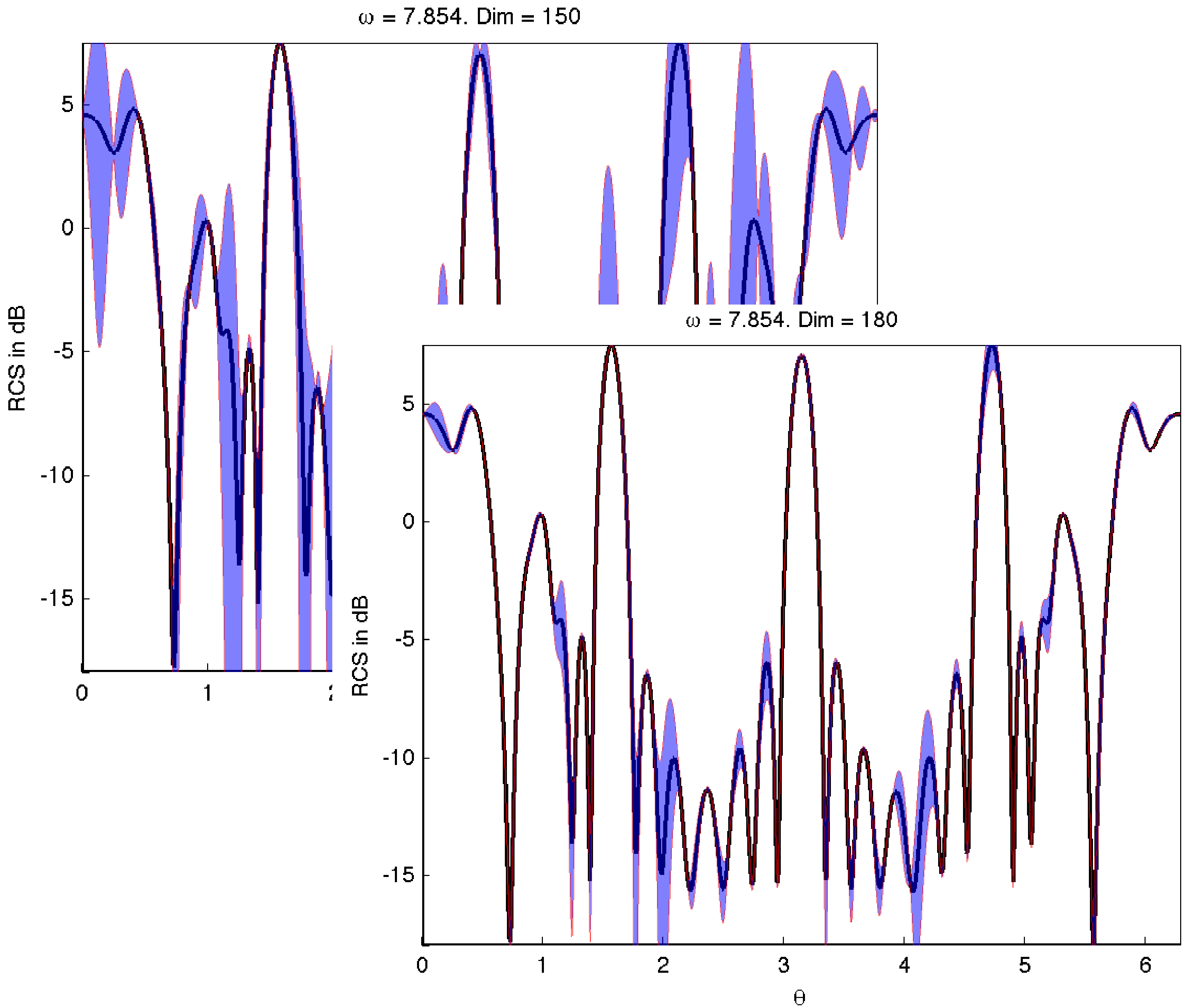


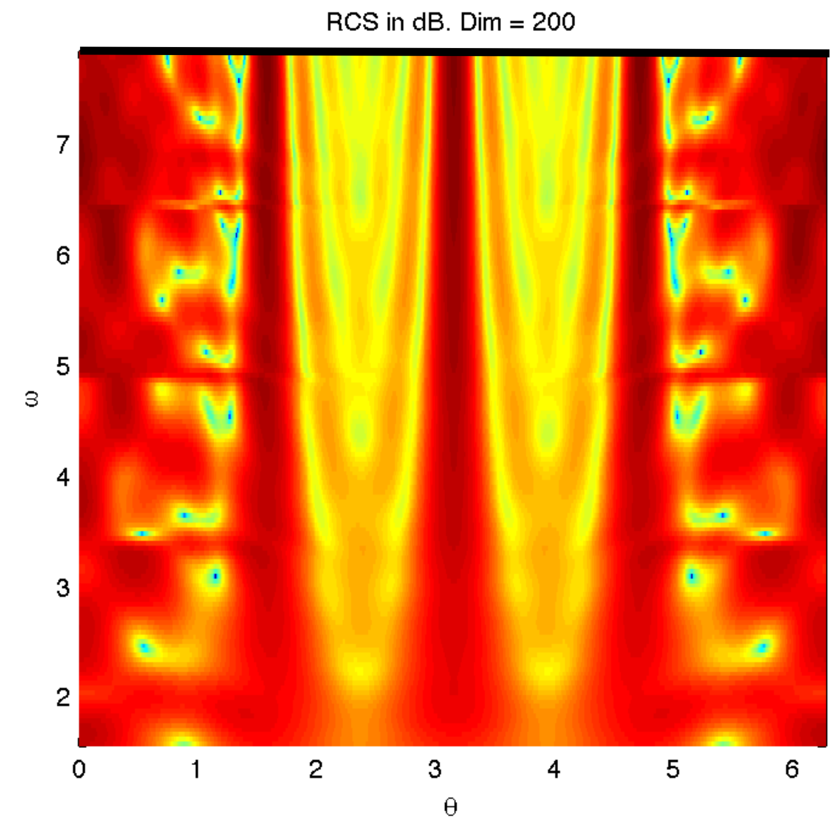
2D Scattering example



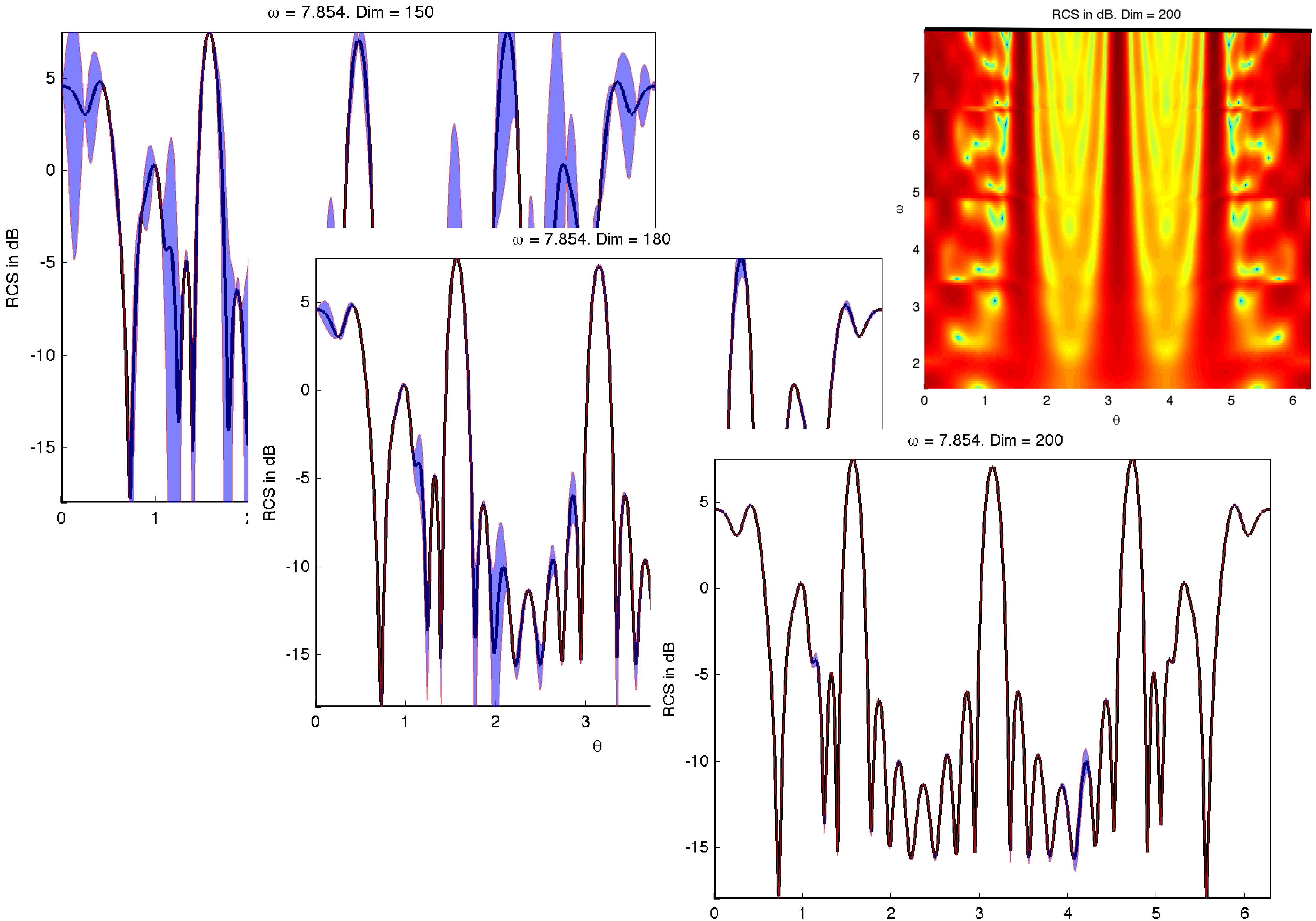


2D Scattering example

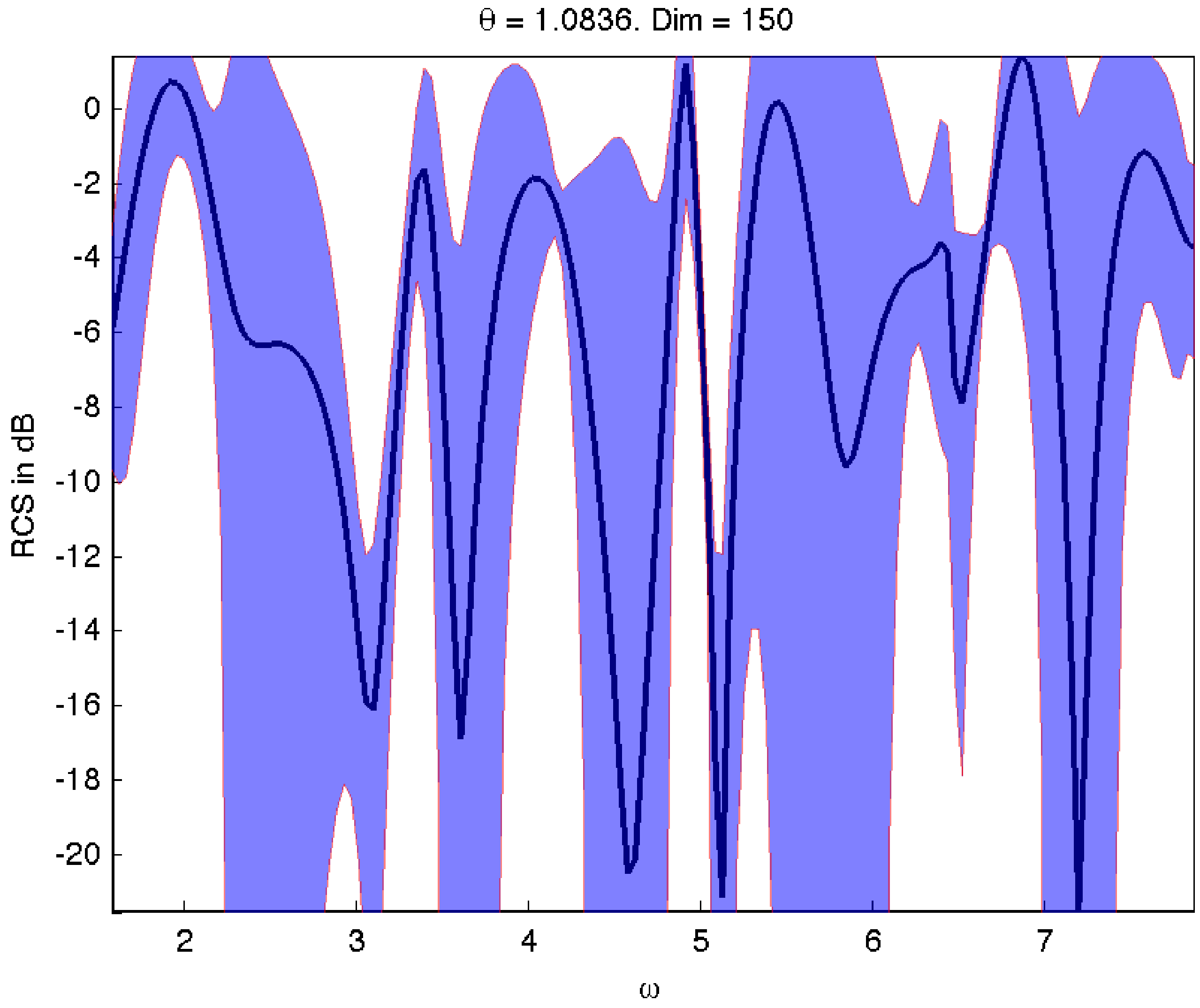


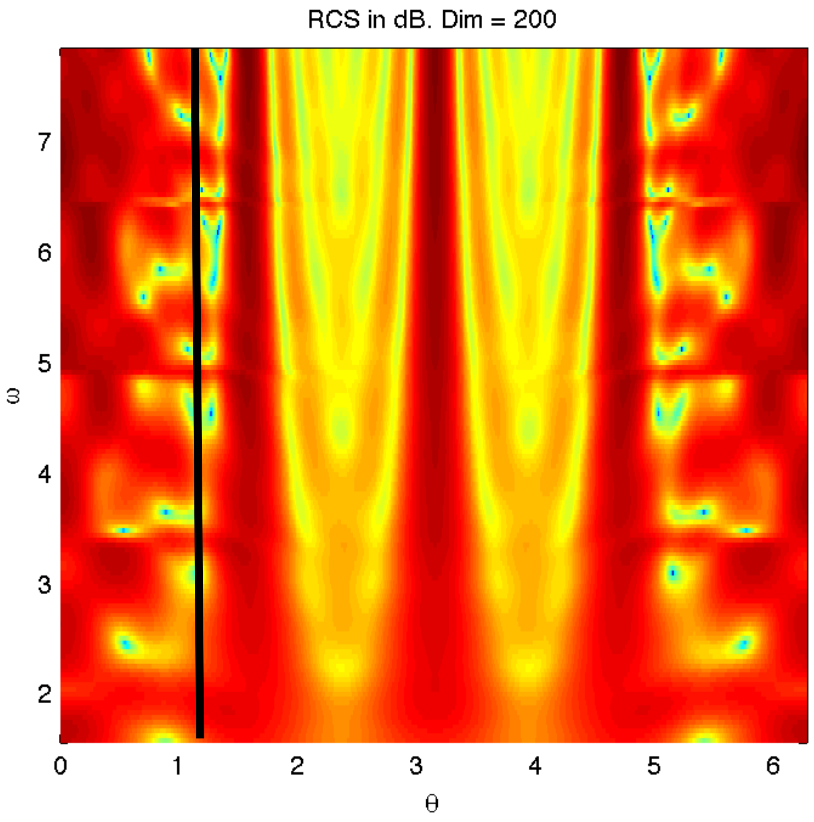


2D Scattering example

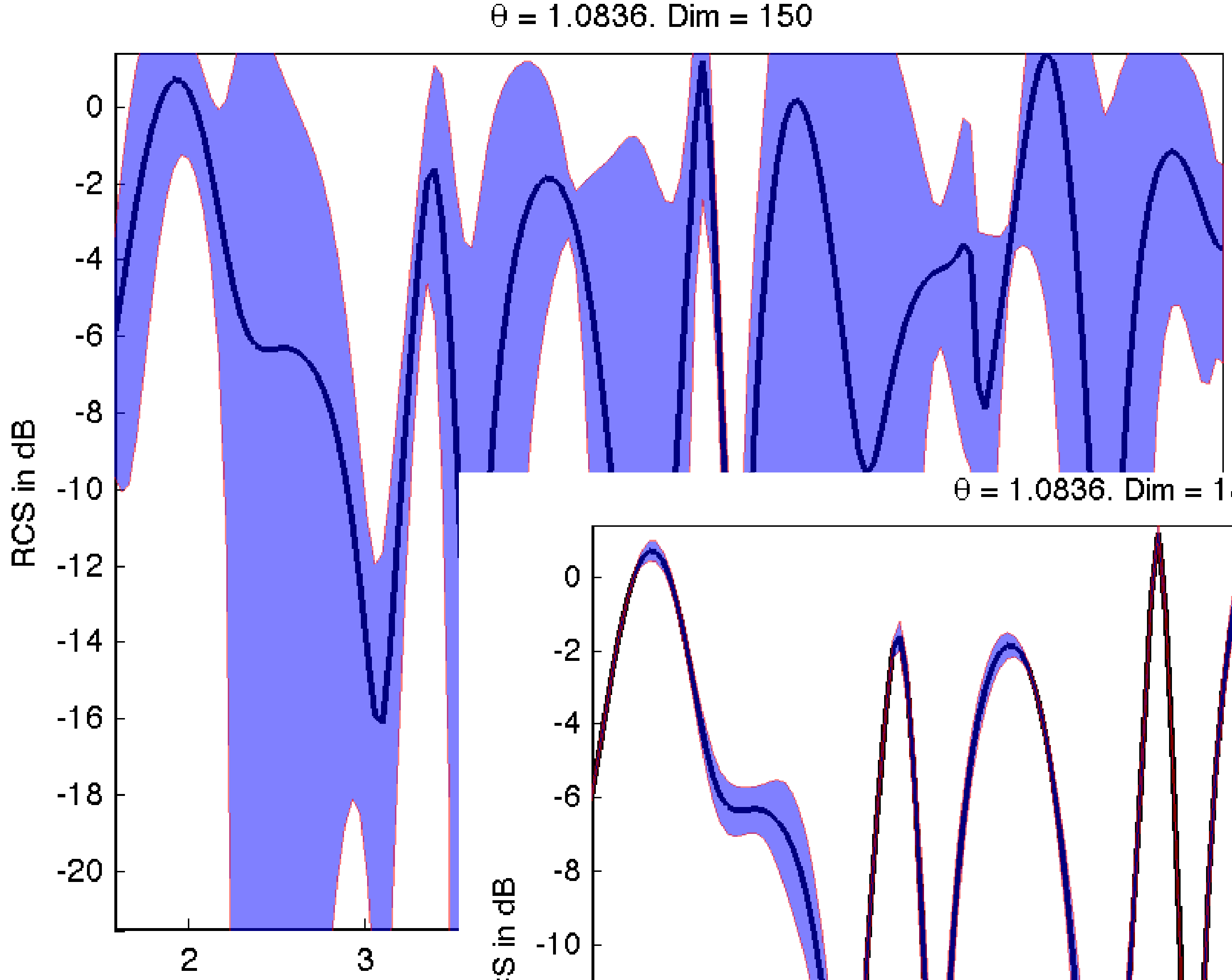


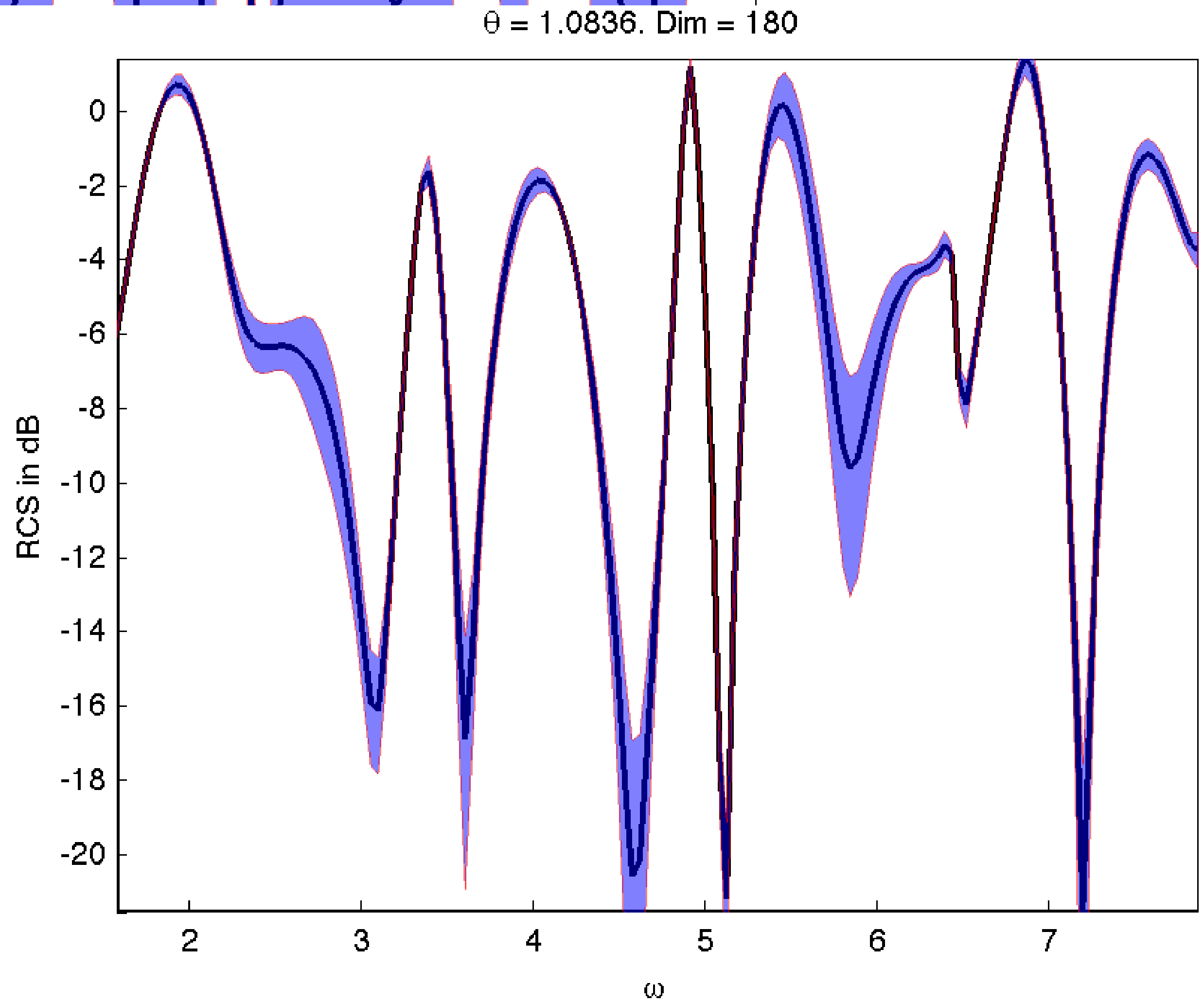
2D Scattering example

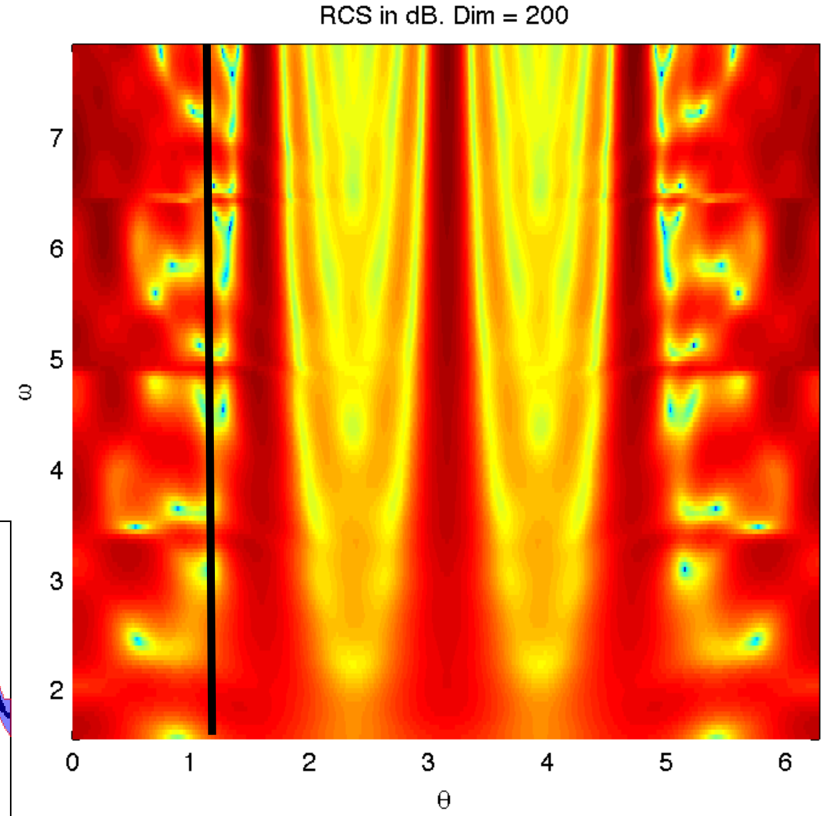




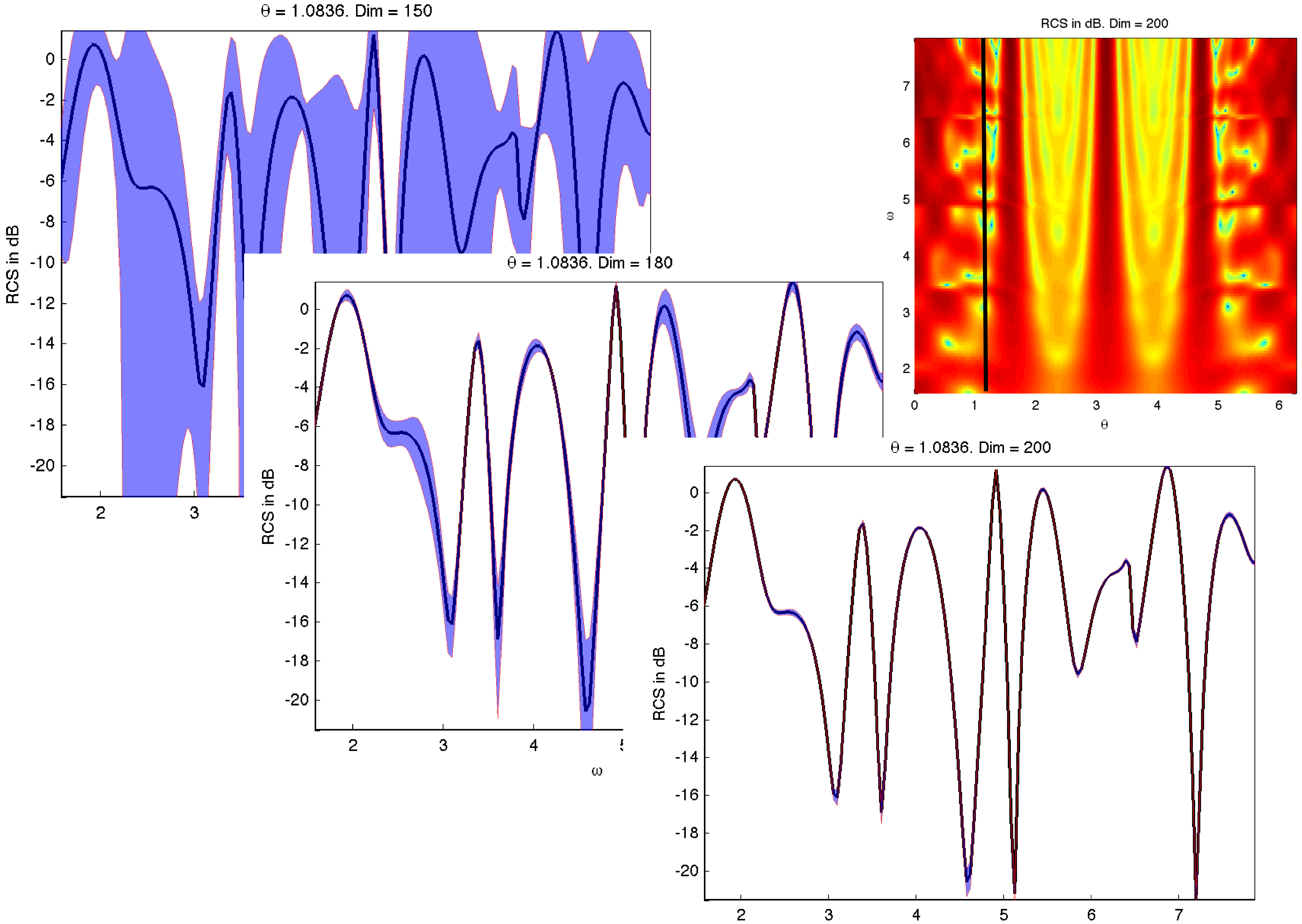
2D Scattering example



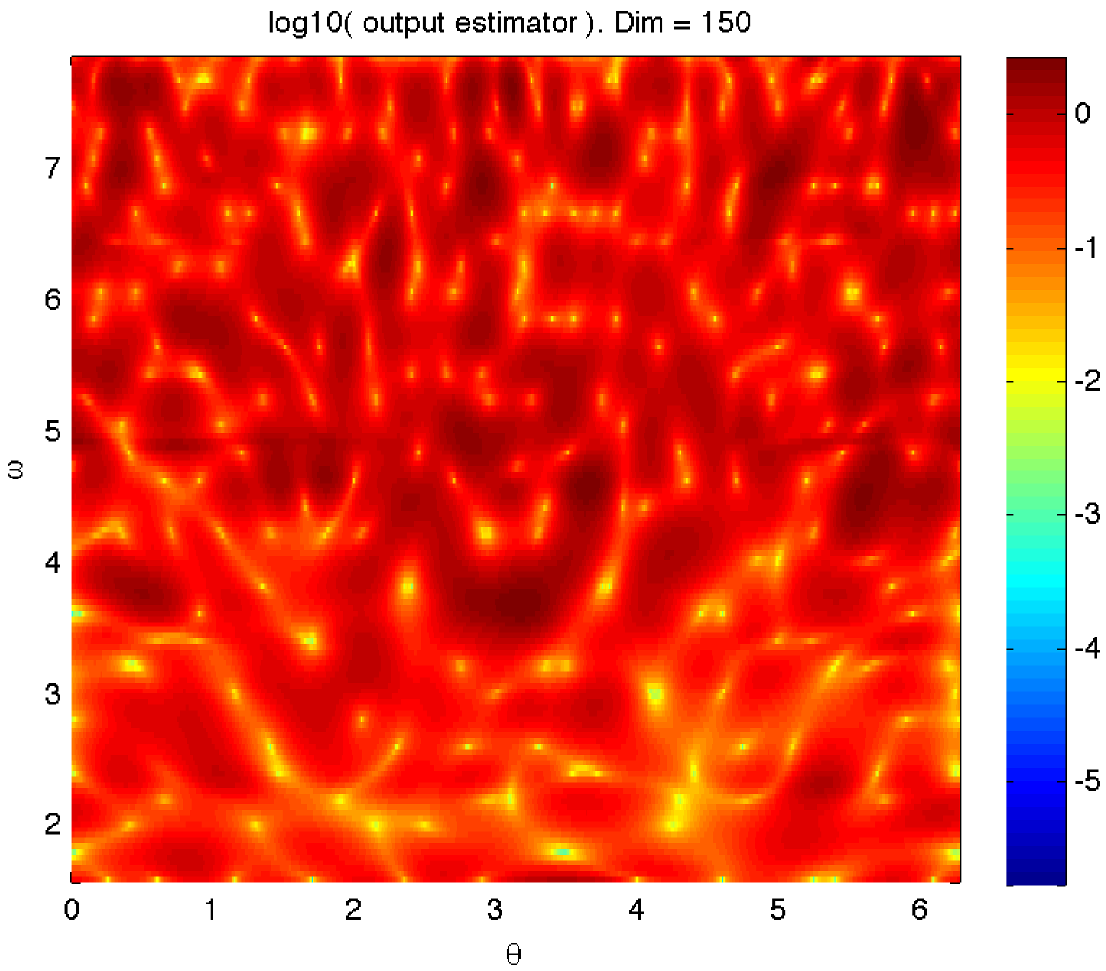




2D Scattering example

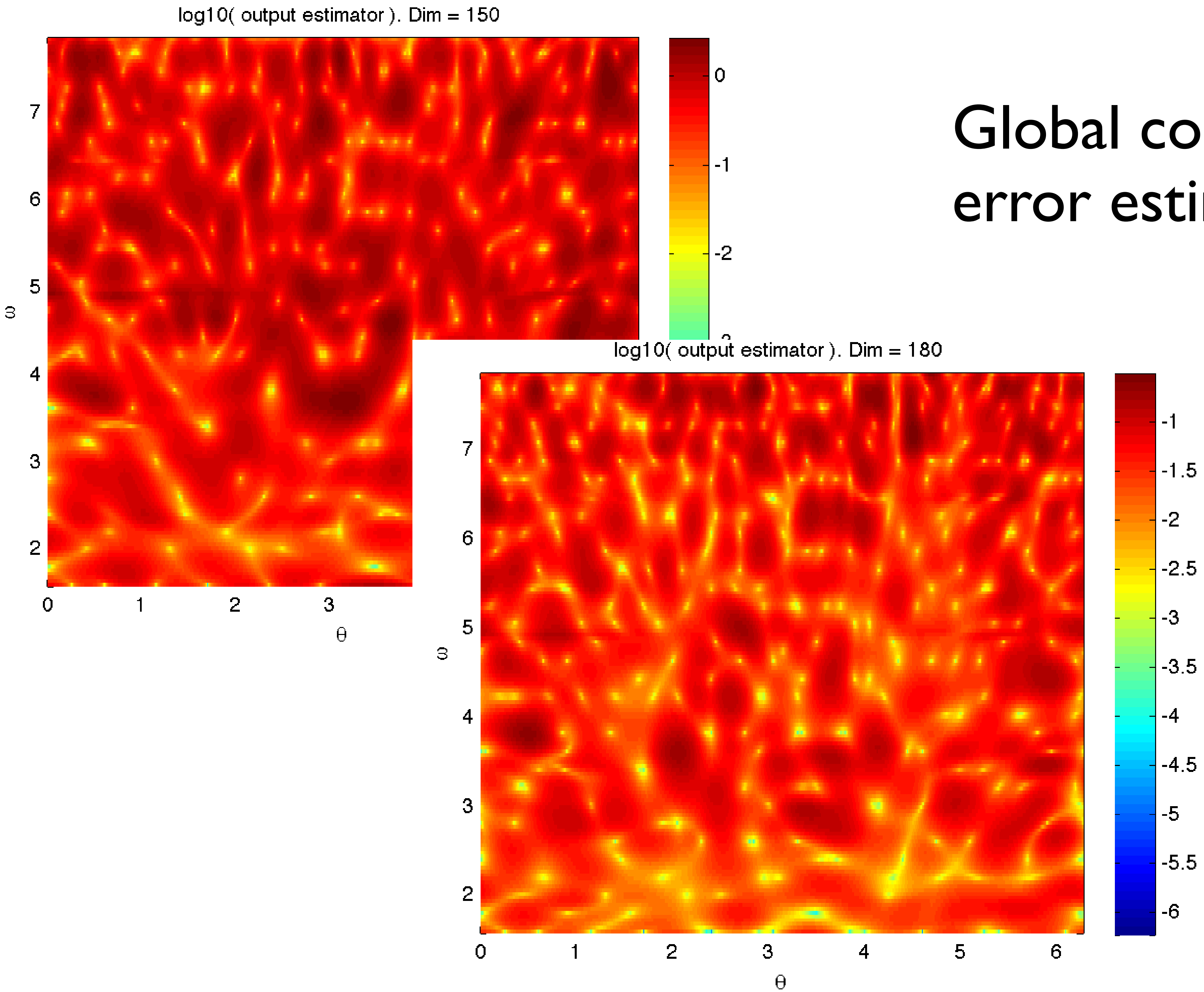


2D Scattering example



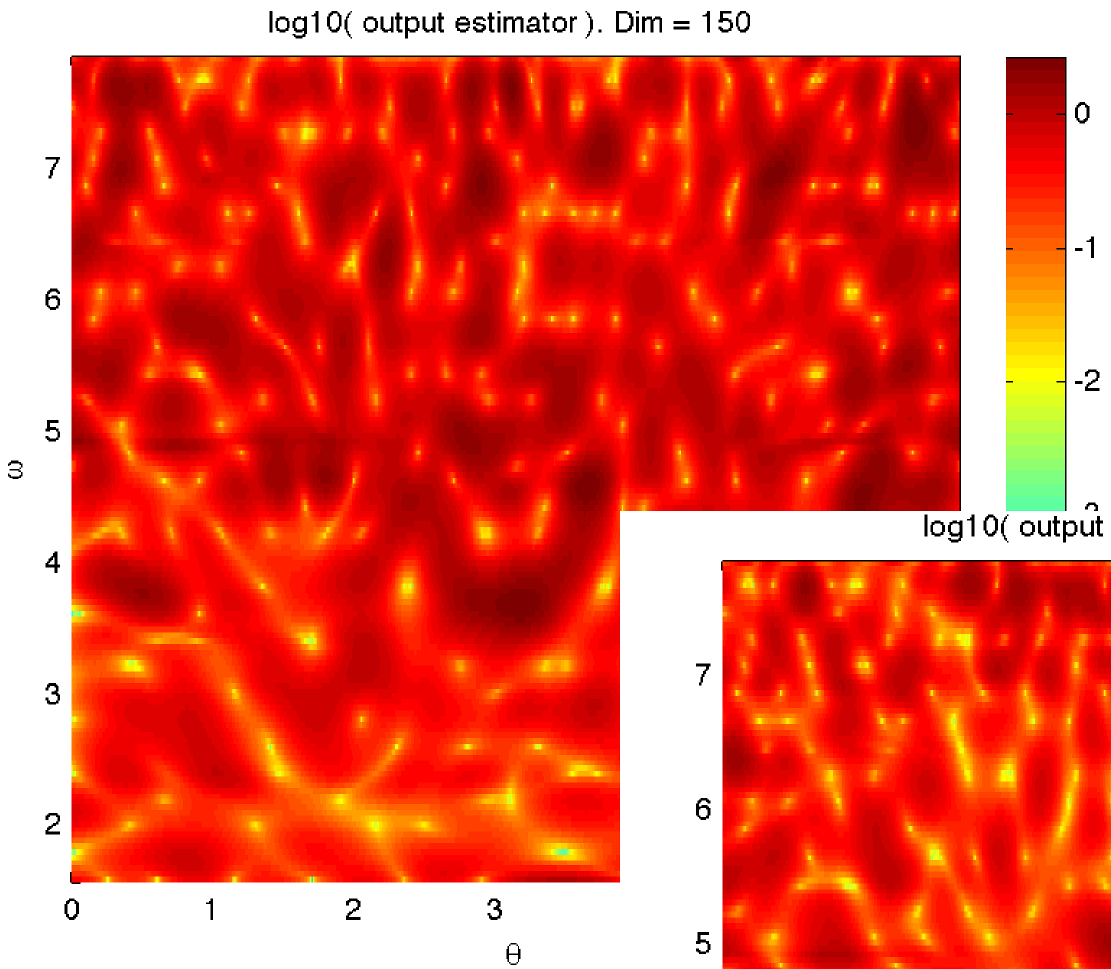
Global convergence of error estimator

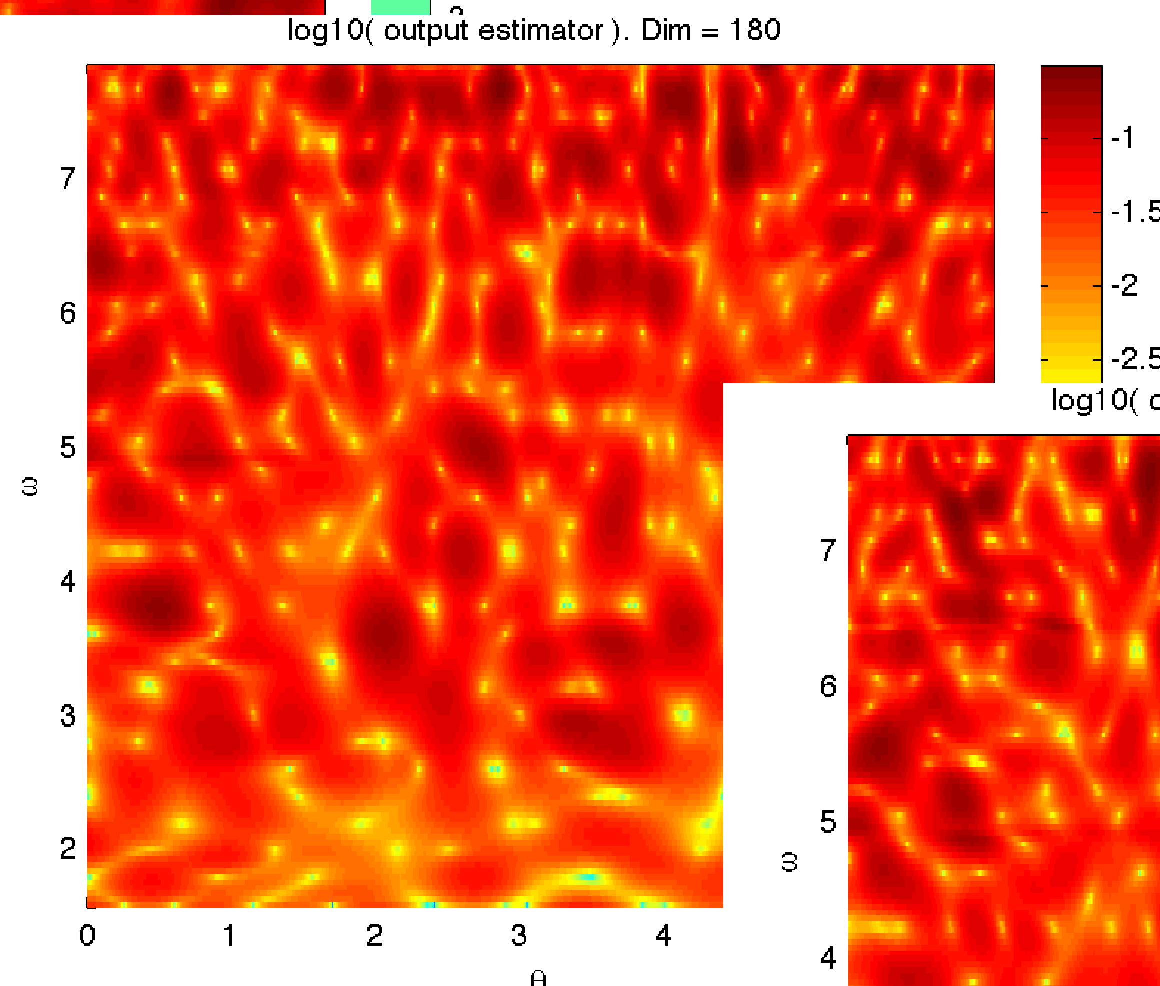
2D Scattering example



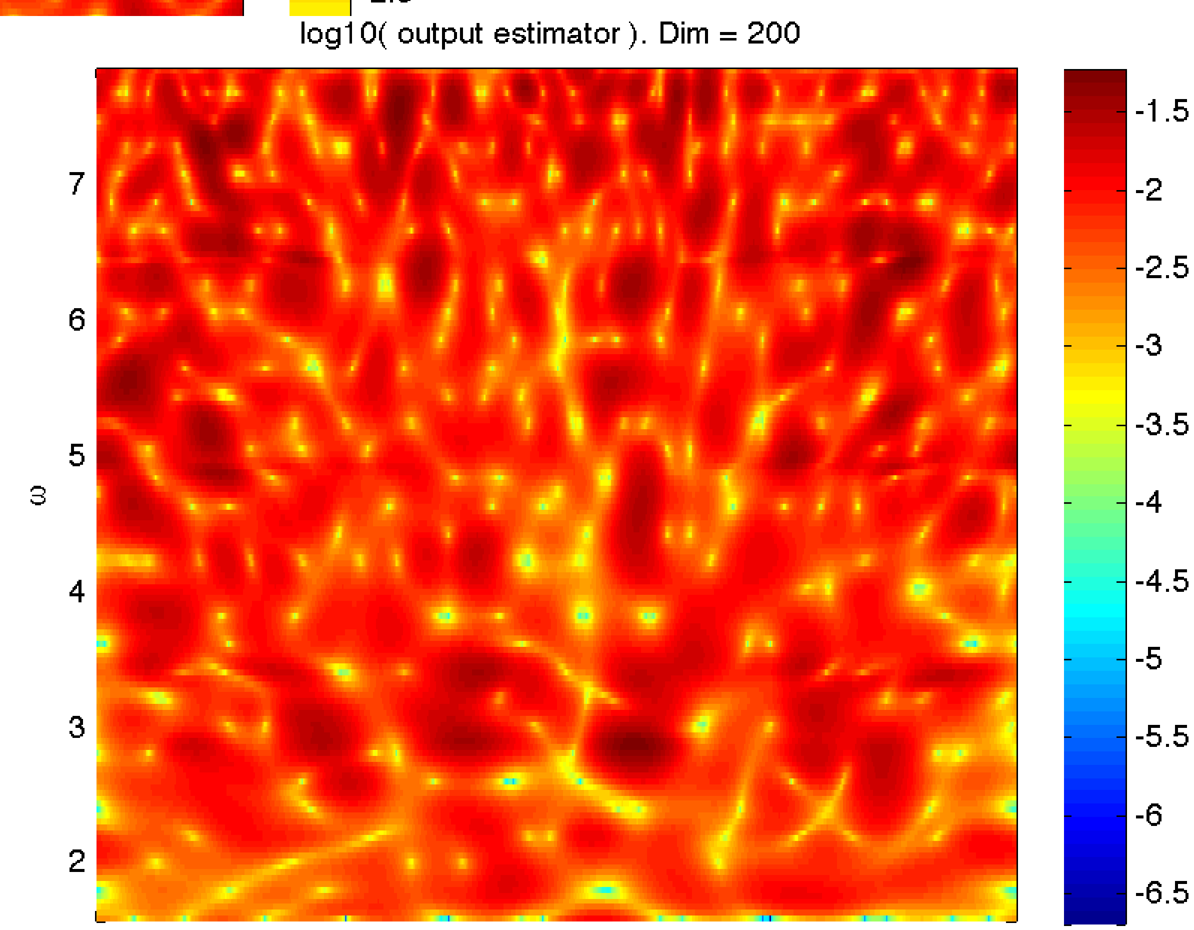
Global convergence of error estimator

2D Scattering example





Global convergence of error estimator



The non-stationary problem

Time-dependent problems

Continuous problem: For any $\mu \in \mathbb{P}$, find for any $t \in [0,T]$ the function $u(\cdot,t;\mu) \in \mathbb{V}$ such that

$$
\begin{aligned}
\frac{d}{dt}(u(\cdot,t;\mu),v)_{\mathbb{V}} + a(u(\cdot,t;\mu),v;\mu) &= f(v,t;\mu), && \forall v \in \mathbb{V}, \\
u(x,0;\mu) &= u_0(x), && \forall x \in \Omega, \\
u(x,t;\mu) &= g(x,t;\mu), && \forall x \in \partial\Omega.
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Full discretization (forward Euler scheme in time for simplicity): For any $\mu \in \mathbb{P}$, find for any $n = 1,\ldots,N_T$ the function $u_\delta^n(\cdot;\mu) \in \mathbb{V}_\delta$ such that

$$\begin{aligned} \frac{1}{\Delta t}(u_\delta^{n+1}(\mu),v_\delta)_{\mathbb{V}} &= \frac{1}{\Delta t}(u_\delta^n(\mu),v_\delta)_{\mathbb{V}} - a(u_\delta^n(\mu),v_\delta;\mu) + f(v_\delta,t_n;\mu), && \forall v_\delta \in \mathbb{V}_\delta,\\ u_\delta^0(x;\mu) &= u_{\delta,0}(x), && \forall x \in \Omega,\\ u_\delta^n(x;\mu) &= g_\delta(x,t_n;\mu), && \forall x \in \partial\Omega, \end{aligned}$$

with $t_n = n\Delta t$.

Time-dependent problems

Suppose: A reduced basis approximation space $\mathbb{V}_{\texttt{rb}}$ is given (it's construction is discussed later).

RBM approximation: For any $\mu \in \mathbb{P}$, find for any $n = 1, \ldots, N_T$ the function $u_{\texttt{rb}}^n(\mu) \in \mathbb{V}_{\texttt{rb}}$ such that

$$
\begin{aligned}
\frac{1}{\Delta t}(u_{\texttt{rb}}^{n+1}(\mu), v_{\texttt{rb}})_{\mathbb{V}} &= \frac{1}{\Delta t}(u_{\texttt{rb}}^{n}(\mu), v_{\texttt{rb}})_{\mathbb{V}} - a(u_{\texttt{rb}}^{n}(\mu), v_{\texttt{rb}}; \mu) + f(v_{\texttt{rb}}, t_n; \mu), && \forall v_{\texttt{rb}} \in \mathbb{V}_{\texttt{rb}},\\
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\end{aligned}
$$

Again: We are mimicking the truth solver but are restricting the solution space from $\mathbb{V}_\delta$ to $\mathbb{V}_{\texttt{rb}}$.

Time-dependent problems

Remaining question: How to construct the reduced basis space $\mathbb{V}_{\mathtt{rb}}$?

POD/Greedy algorithm:

Set $N = 1$, choose $\mu_1 \in \mathbb{P}$ arbitrarily.

1. Compute the time series $u_\delta^n(\mu_N)$ for all $n = 1, \ldots, N_T$ (truth problem: computationally expensive)
2. Define the error trajectory $e_{\mathtt{rb}}^n(\mu) = u_\delta^n(\mu_N) - u_{\mathtt{rb}}^n(\mu_N)$
3. Compute a POD of the error trajectory $e_{\mathtt{rb}}^n(\mu)$ and retain the most important mode ξ_1.
4. Set $\mathbb{V}_{\mathtt{rb}} = \operatorname{span}\{\mathbb{V}_{\mathtt{rb}}, \xi_1\}$
5. Find $\mu_{N+1} = \arg\max_{\mu \in \mathbb{P}} \eta(\mu)$
6. Set $N := N + 1$ and `goto 1.` while $\max_{\mu \in \mathbb{P}} \eta(\mu) > \mathtt{Tol}$

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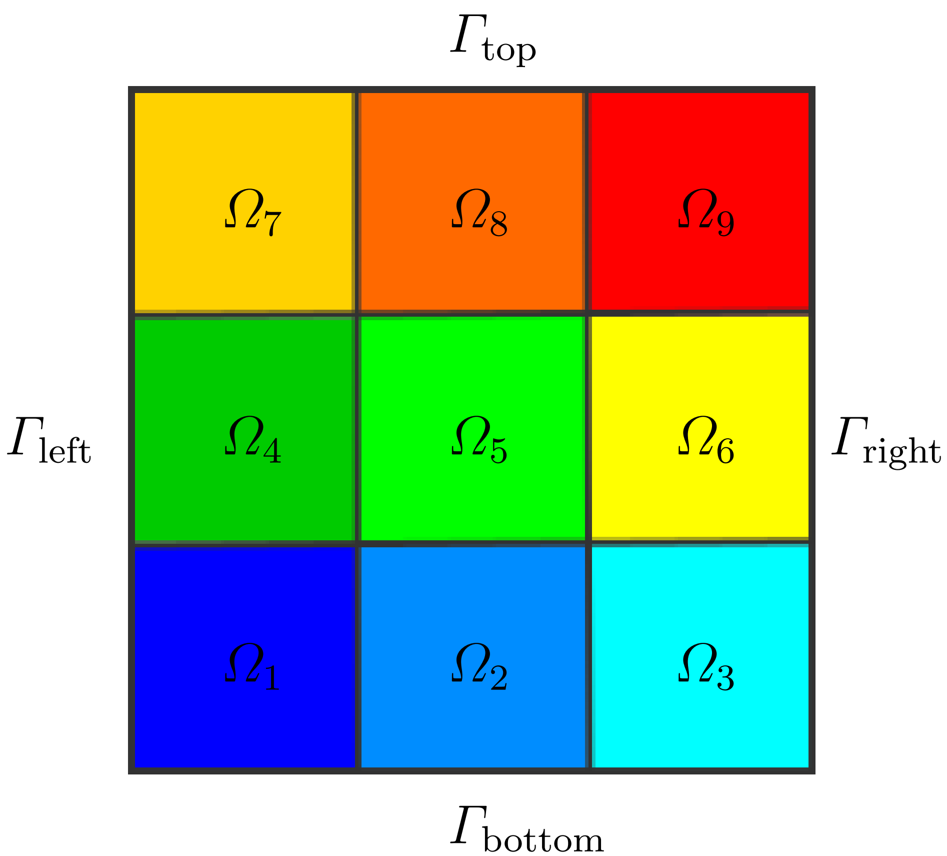
A posteriori estimator: $\eta(\mu) \approx \|u_\delta(\mu) - u_{\texttt{rb}}(\mu)\|_{[0,T]\times\Omega}$:

- Needs to be developed for each type of scheme/equation.
- Sharp estimate is important for good parameter selection in greedy algorithm.
- Is used to certify the error tolerance.

Time-dependent problems

We consider time-dependent heat problem



$$a(w,v;\mu) = \sum_{p=1}^{8} \mu_{[p]} \int_{\Omega_p} \nabla w \cdot \nabla v + \int_{\Omega_9} \nabla w \cdot \nabla v,$$

$$\mu_{[p]} \in [0.1, 10] \qquad \text{for } p = 1, \ldots, 8.$$

$$f(v;\mu) = \mu_{[9]} \int_{\Gamma_{\text{bottom}}} v.$$

$$\mu_{[9]} \in [-1, 1]\,.$$

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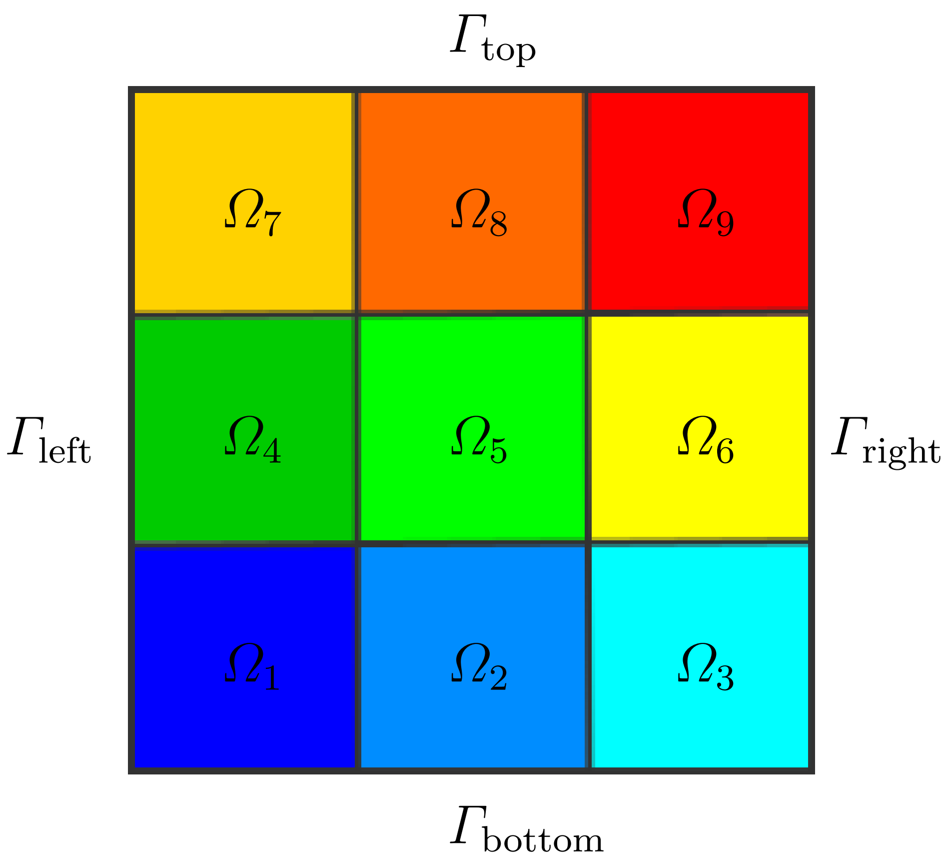

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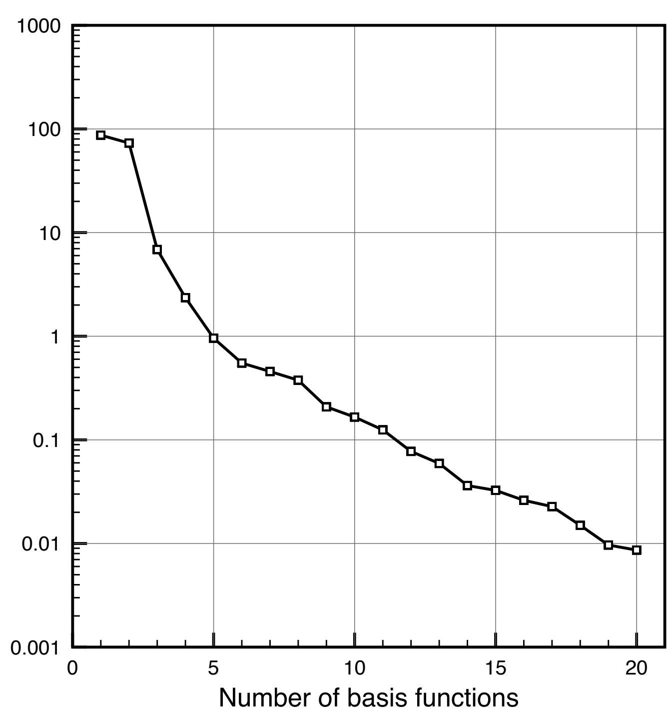
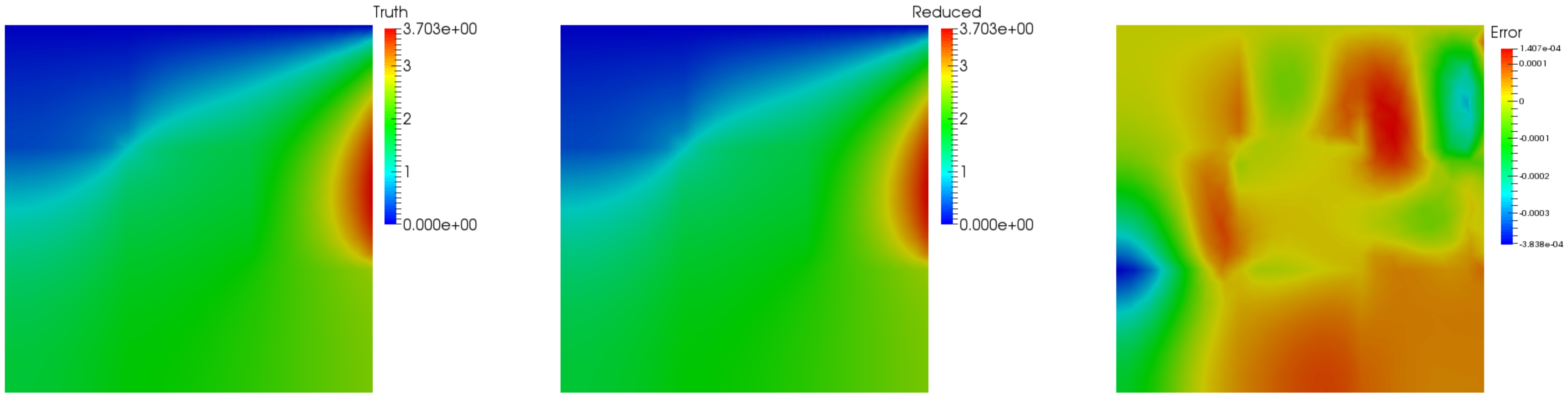


N	$\eta_{\texttt{en,av}}$	$\texttt{eff}_{\texttt{en,max}}$	$\texttt{eff}_{\texttt{en,av}}$
5	0.18	24.67	7.51
10	0.07	26.27	7.69
15	0.03	25.82	6.79
20	0.02	31.63	9.53

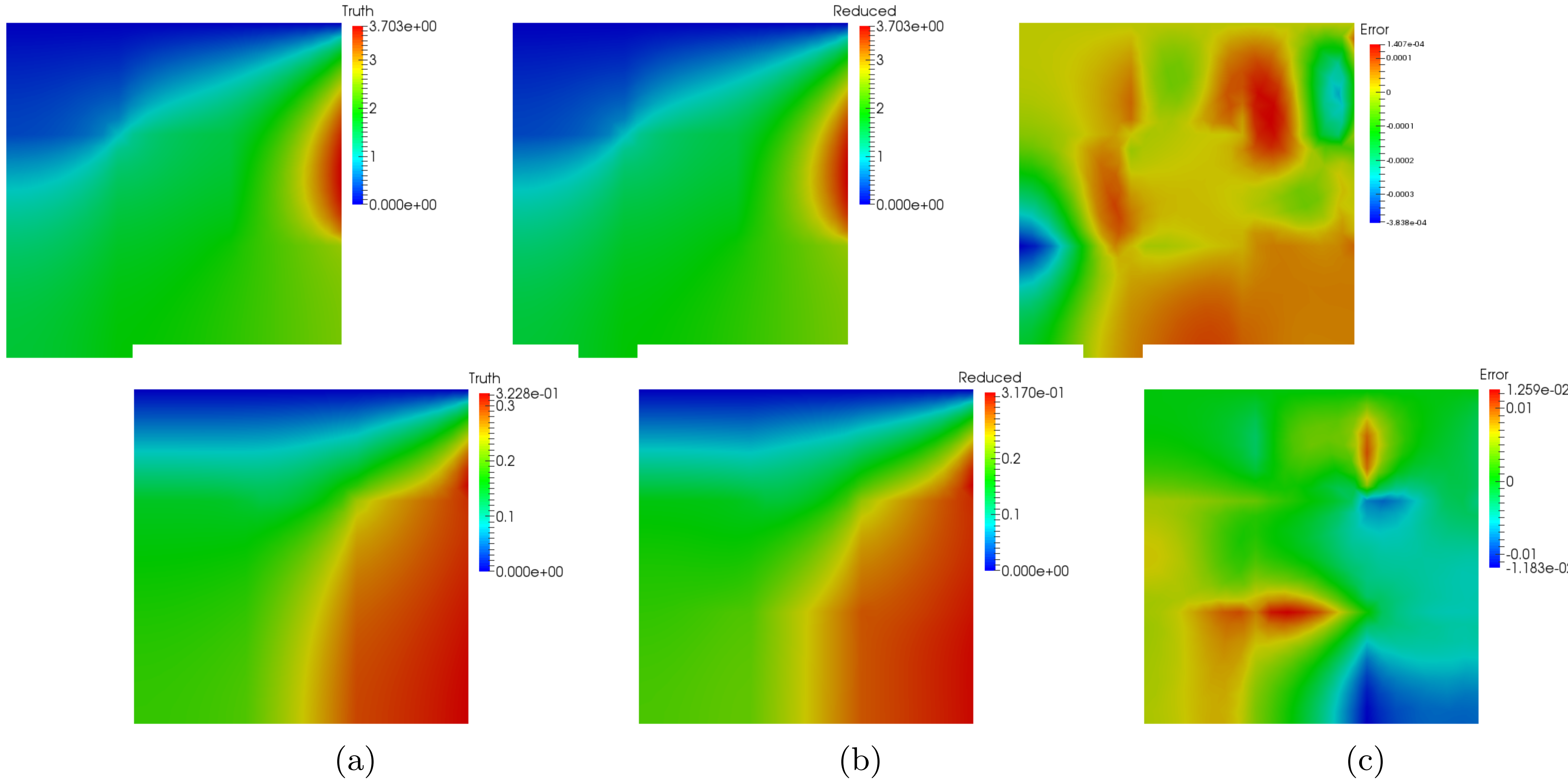
Time-dependent problems



(a) (b) (c)

At T=3

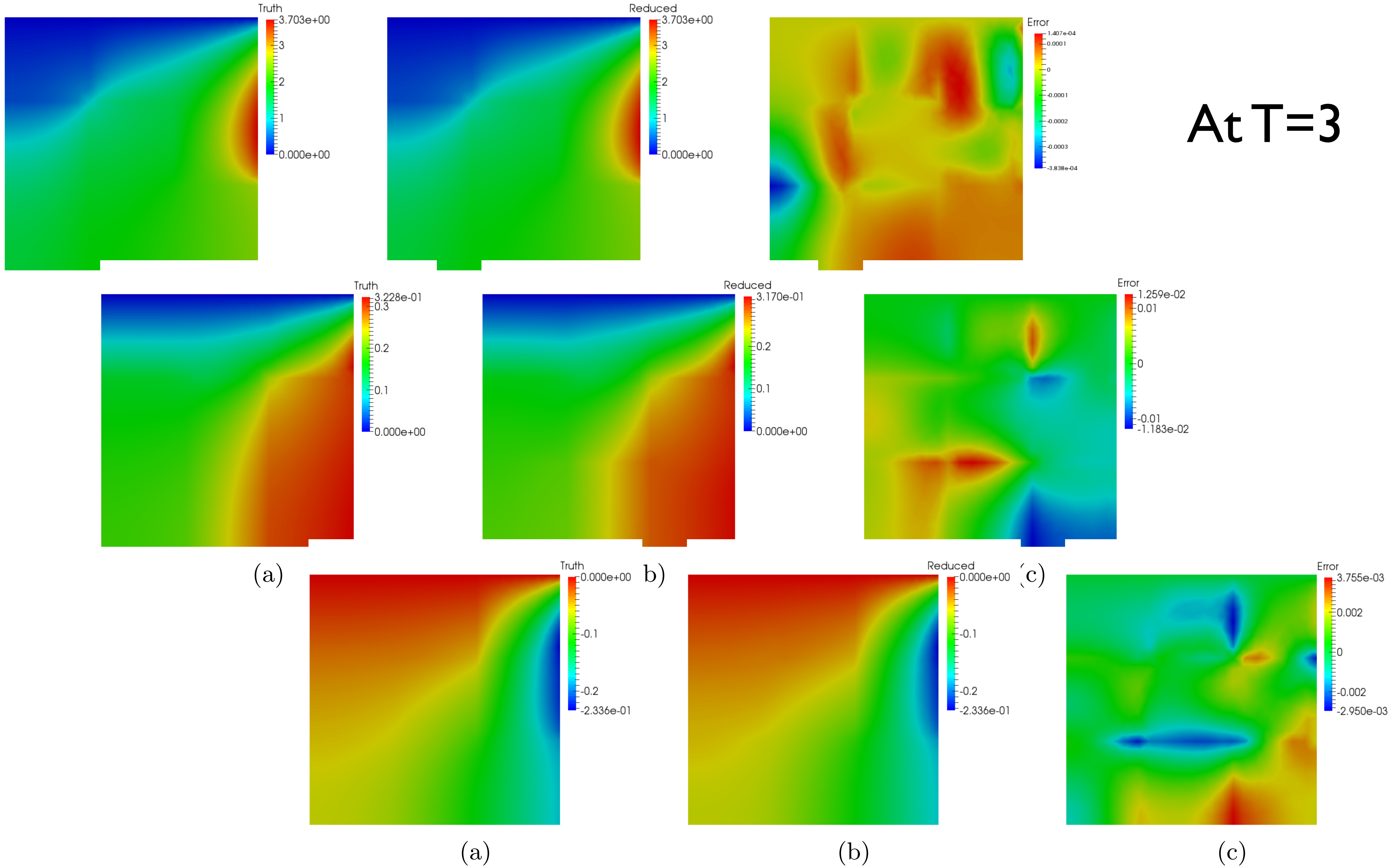
Time-dependent problems



At T=3

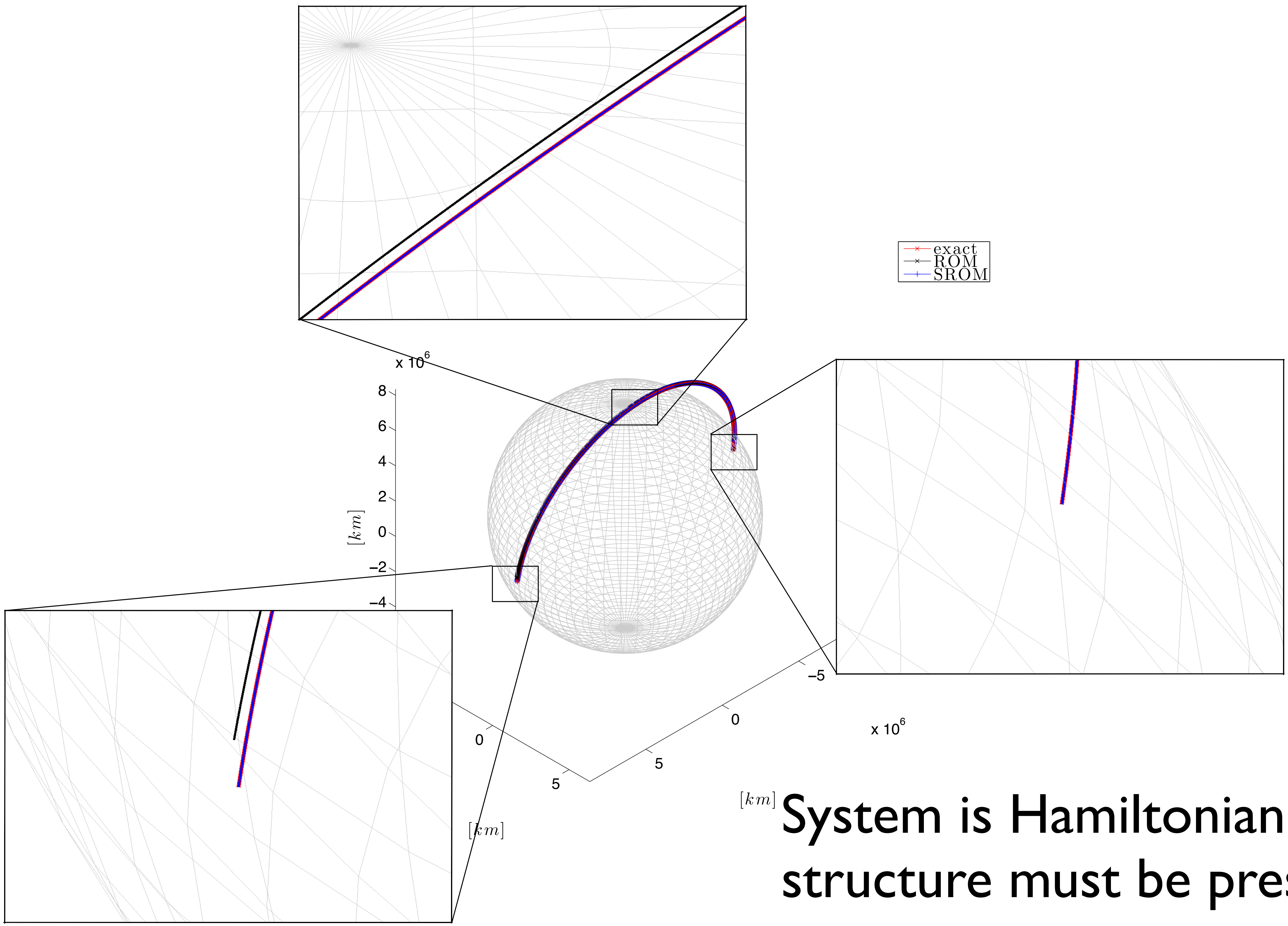
Time-dependent problems

At T=3



Caution is needed

Satellite modeling by reduced basis — careful



System is Hamiltonian - structure must be preserved.

Hamiltonian reduced model

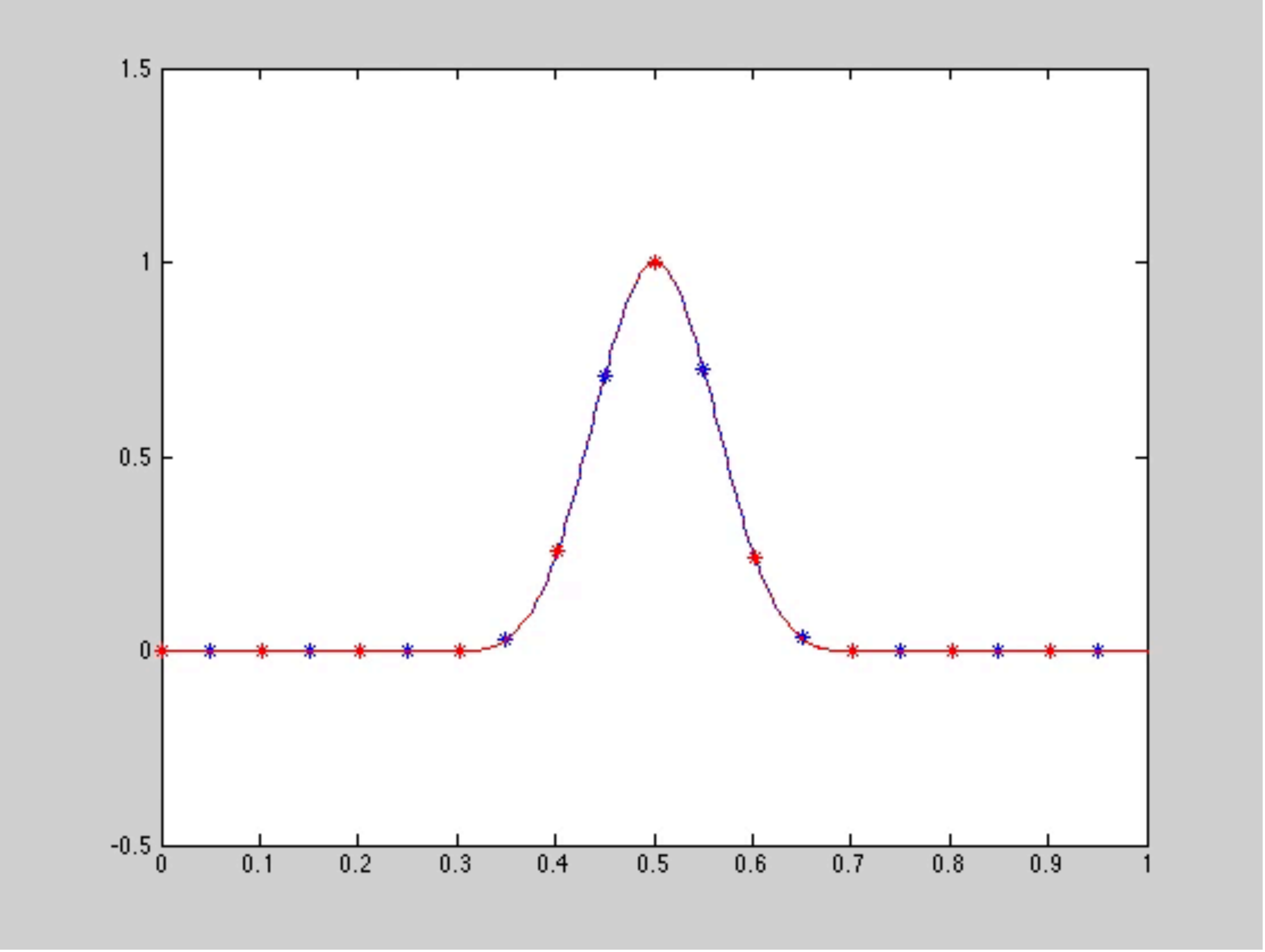
Wave equation:

$$\begin{cases} \dot{q} = p \\ \dot{p} = c^2 q_{xx} \end{cases}$$

Hamiltonian:

$$H(q,p) = \int \left(\frac{1}{2} p^2 + \frac{1}{2} c^2 q_x^2 \right) \, dx$$

- size of original system : 1000
- size of reduced system : 30
- $\Delta H = 5 \times 10^{-4}$.
- $||y - y_r||_{L_2} = 5 \times 10^{-5}$



Stability by construction

Hamiltonian reduced model

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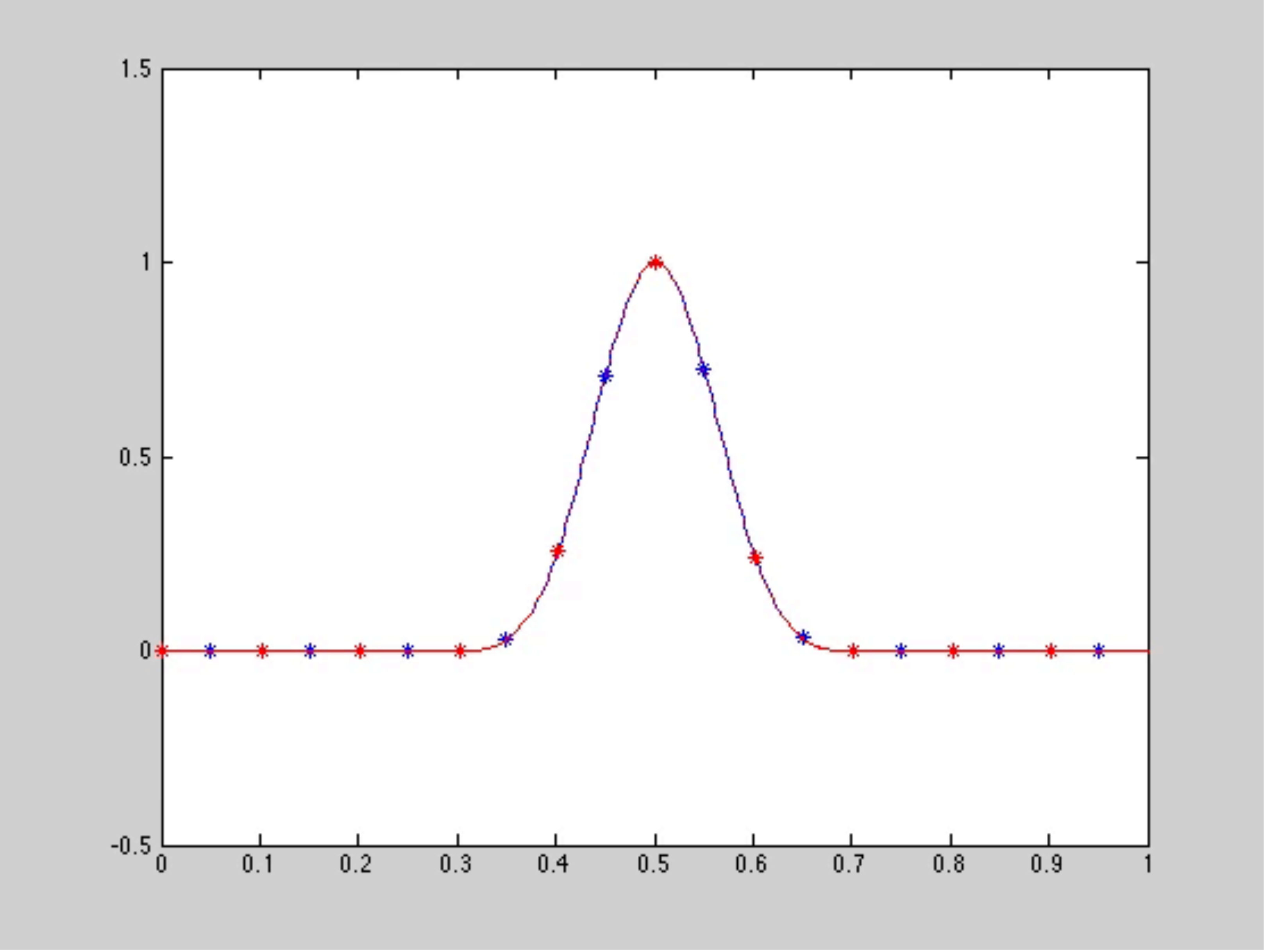

Stability by construction

The non-intrusive problem

Non-intrusive approach

Challenge: All approaches so far requires that we have access to the full solver and all operators

For many problems and solvers this is problematic

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Non-intrusive approach

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For many problems and solvers this is problematic

Question: Can we build reduced models without having access to the solver, i.e., all we have are snapshots ?

Answer: Yes - but… !

Non-intrusive approach

Simple approach: Solve problem at grid in parameter space and interpolate in parameter space

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Problem: High cost

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New Problem: Interpolation on arbitrary grid in parameter space

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Problem: High cost

Solution: Greedy approach based on accuracy of interpolation

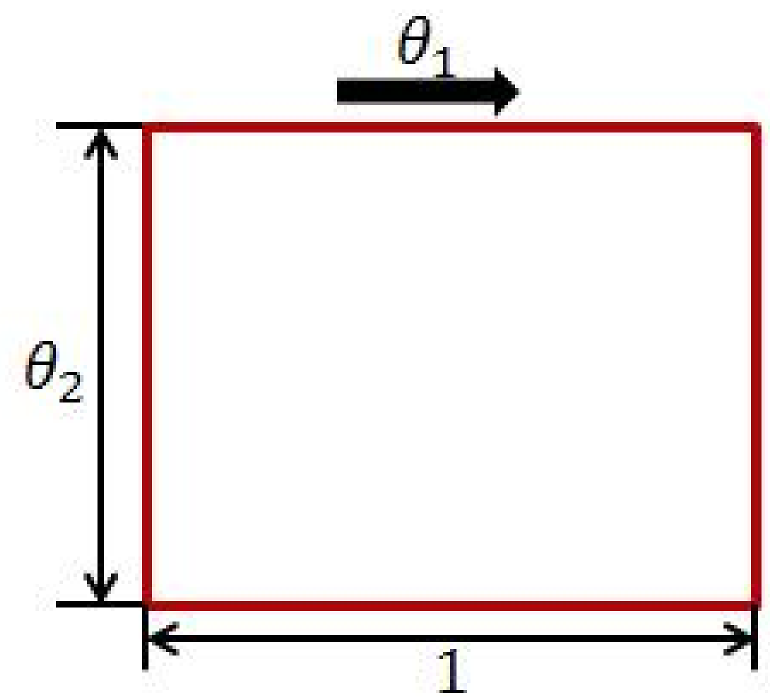
New Problem: Interpolation on arbitrary grid in parameter space

Improved solution: Interpolation based on radial basis functions

$$f(x,\mu) = \sum_{i=1}^{N} f(x,\mu_i)\phi_i(\mu) \qquad \phi_i(\mu) = \phi(\|\mu - \mu_i\|)$$

Example - Driven Cavity Flow

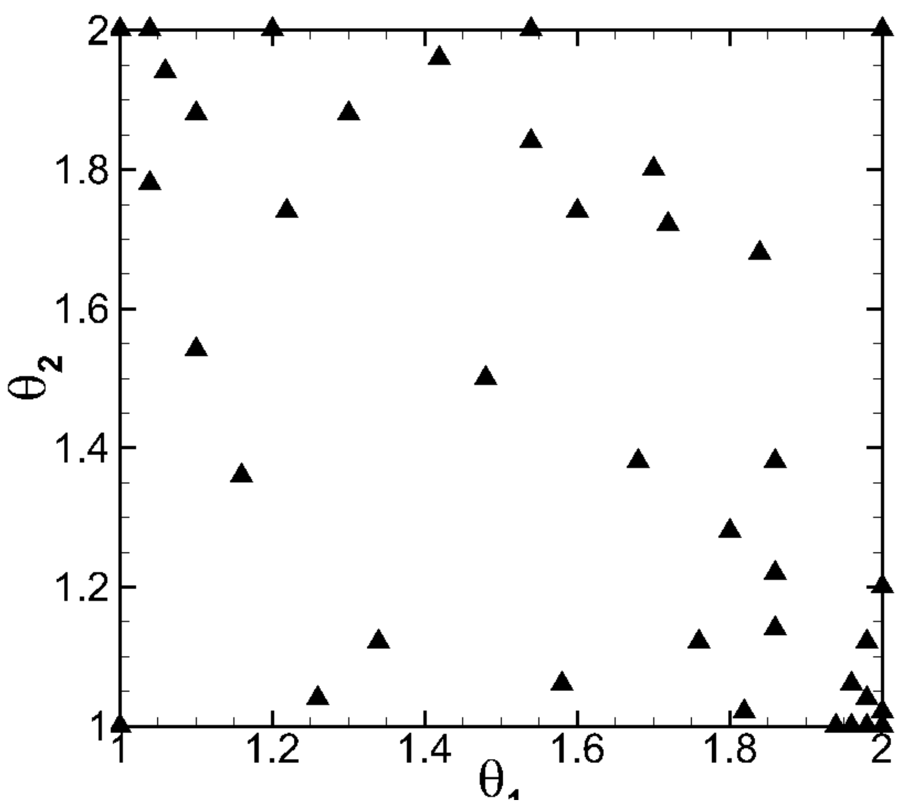
We consider the Navier-Stokes equations in a driven cavity



$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \nabla^2 u = -\nabla(\frac{p}{\rho_0}) + g$$

$$\nabla \cdot u = 0$$

First 34 samples in parameter space



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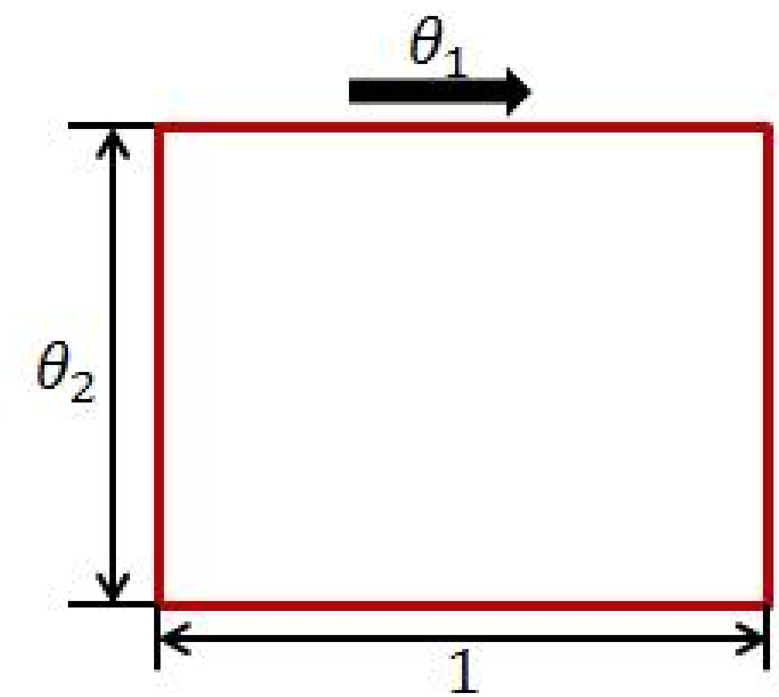

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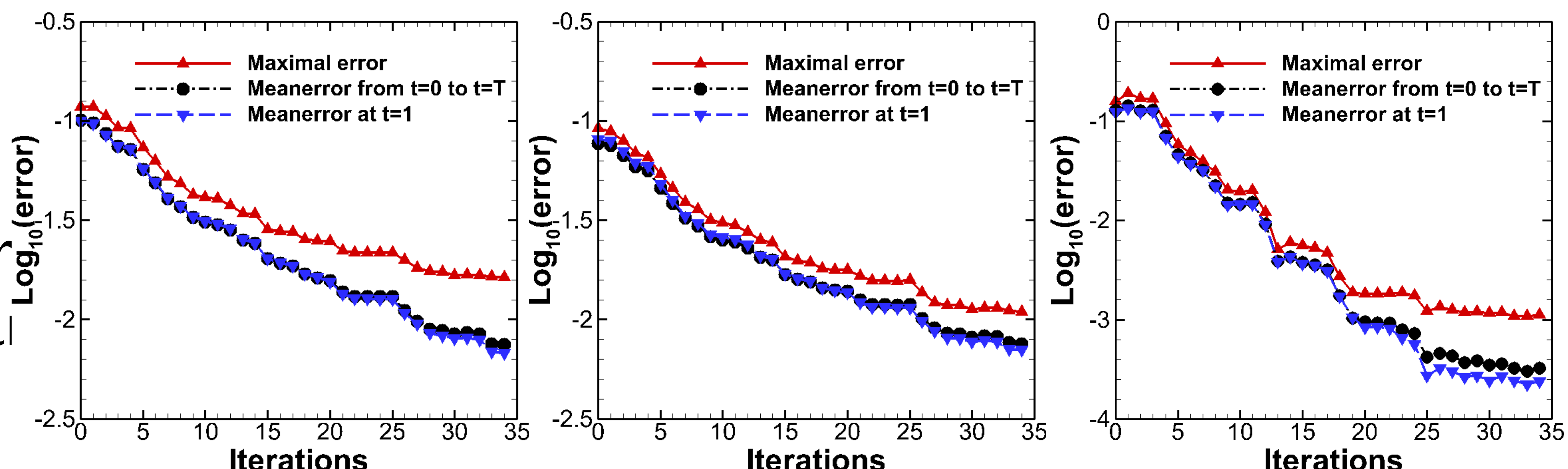
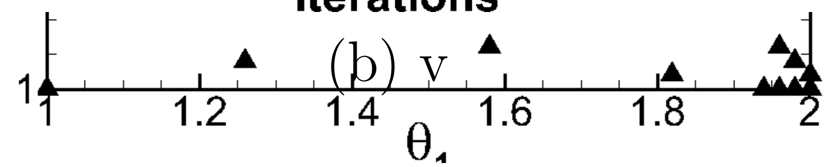
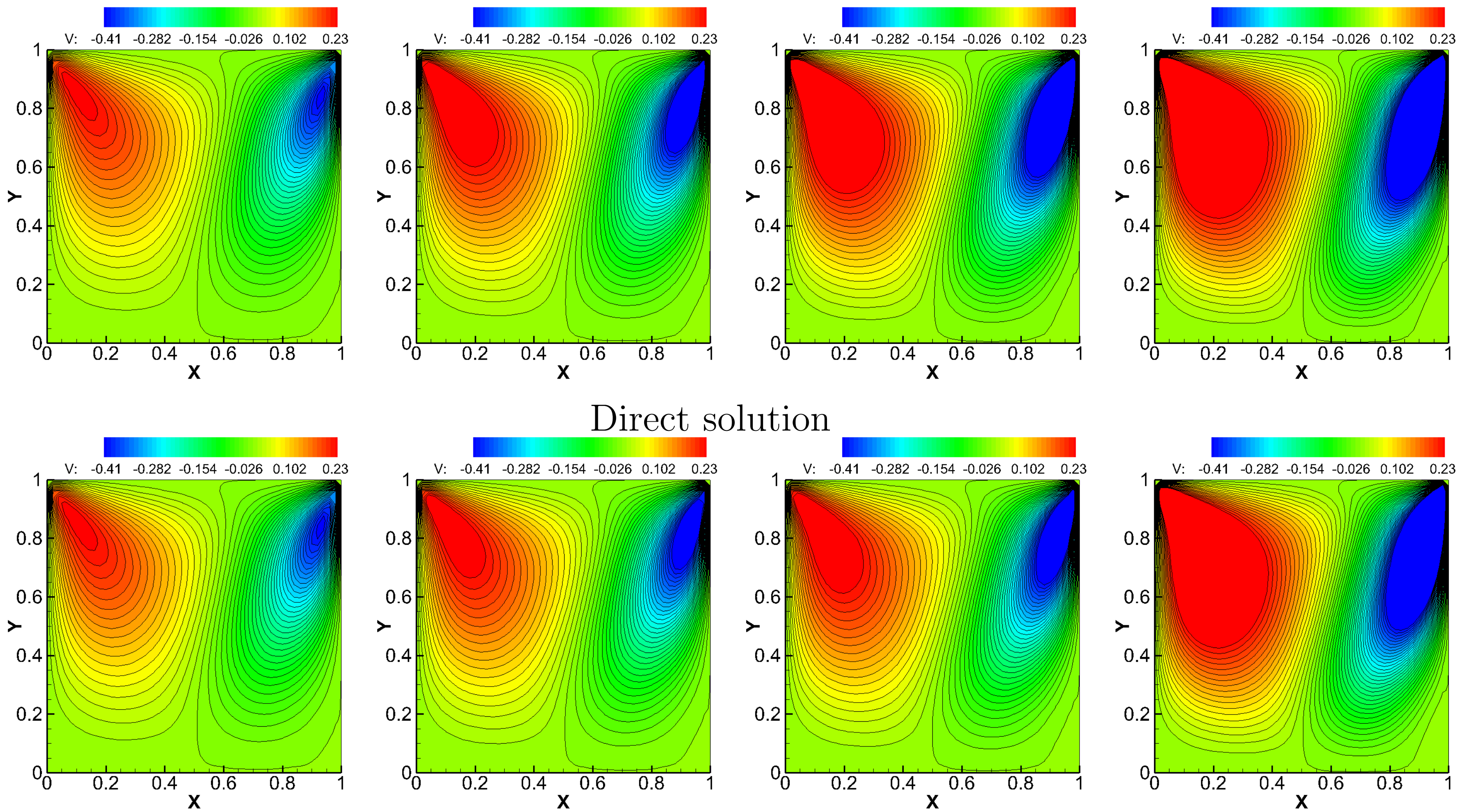


(a) u (b) v (c) p



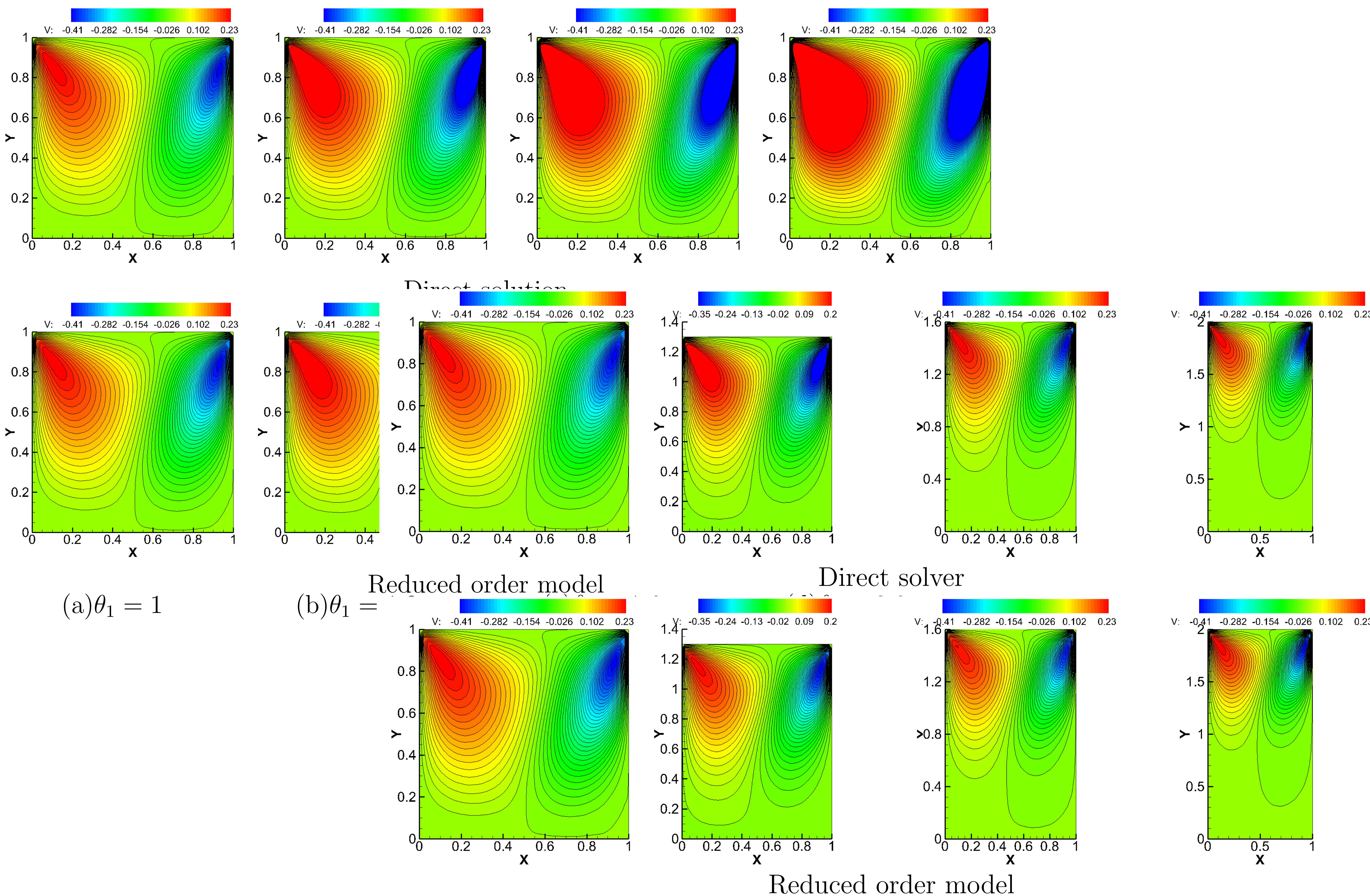
Non-intrusive approach



Reduced order model

(a)$\theta_1 = 1$ (b)$\theta_1 = 1.3$ (c)$\theta_1 = 1.6$ (d)$\theta_1 = 2.0$

Non-intrusive approach



(a)$\theta_1 = 1$ (b)$\theta_1 =$

(a)$\theta_2 = 1$ (b)$\theta_2 = 1.3$ (c)$\theta_2 = 1.6$ (d)$\theta_2 = 2.0$

Non-intrusive approach

Observations:

- Works well
- Simple, in particular for non-linear problems etc

Problem:

- No rigor in error control - but maybe ok ?

The non-standard problems

A summary so far

We have so far discussed how to

- Solve known problems faster
- Doing so with confidence in accuracy
- Minimize off-line cost

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We have so far discussed how to

- Solve known problems faster
- Doing so with confidence in accuracy
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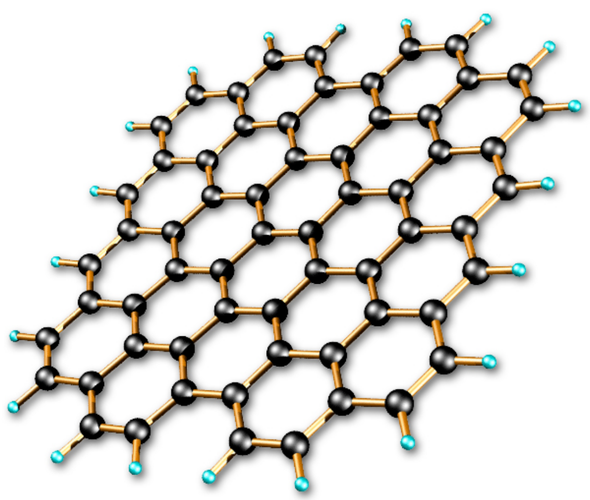
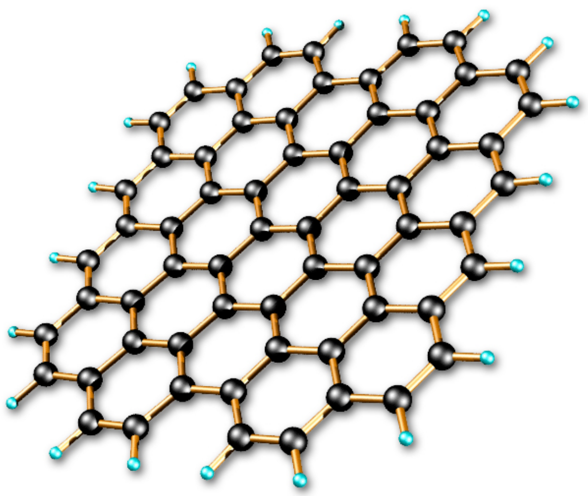
We will now consider how to use the same ideas to solve problems for which we do NOT have a large scale solver



Multiple scattering problems

Exploring related ideas for many body scattering



- Build an RB for each scatterer
- Build an RB for the interaction operation
- Combine through Jacobi-like iteration to enable rapid modeling of complex scatterer configurations

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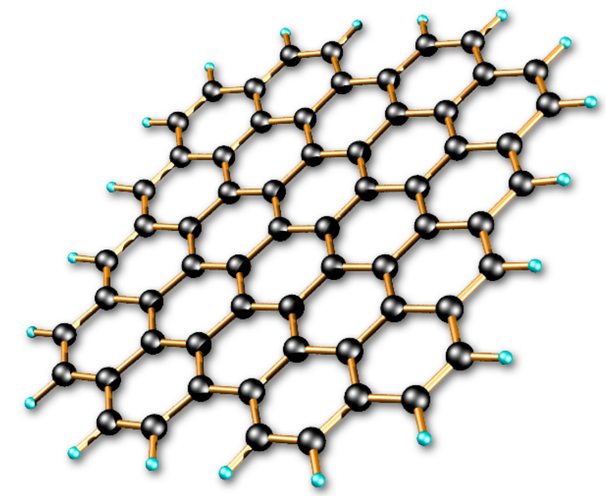

- Build an RB for each scatterer
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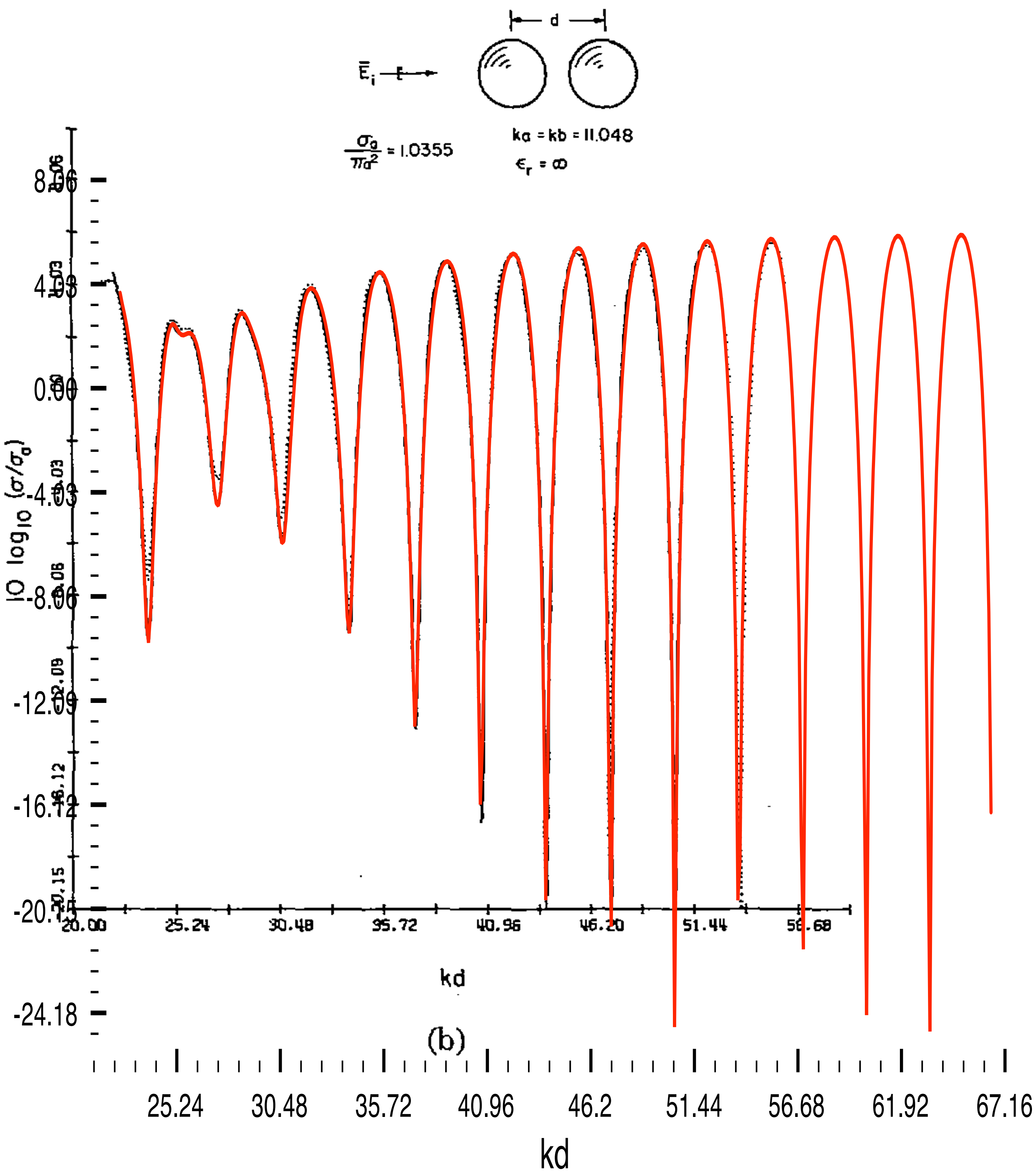
- Combine through Jacobi-like iteration to enable rapid modeling of complex scatterer configurations

This is not a RBM is the classic sense

.. but using RB ideas allows us to solve problems that are otherwise very hard to approach

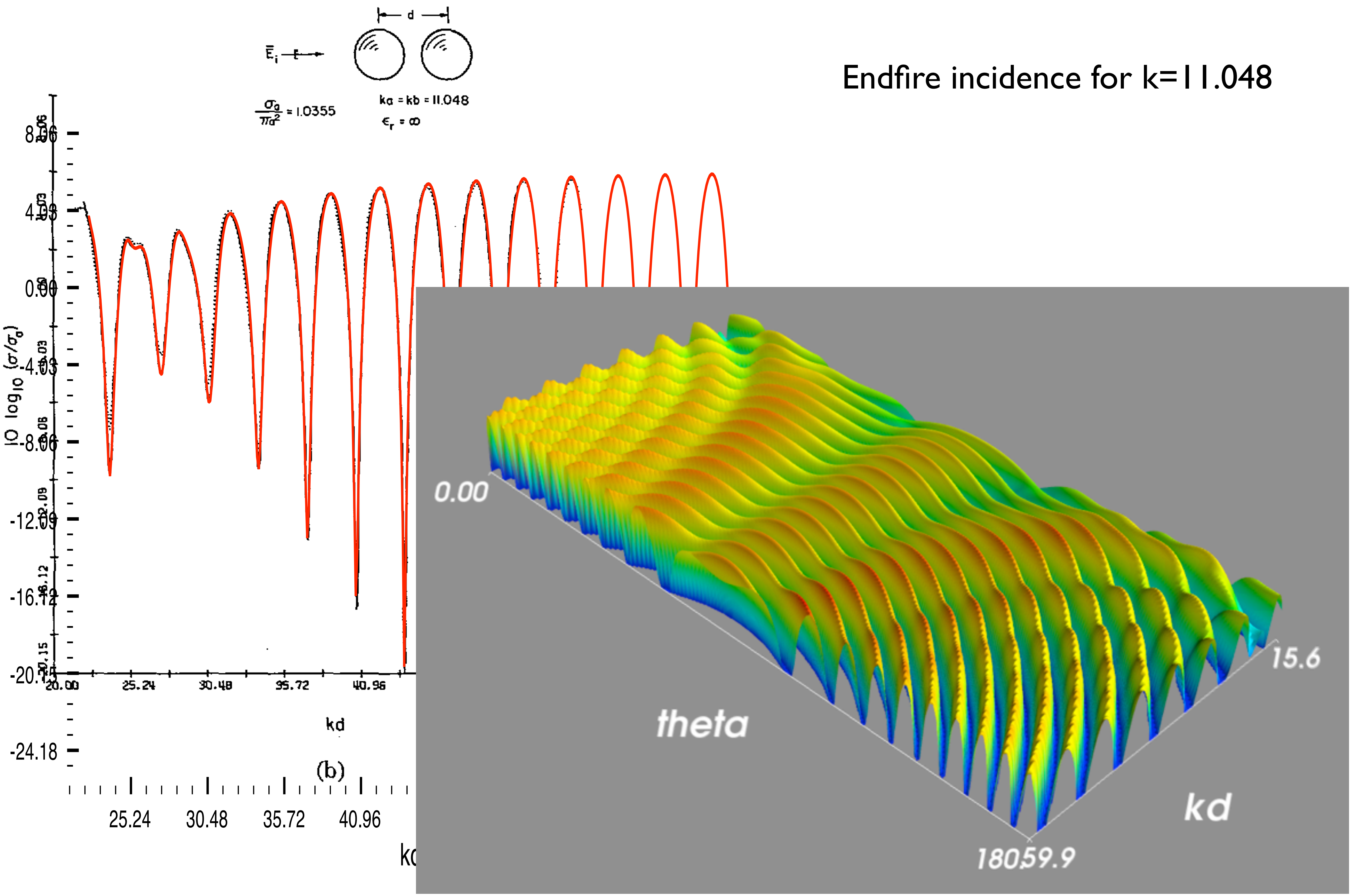
Towards multiple scattering



Endfire incidence for k=11.048

Towards multiple scattering

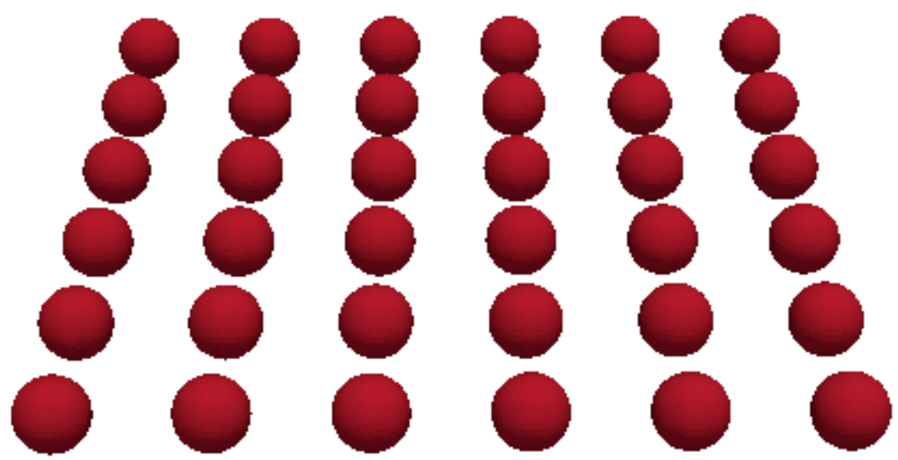
Endfire incidence for k=11.048

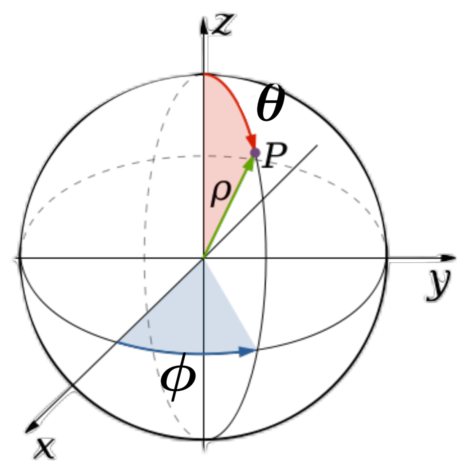


Multiple scattering problems

$$\phi \in [0, 2\pi]; k = 3, \theta = \pi/2$$

$$ka = 1; kd = 4$$





RB for single scatterer has 5 parameters (frequency(1), angle (2), polarization (2))

RB for interaction operator has 8 parameters (frequency(1), relative size(1), distance (2), rotation (2), polarization (2))

Full scattering result computed with iteration

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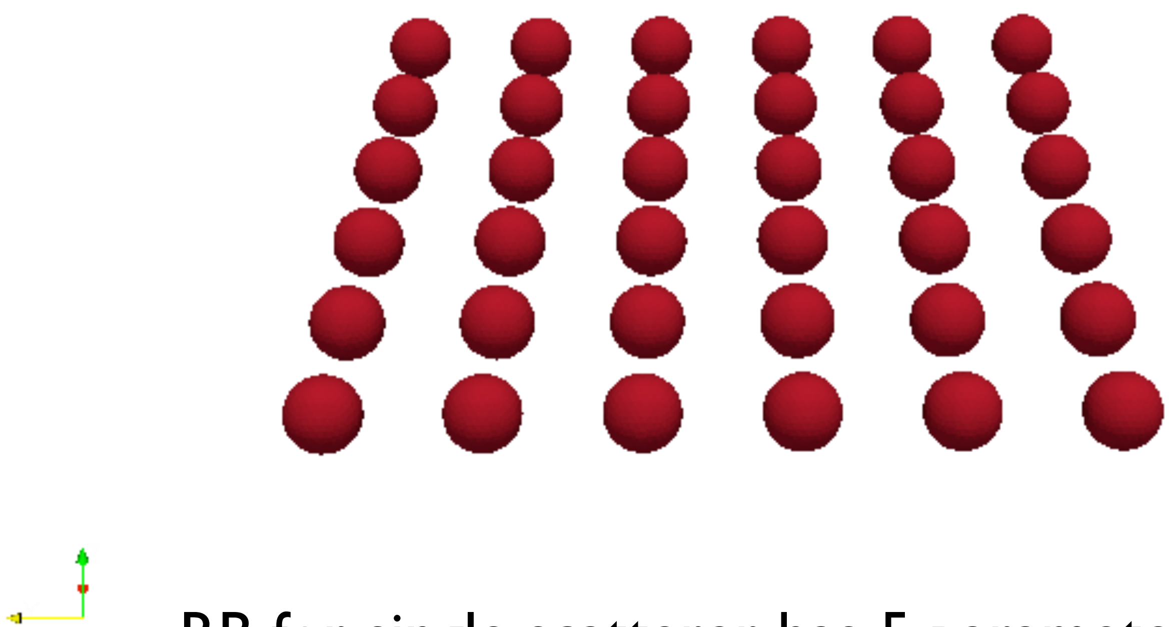

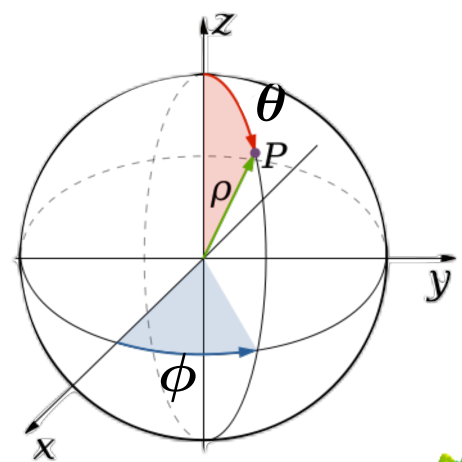

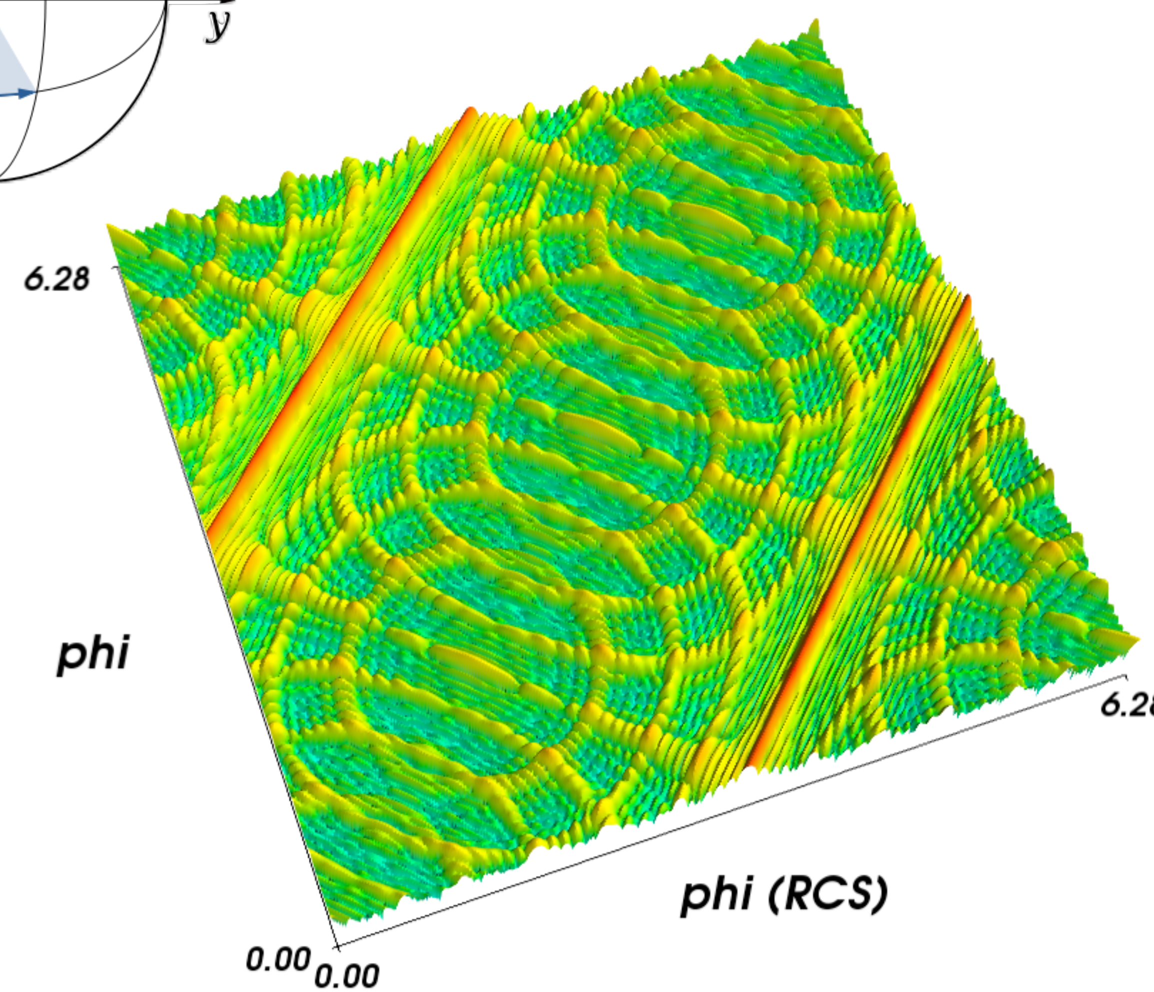

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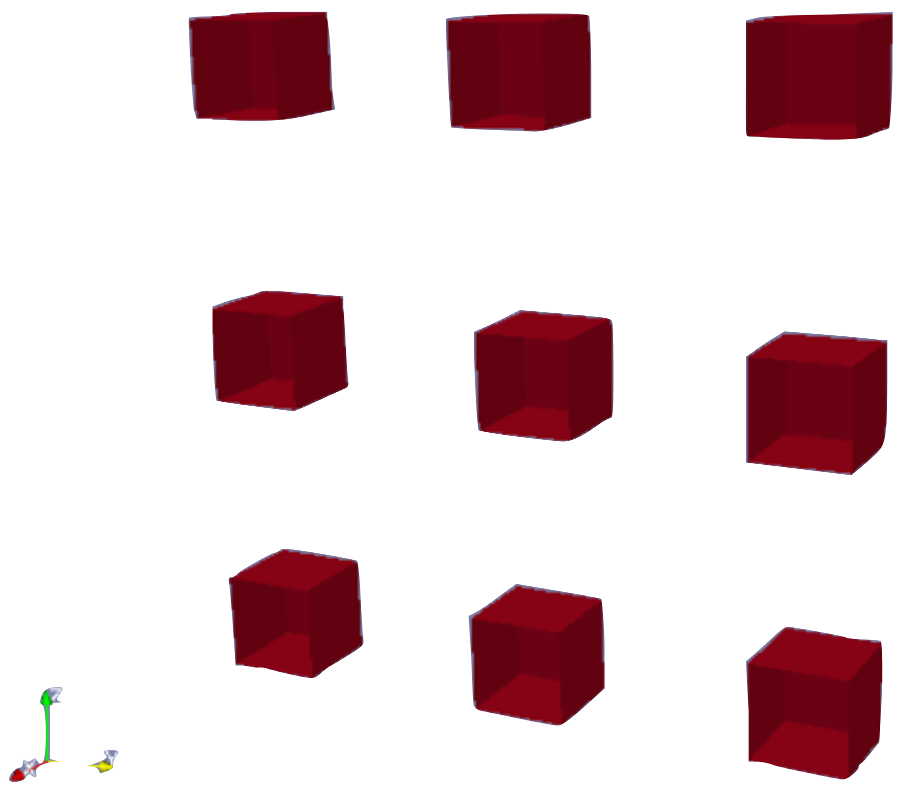
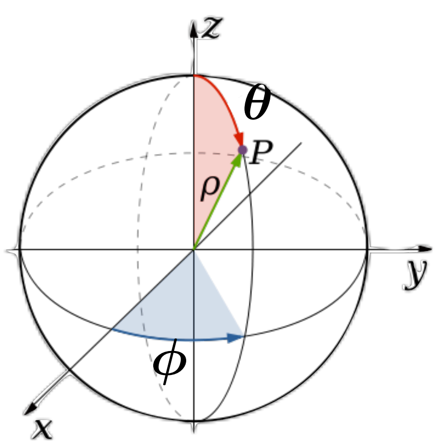
Full scattering result computed with iteration

Full RCS computed in less than 3 minutes for 36 spheres

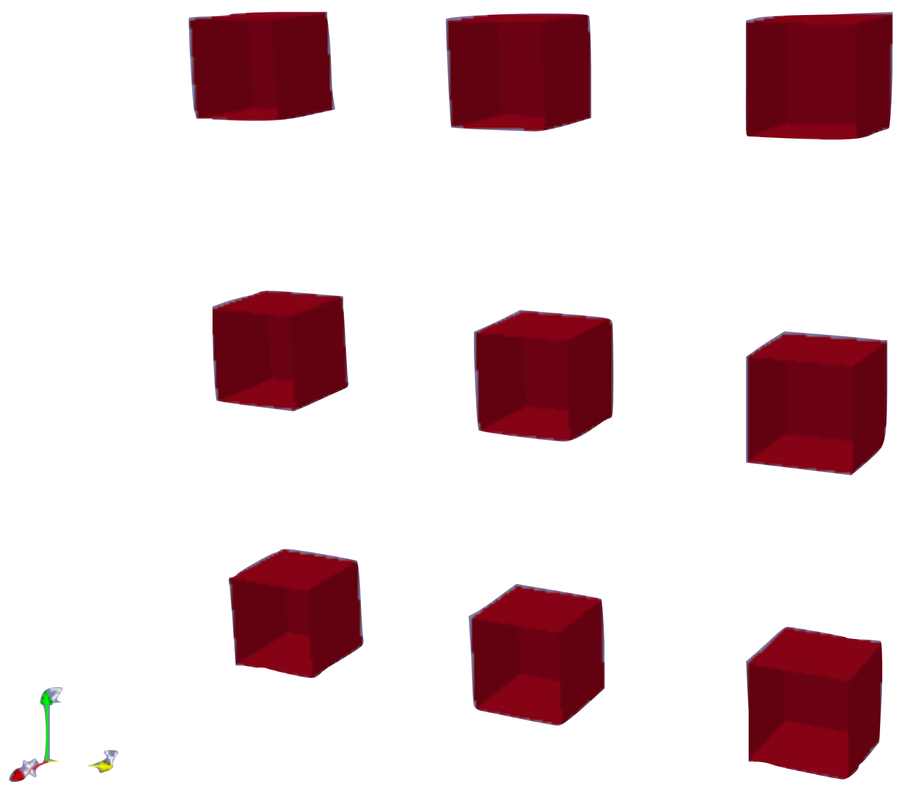
Multiple scattering problem

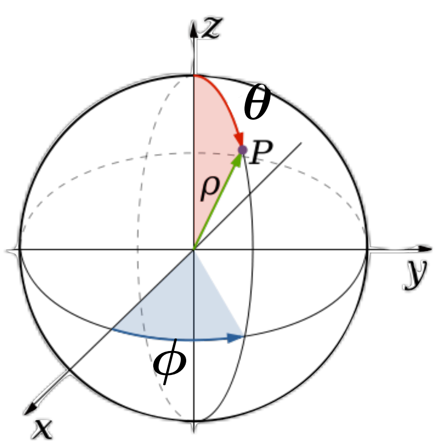




Vertical position of middle cavity uniformly distributed within [-1,1]

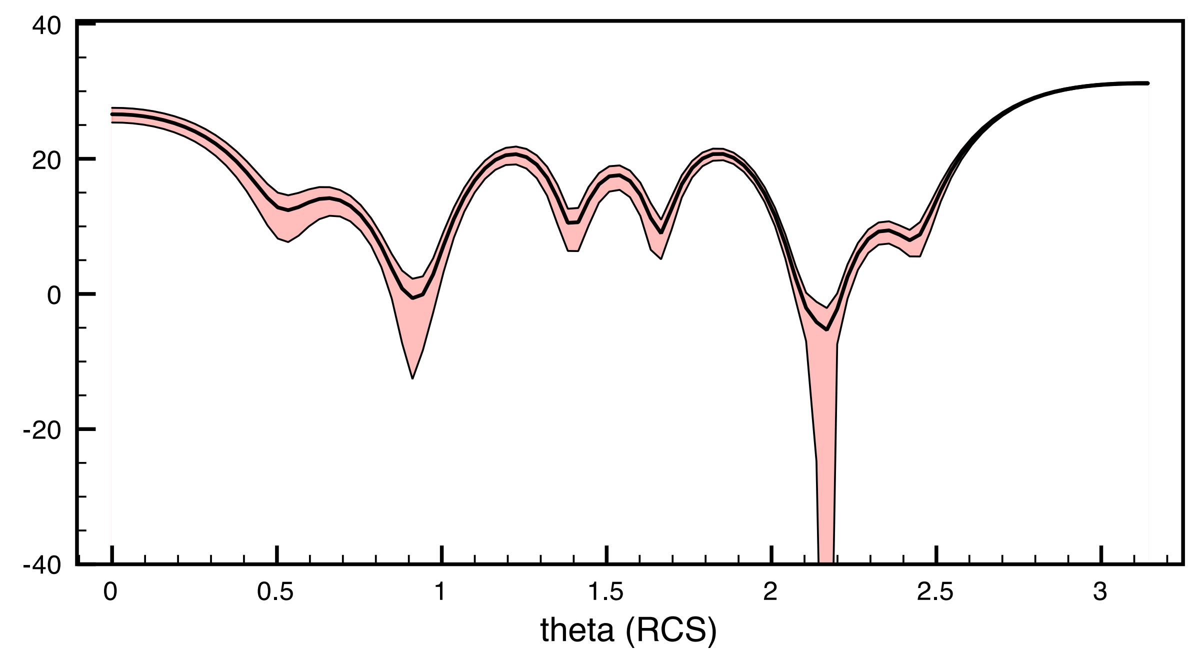
Multiple scattering problem





$$k = 3, \phi^i = 0, \theta^i = 0, 90$$

$$\phi^o = 0, \theta^o = 0 - 180$$



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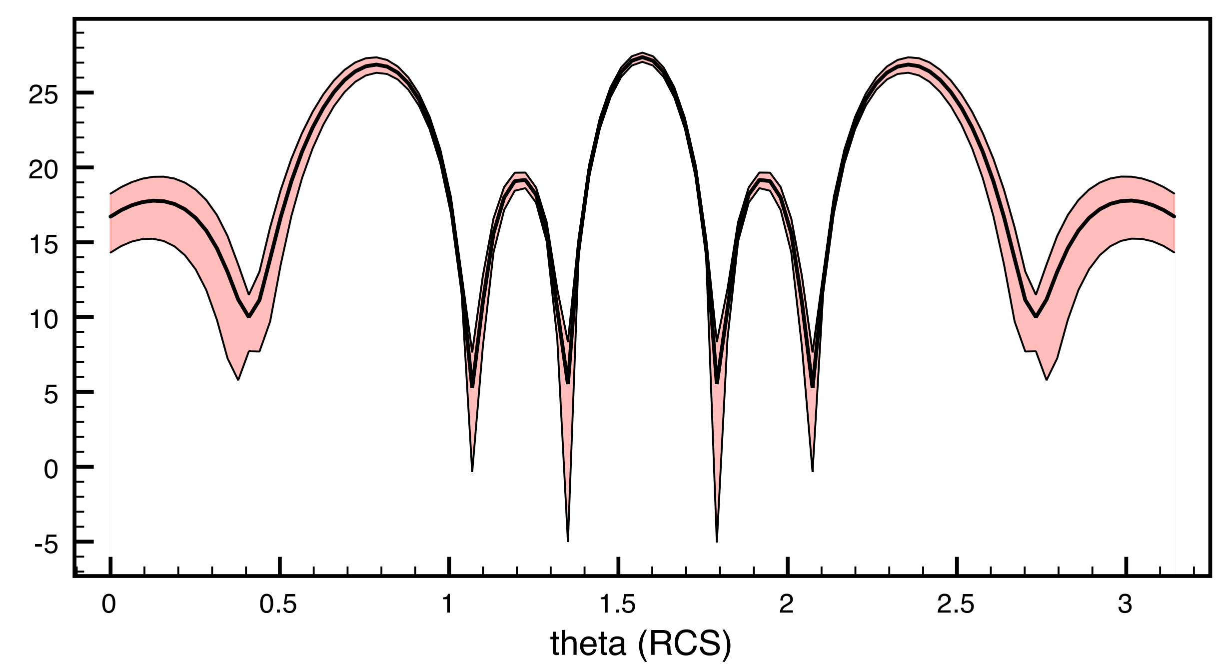
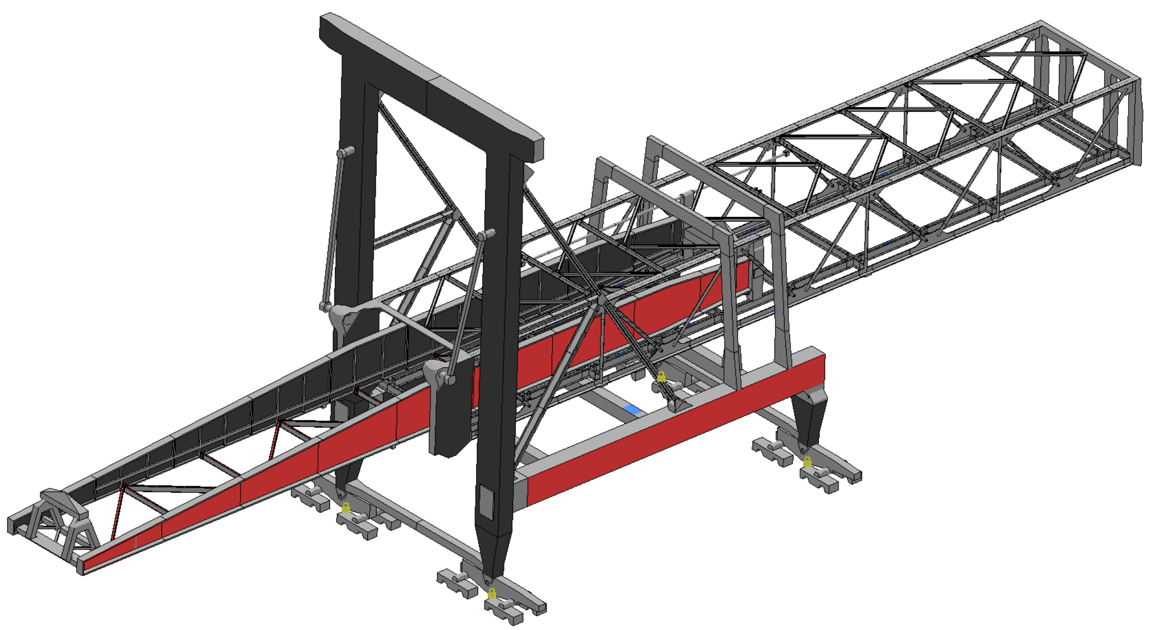


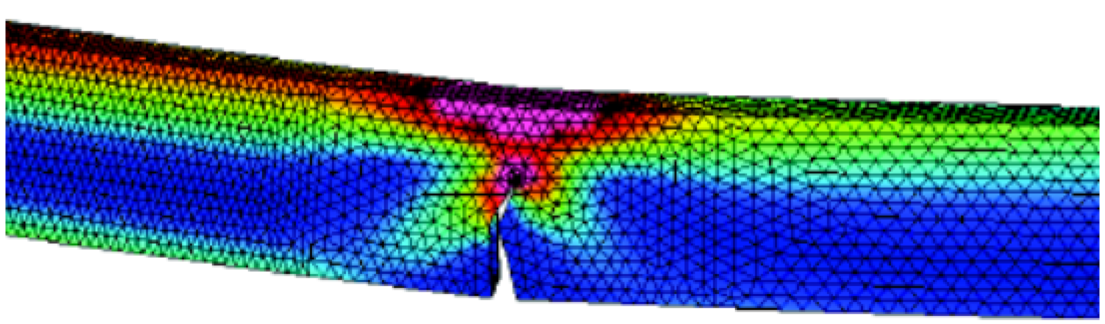
In a similar spirit

A company - Akselos (CH) - is making a business of this

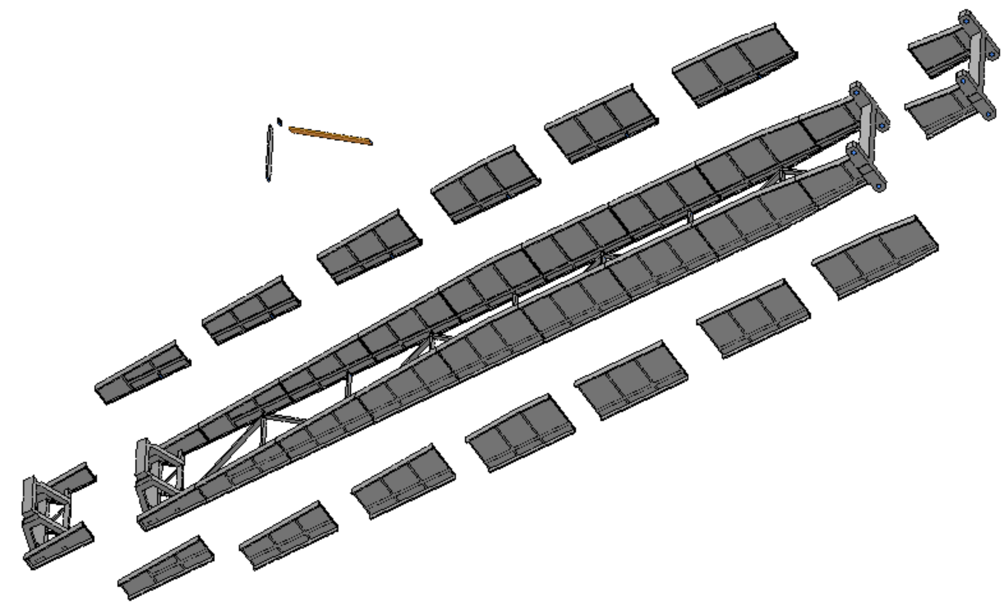


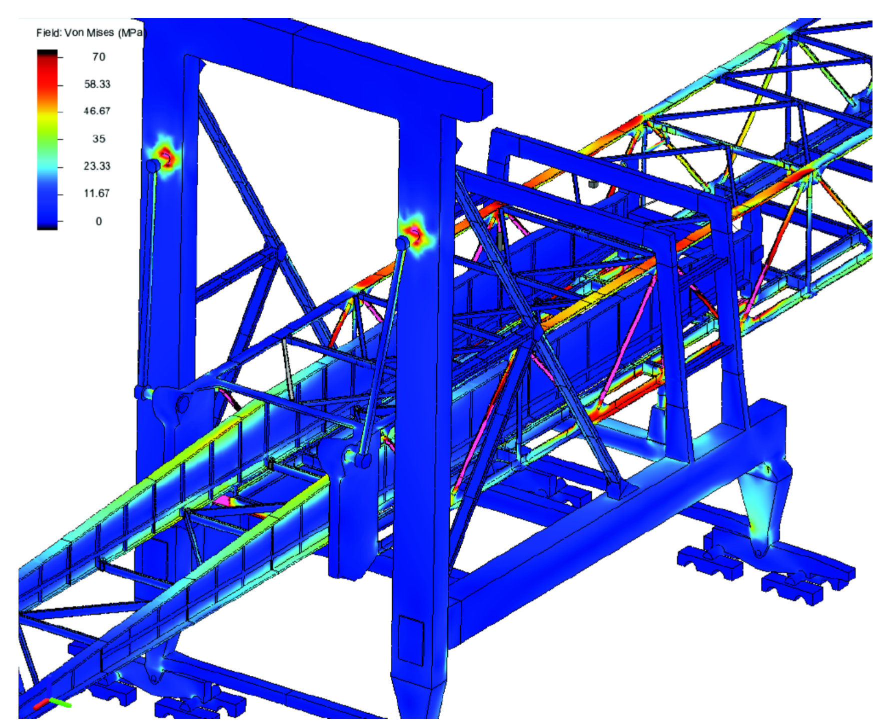
(a) Shiploader model.

Figures by Akselos, S.A.



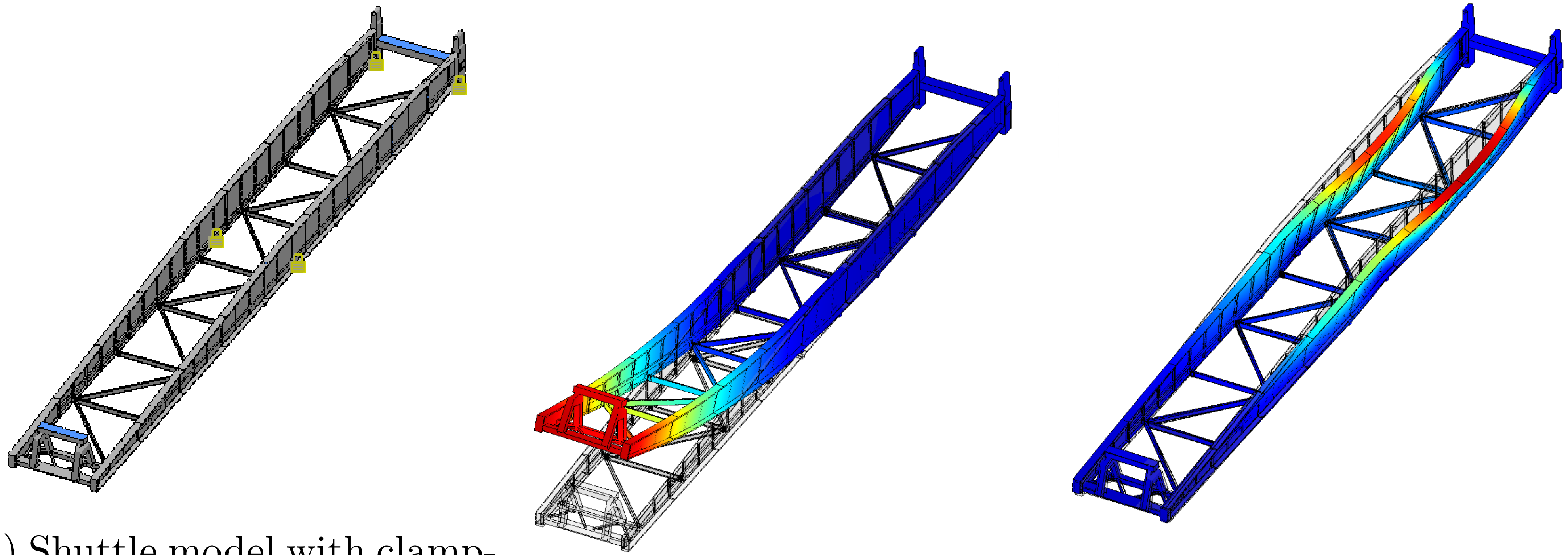
(d) Parametrized crack component.





(b) Stress visualization.

In a similar spirit



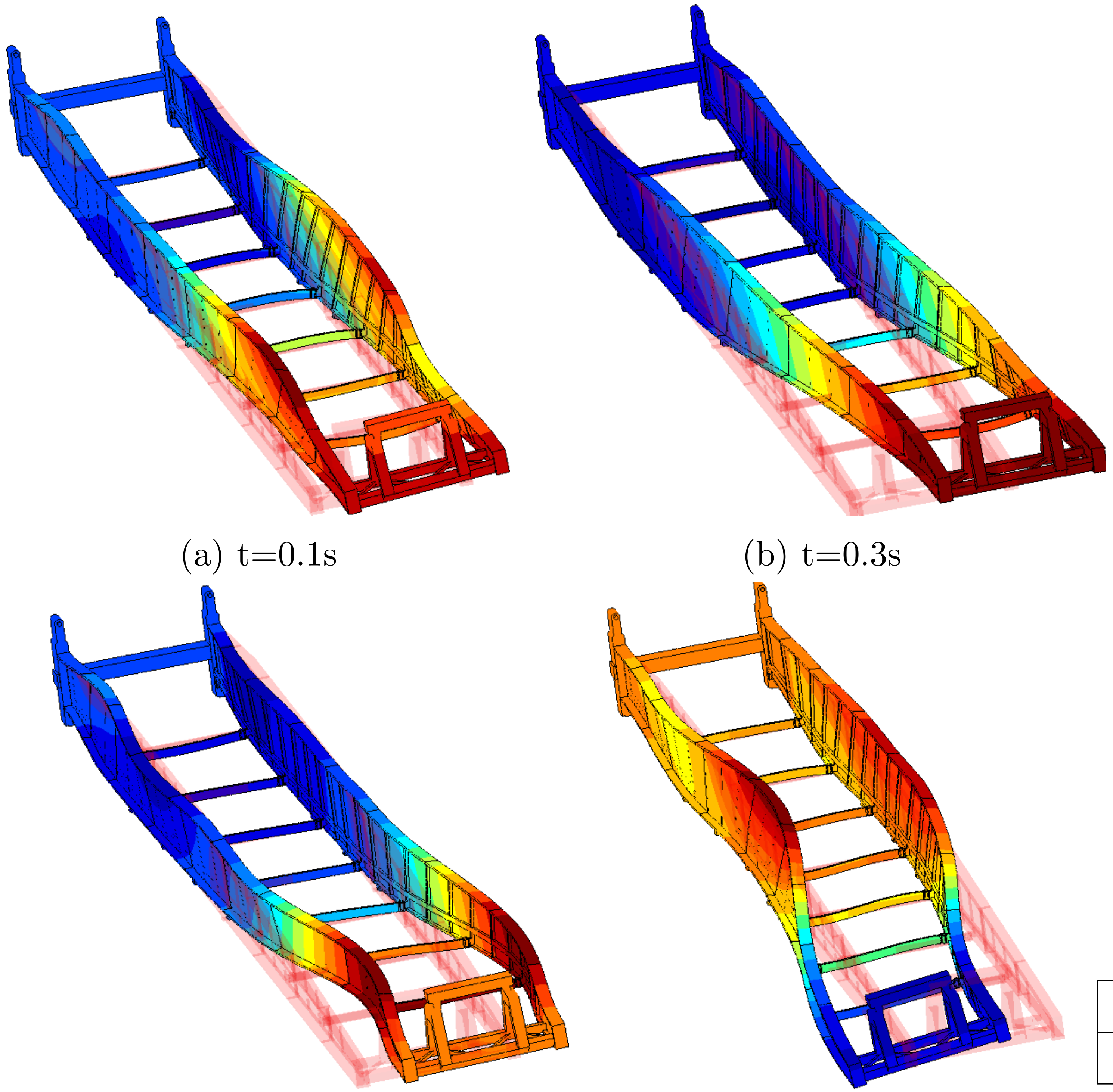
(a) Shuttle model with clamping locations.

(b) First mode.

(c) Fifth mode.

Figures by Akselos, S.A.

In a similar spirit



(a) t=0.1s (b) t=0.3s

(c) t=0.6s (d) t=1s

Figure 7: A local lateral shock is applied at initial time t=0s.

Speed up vs. FEA	700
Maximum error vs. FEA	1%

Figures by Akselos, S.A.

Software

You do not have to do it all yourself

rbMIT - MATLAB based
http://augustine.mit.edu/methodology/methodology_rbMIT_SystemPackage.htm

RBniCS - python based with FEniCS link
http://mathlab.sissa.it/rbnics

pyMOR - python based with FEniCS/DUNE link
http://pymor.org

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Questions ?

Thank you !