LECTURES ON MICROLOCAL CHARACTERIZATIONS IN LIMITED-ANGLE TOMOGRAPHY

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4 LECTURES

1. *Nov. 11*: Introduction to the mathematics of computerized tomography
2. *Nov. 18:* Introduction to the basic concepts of microlocal analysis
3. **Today:** Microlocal analysis of limited angle reconstructions in tomography
4. *Dec. 02:* Wrap up & Discussion (possible Synergies, Projects, Grants) I

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References:

F. Natterer, *The mathematics of computerized tomography*. Stuttgart: B. G. Teubner, 1986.

L. Hörmander, *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*, vol. 256. Berlin: Springer-Verlag, 2003.

JF and E. T. Quinto, *Characterization and reduction of artifacts in limited angle tomography*, Inverse Problems 29(12):125007, December 2013.

In limited angle tomography, the projectios $g_\theta = \mathcal{R}_\theta f$ are known only for certain directions $\theta \in S^1_\Phi \subsetneq S^1$, for other directions θ the projections g_θ are unknown. In other words, in limited angle tomography we are given truncated data

$$g_\Phi(\theta, s) = \chi_{S^1_\Phi \times \mathbb{R}} \cdot \mathcal{R}f(\theta, s).$$

FBP inversion formula applied to limited angle data

$$\mathcal{R}^\dagger g_\Phi(x) = \frac{1}{4\pi} \int_{S^1_\Phi} [\psi * g_\theta](x \cdot \theta)\,\mathrm{d}\theta \quad = \quad ???$$

What do we reconstruct?

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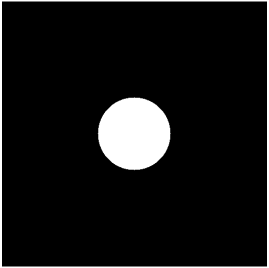
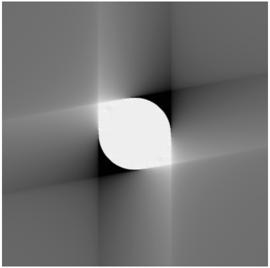
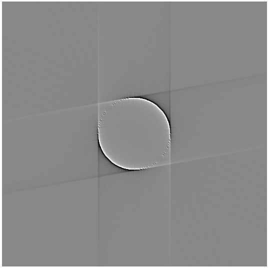

Original


FBP


Lambda

Reconstructions for an angular range $[0^\circ, 100^\circ]$

Observations at a first glance:

- Only certain singularities of the original object can be reconstructed
- Artifacts (new singularities) are generated

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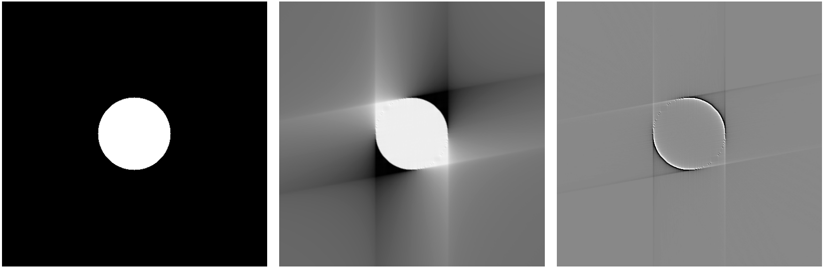
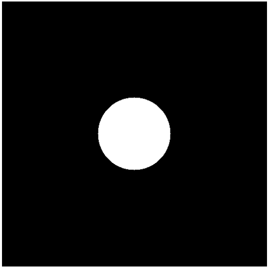
Goal: Use **microlocal analysis** to

- Characterize singularities that can be reliably reconstructed,
- Develop strategy to reduce artifacts.



Original FBP Lambda

Reconstructions for an angular range $[0^\circ, 100^\circ]$



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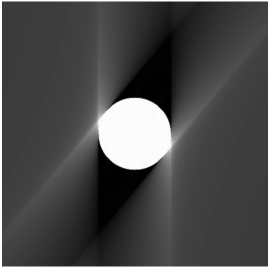
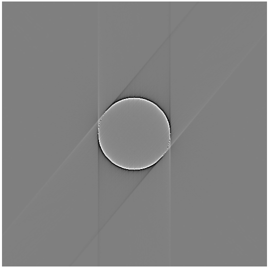


FBP



Lambda

Reconstructions for an angular range $[0^\circ, 140^\circ]$

WHICH OF THE ORIGINAL SINGULARITIES ARE RELIABLY RECONSTRUCTED?

Original

FBP

Lambda

Reconstructions for an angular range $[0^\circ, 100^\circ]$

- A formula for limited angle FBP reconstructions
- Characterization of visible and invisible singularities
- Severe ill-posedness of limited angle tomography
- Characterization & reduction of artifacts

We study the restricted or limited angle Radon transform

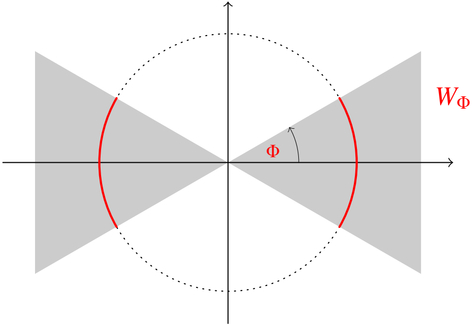
$$\mathcal{R}_\Phi f(\theta, s) = \chi_{S^1_\Phi \times \mathbb{R}} \cdot \mathcal{R}f(\theta, s),$$

where $0 < \Phi < \pi/2$ and

$$S^1_\Phi = \left\{\theta \in S^1 : \ \theta = \pm(\cos\varphi, \sin\varphi), \ \ |\varphi| < \Phi\right\}.$$

Moreover, we define the polar wedge

$$W_\Phi = \mathbb{R} \cdot S^1_\Phi = \left\{r\theta : \ r \in \mathbb{R}, \ \theta \in S^1_\Phi\right\}.$$

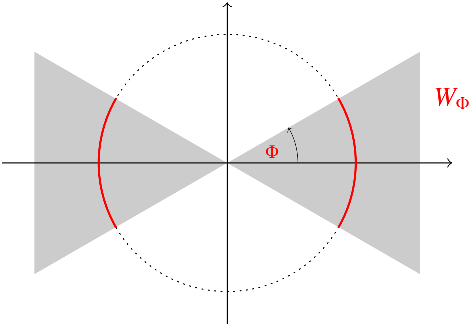


Theorem (F., Quinto (2013))

Let $f \in \mathcal{S}(\mathbb{R}^2)$. Then, the FBP reconstruction formula $\mathcal{R}_\Phi^\dagger g = \frac{1}{4\pi}\mathcal{R}_\Phi^* \Lambda g$ and the Lambda reconstruction formula $\mathcal{L}_\Phi g = \frac{1}{4\pi}\mathcal{R}_\Phi^*\left(-\frac{d^2}{ds^2} g\right)$ satisfy

$$P_\Phi f = \mathcal{R}_\Phi^\dagger(\mathcal{R} f) \quad \text{and} \quad P_\Phi(\Lambda f) = \Lambda(P_\Phi f) = \mathcal{L}(\mathcal{R}_\Phi f),$$

where $P_\Phi f = \mathcal{F}^{-1}\left(\chi_{W_\Phi} \hat{f}\right)$. This formula is also valid for $f \in \mathcal{E}'(\mathbb{R}^2)$. Furthermore, the maps $\mathcal{R}_\Phi^\dagger \mathcal{R}$ and $\mathcal{L}_\Phi^\dagger \mathcal{R}$ are weakly continuous from $\mathcal{E}'(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$.



Remarks

- The theorem shows that a perfect reconstruction of a function f is only possible if

$$\operatorname{supp} \hat{f} \subset W_{\Phi}$$

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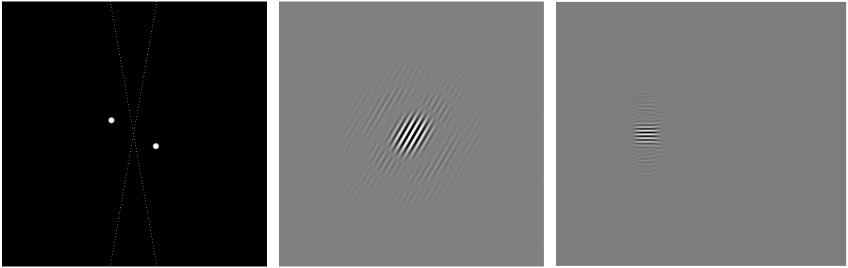

$\hat{f}$ f $\mathcal{R}_\Phi f$

Reconstructions for an angular range $[-80°, 80°]$ $(\Phi = 80°)$

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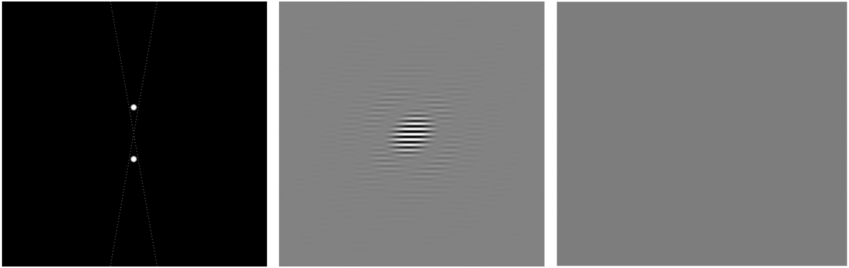


$\hat{f}$ $\qquad$ f $\qquad$ $\mathcal{R}_\Phi f$

Reconstructions for an angular range $[-80^\circ, 80^\circ]$ ($\Phi = 80^\circ$)

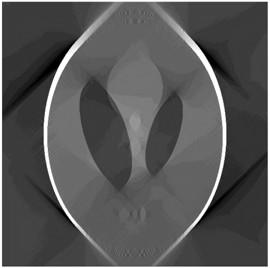
Corollary (Quinto (1993); F., Quinto (2013))

Let $f \in \mathcal{E}'(\mathbb{R}^2)$. *Then*

$$\mathrm{WF}(\Lambda(P_\Phi f)) = \mathrm{WF}(P_\Phi f) \subset \mathbb{R}^2 \times W_\Phi.$$

Reconstruction at the angular range $[-45^\circ, 45^\circ]$

f



$f_{\mathrm{FBP}} = P_\Phi f$

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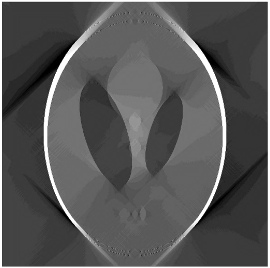
We can only expect to reconstruct singularities (x, ξ) where $\xi \in W_\Phi$

Visible singularities (red) at the angular range $[-45°, 45°]$

f



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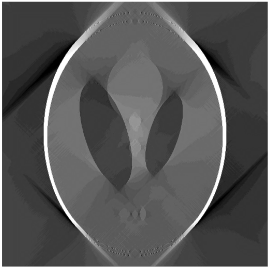
Visible singularities

$$\mathrm{WF}_\Phi(f) := \{(x,\xi) \in \mathrm{WF}(f) : \xi \in W_\Phi\}$$

Visible singularities (red) at the angular range $[-45^\circ, 45^\circ]$

f



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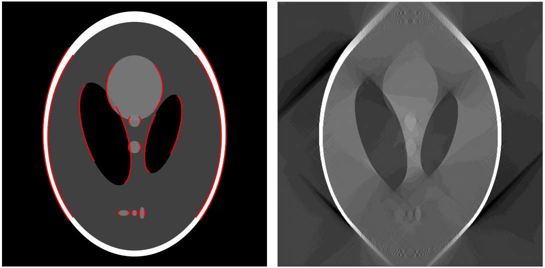
Invisible singularities, (x,ξ) with $\xi \in \mathbb{R}^2 \setminus \overline{W_\Phi}$, are smeared or distorted

Visible singularities (red) at the angular range $[-45^\circ, 45^\circ]$



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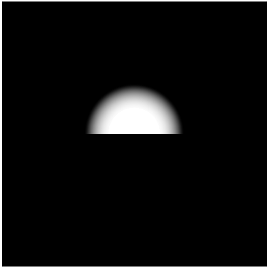
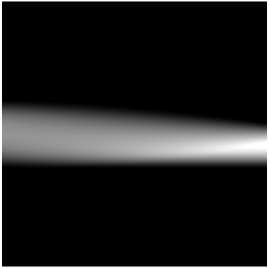
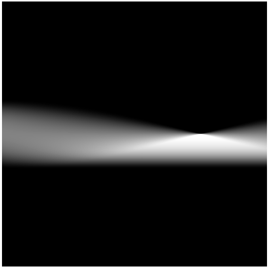


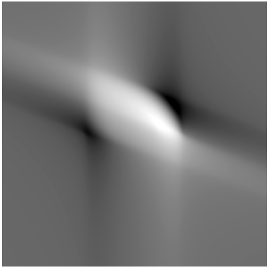
Original



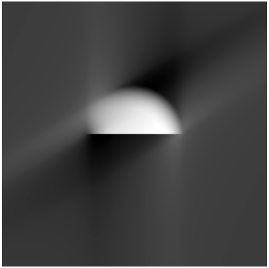
Sinogram for $[0°, 70°]$



Sinogram for $[0°, 120°]$



FBP for $[0°, 70°]$



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Remarks about ill-posedness

Recall: In case of full data we have the Sobolev-space estimates

$$c\,\|f\|_{H_0^\alpha} \leq \|\mathcal{R}f\|_{H^{\alpha+1/2}} \leq C\,\|f\|_{H_0^\alpha}$$

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- We don't have control over the Fourier region outside of W_Φ! Here, anything can happen and that's where the severe instabilities come from.
- The existence of invisible singularities makes the problem severely (or exponentially) ill-posed
- If one would use $\operatorname{supp} \hat{f} \subset W_\Phi$ or $\operatorname{WF}(f) \subset \mathbb{R}^2 \times W_\Phi$ as a-priori information, then we could get the same stability as in the case of full data, i.e., we can show that

$$c\,\|P_\Phi f\|_{H_0^\alpha} \le \|\mathcal{R}_\Phi f\|_{H^{\alpha+1/2}}$$

Theorem (F., Quinto (2013); Katsevich (1997))

Let $\Phi \in [0, \pi/2)$ *and let* $f \in \mathcal{E}'(\mathbb{R}^2)$. *Let* $\mathcal{R}^\dagger$ *be the FBP reconstruction operator. Then*

$$\mathrm{WF}_\Phi(f) \subset \mathrm{WF}\left(\mathcal{R}^\dagger(\mathcal{R}_\Phi f)\right) \subset \mathrm{WF}_{\bar{\Phi}}(f) \cup \mathcal{A}_\Phi(f),$$

where

$$\mathcal{A}_\Phi = \left\{(x + r\theta(\varphi)^\perp, \alpha\theta(\varphi)) : (x, \theta(\varphi)) \in \mathrm{WF}(f),\ r, \alpha \in \mathbb{R} \setminus \{0\},\ \varphi = \pm\Phi\right\}$$

is the set of added singularities. Here $\theta(\varphi) = (\cos\varphi, \sin\varphi)$ *for* $\varphi \in [-\pi, \pi)$.

Artifacts are located on straight lines with normal directions $\theta(\pm\Phi)$

Added singularities

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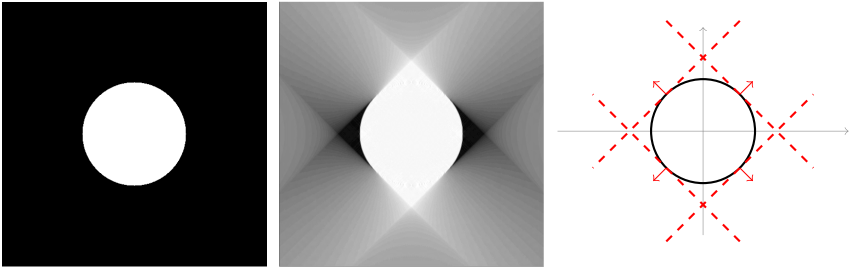

Original

Reconstruction for $[-\frac{\pi}{4}, \frac{\pi}{4}]$

Set of added singularities $\mathcal{A}_{\frac{\pi}{4}}$

- First note that

$$\mathcal{R}^{\dagger}(\mathcal{R}_{\Phi} f) = P_{\Phi} f = \mathcal{F}^{-1}(\chi_{W_{\Phi}} \cdot \hat{f}) = \frac{1}{2\pi} \check{u}_{\Phi} * f,$$

where

$$\check{u}_{\Phi} = \mathcal{F}^{-1}(\chi_{W_{\Phi}}).$$

Therefore

$$\mathrm{WF}(\mathcal{R}^{\dagger}(\mathcal{R}_{\Phi} f)) = \mathrm{WF}(\check{u}_{\Phi} * f)$$

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- Then, use the following general result from microlocal analysis: If either f or g or both have compact support (as distributions) then

$$\mathrm{WF}(f * g) \subset \left\{(x + y, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \ (x, \xi) \in \mathrm{WF}(f), \ (y, \xi) \in \mathrm{WF}(g)\right\}$$

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- Applied to our situation, we get

$$\mathrm{WF}(\mathcal{R}^\dagger\mathcal{R}_\Phi f) \subset \Big\{(x+y,\xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \ (x,\xi) \in \mathrm{WF}(f), \ (y,\xi) \in \mathrm{WF}(\check{u}_\Phi)\Big\}$$

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- Need to calculate $\mathrm{WF}(\check{u}_\Phi)$: To that end, note that $\check{u}_\Phi$ is a homogeneous distribution as the (inverse) Fourier transform of the homogeneous distribution $u_\Phi = \chi_{W_\Phi}$. Then, we can use the following general result for homogeneous distributions u:

$$\begin{aligned} (x, \xi) \in \mathrm{WF}(u) \quad &\Leftrightarrow \quad (\xi, -x) \in \mathrm{WF}(\hat{u}) \quad \text{for } x \neq 0, \ \xi \neq 0 \\ (0, \xi) \in \mathrm{WF}(u) \quad &\Leftrightarrow \quad \xi \in \text{sing supp}(\hat{u}) \end{aligned}$$

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- Applied to our situation, we get

$$\mathrm{WF}(\mathcal{R}^\dagger \mathcal{R}_\Phi f) \subset \left\{(x+y,\xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \; (x,\xi) \in \mathrm{WF}(f), \; (y,\xi) \in \mathrm{WF}(\check{u}_\Phi)\right\}$$

- Need to calculate $\mathrm{WF}(\check{u}_\Phi)$: To that end, note that $\check{u}_\Phi$ is a homogeneous distribution as the (inverse) Fourier transform of the homogeneous distribution $u_\Phi = \chi_{W_\Phi}$. Then, we can use the following general result for homogeneous distributions u:

$$(x,\xi) \in \mathrm{WF}(u) \quad \Leftrightarrow \quad (\xi,-x) \in \mathrm{WF}(\hat{u}) \quad \text{for } x \neq 0, \; \xi \neq 0$$
$$(0,\xi) \in \mathrm{WF}(u) \quad \Leftrightarrow \quad \xi \in \operatorname{sing\,supp}(\hat{u})$$

- To calculate $\mathrm{WF}(\check{u}_\Phi)$ we therefore first calculate

$$\mathrm{WF}(\chi_{W_\Phi}) = \left\{(\alpha\theta(\varphi), r\theta^\perp(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \; \alpha, r \in \mathbb{R} \setminus 0, \; \varphi = \pm\Phi\right\} \cup (\{0\} \times \overline{W_\Phi})$$

OUTLINE OF THE PROOF

- To calculate $\mathrm{WF}(\check{u}_\Phi)$ we first observe that outside of the origin ($x \neq 0$) we have

$$(\alpha\theta(\varphi), r\theta^\perp(\varphi)) \in \mathrm{WF}(\chi_{W_\Phi}) \quad \text{for } \alpha, r \in \mathbb{R} \setminus 0,\ \varphi = \pm\Phi$$

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- We now apply the result about the wavefront set of convolutions (see previous slide) and obtain the assertion

$$\begin{aligned}\mathrm{WF}(\mathcal{R}^\dagger\mathcal{R}_\Phi f) &\subset \left\{(x+y,\xi) \in \mathbb{R}^2 \times (\mathbb{R}^2\setminus 0) :\ (x,\xi) \in \mathrm{WF}(f),\ (y,\xi) \in \mathrm{WF}(\check{u}_\Phi)\right\} \\ &\subset \left\{(x + r\theta^\perp(\varphi), \alpha\theta(\varphi)) \in \mathbb{R}^2 \times (\mathbb{R}^2\setminus 0) :\ \alpha, r \in \mathbb{R}\setminus 0, (x,\alpha\theta(\varphi)) \in \mathrm{WF}(f)\ \varphi = \pm\Phi\right\} \\ &\qquad \cup \left\{(x,\xi) \in \mathbb{R}^2 \times (\mathbb{R}^2\setminus 0) :\ (x,\xi) \in \mathrm{WF}(f),\ \xi \in \overline{W_\Phi}\right\} \\ &= \mathcal{A}_\Phi \cup \mathrm{WF}_{\overline{\Phi}}(f)\end{aligned}$$

□

What is the cause of artifacts?

- First observe that if we had

$$\mathcal{A}_\Phi = \Big\{(x+y,\xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \ (x,\xi) \in \mathrm{WF}(f), \ (y,\xi) \in \mathrm{WF}(\check{u}_\Phi), y \neq 0\Big\},$$

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DTU

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- To avoid the generation of additional artifacts, the idea is to develop an FBP type reconstruction formula

$$\mathcal{R}^* P \mathcal{R} f = \frac{1}{4\pi} f * \check{\kappa}_\Phi,$$

  such that $\check{\kappa}_\Phi$ is a homogeneous distribution with a smooth Fourier transform away from origin (then $\operatorname{sing\,supp} \check{\kappa}_\Phi = \{0\}$).

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- Alternatively, since we know that pseudodifferential operators do not increase wavefront sets, we could formulate the artifact reduction strategy in a more abstract way as follows: Design an FBP reconstruction operator $\mathcal{R}^* P$ such that $\mathcal{R}^* P \mathcal{R}$ is a standard pseudodifferential operator, then $\mathrm{WF}(\mathcal{R}^* P \mathcal{R} f) \subset \mathrm{WF}(f)$.

Theorem (F., Quinto (2013))

Let $\kappa : S^1 \to \mathbb{R}$ *be a smooth function with* $\operatorname{supp}(\kappa) \subset \operatorname{cl}(S^1_\Phi)$ *and assume* $\kappa = 1$ *on* $S^1_{\Phi'}$*, for some* $\Phi' \in (0, \Phi)$*. Let* $\mathcal{K}$ *be the operator that multiplies by* κ

$$\mathcal{K}g(\theta, s) = \kappa(\theta)g(\theta, s)\,.$$

Then, the operator

$$\mathcal{R}^\dagger \mathcal{K} \mathcal{R}_\Phi$$

is a standard pseudodifferential operator and for $f \in \mathcal{E}'(\mathbb{R}^2)$*,*

$$\mathrm{WF}_{\Phi'}(f) \subset \mathrm{WF}(\mathcal{R}^\dagger \mathcal{K}(\mathcal{R}_\Phi f)) \subset \mathrm{WF}_\Phi(f).$$

The reconstruction formula $\mathcal{R}^\dagger \mathcal{K}(\mathcal{R}_\Phi)$ does not produce additional artifacts!

Remarks

$$\mathcal{R}^{\dagger}\mathcal{K}\mathcal{R}_{\Phi}f = \frac{1}{4\pi}\mathcal{R}^{*}\mathcal{I}^{-1}\mathcal{K}\mathcal{R}_{\Phi}f$$

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- Preprocessing of limited angle data data $g_{\Phi} = \mathcal{R}_{\Phi}f$:

$$\bar{g}_{\Phi}(\theta, s) = \kappa_{\Phi}(\theta) \cdot g_{\Phi}(\theta, s)$$

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- Modification of the FBP filter in the Fourier domain:

$$\hat{\psi}(\theta, r) = |r| \quad \mapsto \quad \hat{\psi}_{\Phi}(\theta, r) = \kappa_{\Phi}(\theta)\,|r|$$

REMARKS

$$\mathcal{R}^{\dagger}\mathcal{K}\mathcal{R}_{\Phi}f = \frac{1}{4\pi}\mathcal{R}^{*}\mathcal{I}^{-1}\mathcal{K}\mathcal{R}_{\Phi}f$$

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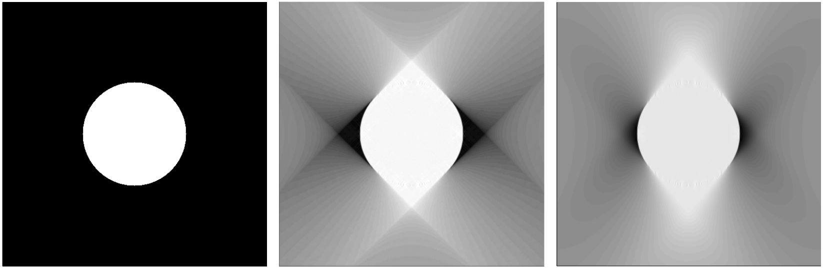
$$\bar{g}_\Phi(\theta, s) = \kappa_\Phi(\theta) \cdot g_\Phi(\theta, s)$$

- Modification of the FBP filter in the Fourier domain:

$$\hat{\psi}(\theta, r) = |r| \quad \mapsto \quad \hat{\psi}_\Phi(\theta, r) = \kappa_\Phi(\theta)\,|r|$$

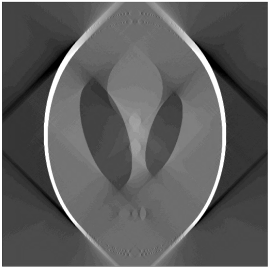
- Preconditioner for the limited angle Radon transform:

$$\mathcal{R}_\Phi \quad \mapsto \quad \mathcal{K}\mathcal{R}_\Phi$$

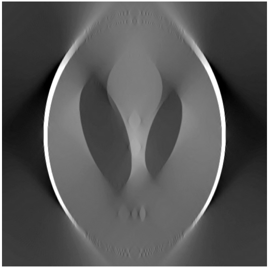


Original FBP artifact reduced FBP

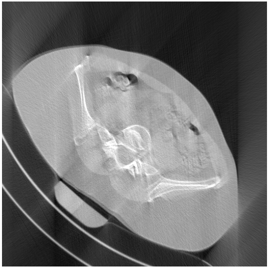
Original



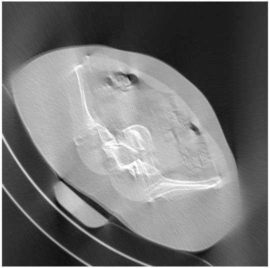
FBP



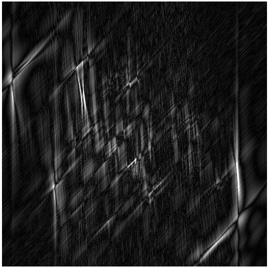
artifact reduced FBP



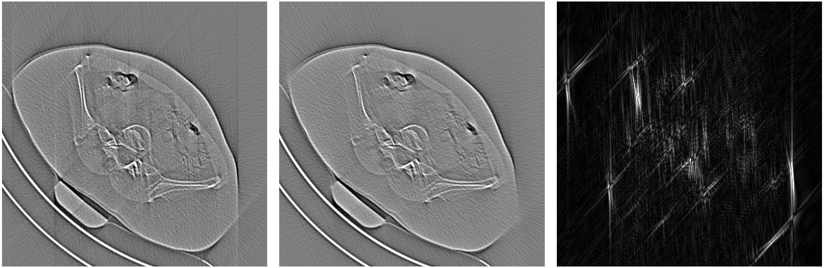
FBP



artifact reduced FBP



Difference



Lambda | artifact reduced Lambda | Difference

Thanks!