Lectures on microlocal characterizations in limited-angle tomography

Jürgen Frikel

DTU Compute
Department of Applied Mathematics and Computer Science

4 Lectures

1. *Nov. 11*: Introduction to the mathematics of computerized tomography
2. **Today:** Introduction to the basic concepts of microlocal analysis
3. *Nov. 25:* Microlocal analysis of limited angle reconstructions in tomography I
4. *Dec. 02:* Microlocal analysis of limited angle reconstructions in tomography I

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References:

L. Hörmander, *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*, vol. 256. Berlin: Springer-Verlag, 2003.

JF and E. T. Quinto, *Characterization and reduction of artifacts in limited angle tomography*, Inverse Problems 29(12):125007, December 2013.

We consider the Radon transform $\mathcal{R}$ in 2D and filtered backprojection inversion formulas.

Data: $g = \mathcal{R}f$.

- **Classical reconstruction:**

$$f(x) = \frac{1}{4\pi}\mathcal{R}^* I_s^{-1} g = \frac{1}{4\pi}\mathcal{R}^*(-\partial_s^2)^{1/2} g$$
$$= \frac{1}{4\pi}\int_{S^1} I_s^{-1} g(\theta, x\cdot\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{4\pi}\int_{S^1} [\psi *_s g](\theta, x\cdot\theta)\,\mathrm{d}\theta,$$

where the filter ψ is defined in the Fourier domain via $\widehat{\psi}(\sigma) = |\sigma|$.

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  where the filter ψ is defined in the Fourier domain via $\widehat{\psi}(\sigma) = |\sigma|$.

- **Lambda reconstruction:**

$$\Lambda f(x) = \frac{1}{4\pi}\mathcal{R}^*(-\partial_s^2) g = \frac{1}{4\pi}\int_{S^1} [\psi_\Lambda *_s g](\theta, x\cdot\theta)\,\mathrm{d}\theta$$

  where the filter ψ is defined in the Fourier domain via $\widehat{\psi}_\Lambda(\sigma) = |\sigma|^2$.

- In Lambda reconstruction a differential operator $(-\partial_s^2)$ is used as filter, which can be written in terms of the Fourier transform as follows (recall $\mathcal{F}(\partial_s g) = i\sigma \cdot \hat{g}$):

$$(-\partial_s^2)g(s) = \mathcal{F}^{-1}\left(|\sigma|^2\,\hat{g}(\sigma)\right) = (2\pi)^{-1/2}\int_{\mathbb{R}} |\sigma|^2\,\hat{g}(\sigma)e^{is\sigma}\,\mathrm{d}s.$$

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- In classical reconstructions a fractional power of a differential operator $(-\partial_s^2)^{-1/2}$ is used as filter. We have a similar representation in the Fourier domain

$$(-\partial_s^2)^{-1/2}g(s) = \mathcal{F}^{-1}\left(|\sigma|\,\hat{g}(\sigma)\right) = (2\pi)^{-1/2}\int_{\mathbb{R}} |\sigma|\,\hat{g}(\sigma)e^{is\sigma}\,\mathrm{d}s.$$

  However, the difference is huge as this operator is not local any more (involves Hilbert transforms).

  This is a so-called pseudodifferential operator.

General form FBP type inversion formulas

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  to data $g = \mathcal{R}f$.

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- **General FBP type reconstruction formulas use general pseudodifferential operators P as filters.**

Since FBP type reconstructions are so easy to implement and since the yield very fast reconstructions, the are applied to all kinds of tomographic data, in particular to incomplete data

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- Limited angle data
- Sparse angle data
- Interior data
- Exterior data
- . . .

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In limited angle tomography, the projectios $g_\theta = \mathcal{R}_\theta f$ are known only for certain directions $\theta \in S^1_\Phi \subsetneq S^1$, for other directions θ the projections $g_\theta = \mathcal{R}_\theta f$ are unknown.

FBP Reconstruction of limited angle data

$$\mathcal{R}^\dagger(\mathcal{R}_\Phi f)(x) = \frac{1}{4\pi}\int_{S^1_\Phi} [\psi *_s g](\theta, x\cdot\theta)\,\mathrm{d}\theta \quad = \quad ???$$

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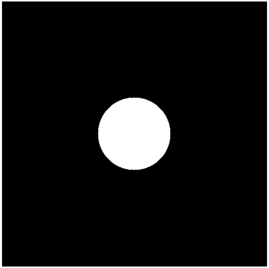
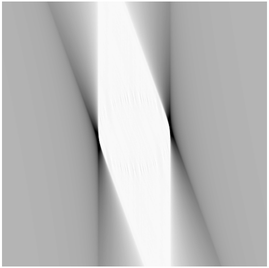
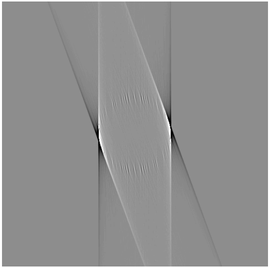
What do we reconstruct?



Original

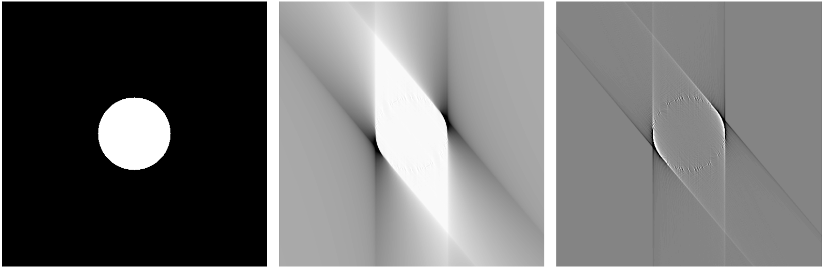


FBP



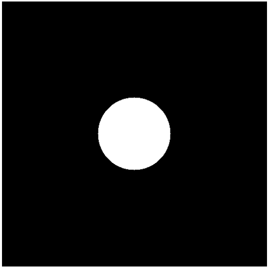
Lambda

$[0^\circ, 20^\circ]$

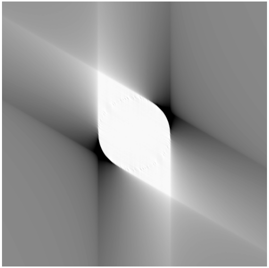


Original FBP Lambda

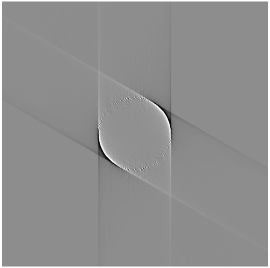
$[0^\circ, 40^\circ]$



Original

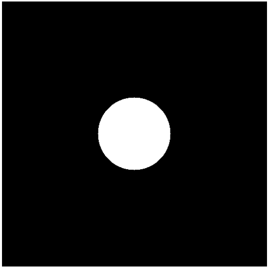


FBP



Lambda

$[0°, 60°]$



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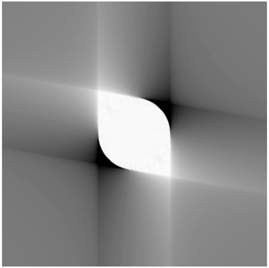

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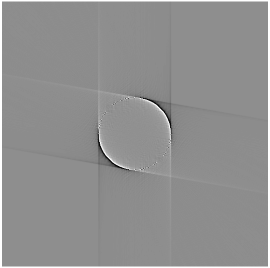
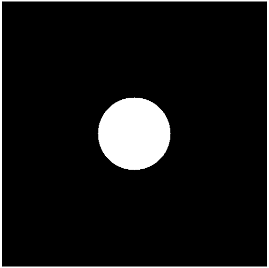


Lambda

$[0°, 80°]$



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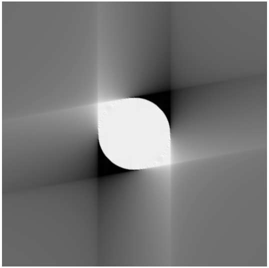
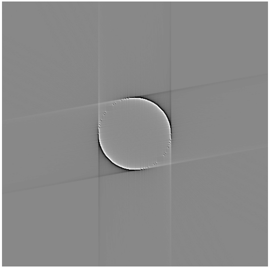
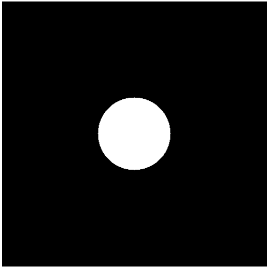


FBP



Lambda

$[0°, 100°]$



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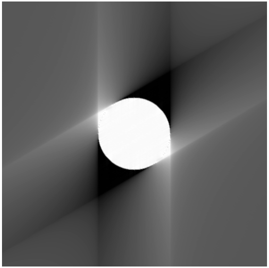
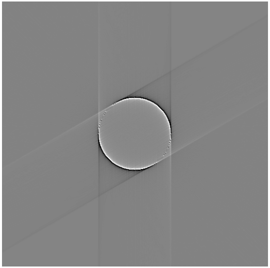
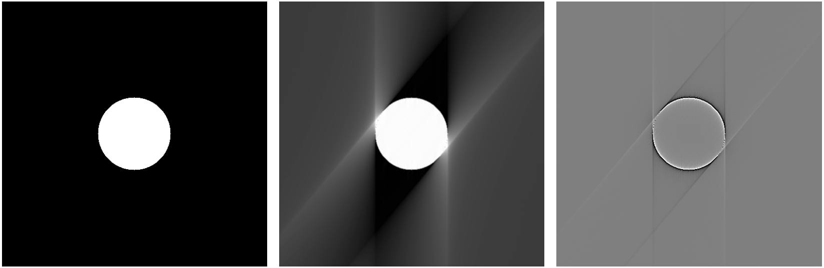


FBP



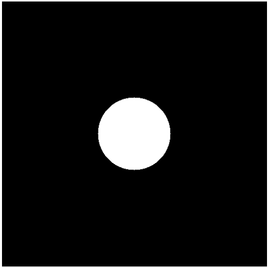
Lambda

$[0°, 120°]$



Original FBP Lambda

$[0°, 140°]$



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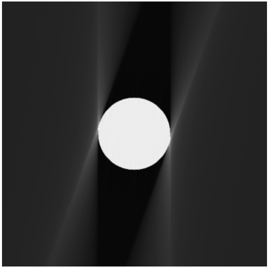
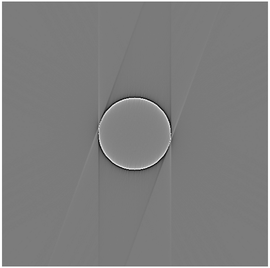
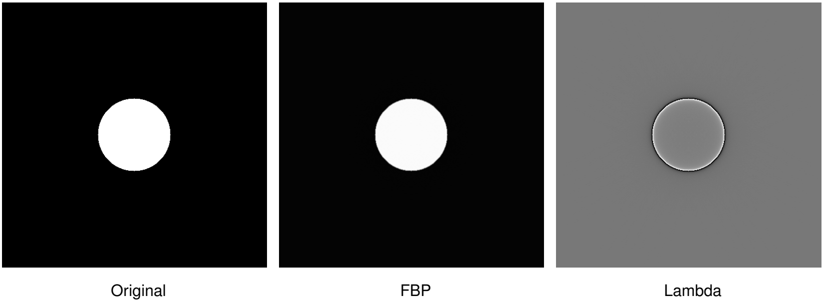


FBP



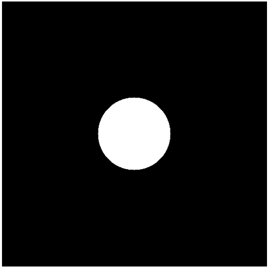
Lambda

$[0°, 160°]$



$[0°, 180°]$

Limited angle reconstructions: Experiment 2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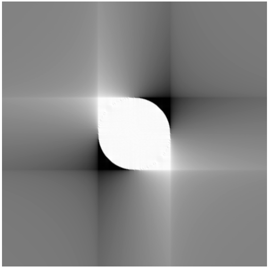
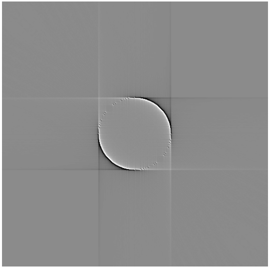
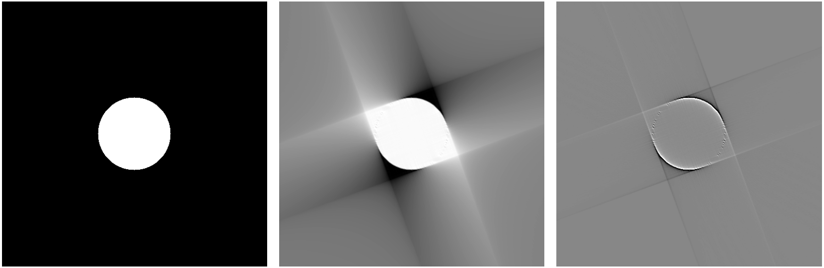

FBP

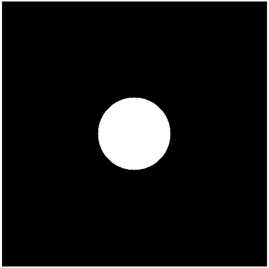

Lambda

$[0^\circ, 90^\circ]$

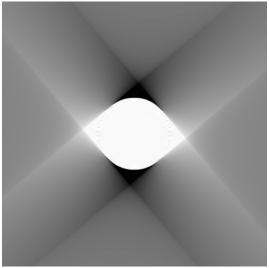


Original FBP Lambda

$[20^\circ, 110^\circ]$



Original



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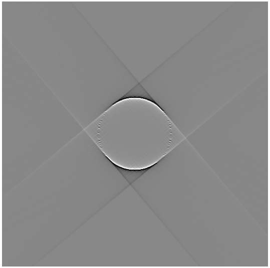
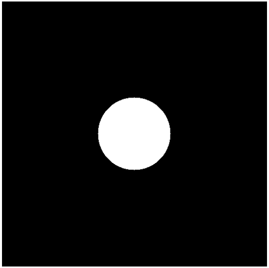


Lambda

$[40°, 130°]$



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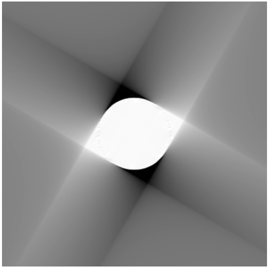
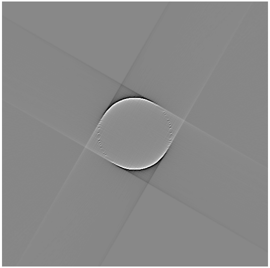
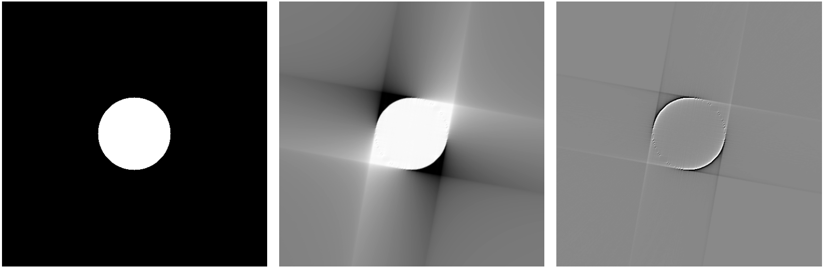


FBP



Lambda

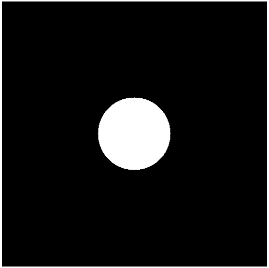
$[60°, 150°]$



Original FBP Lambda

$[80°, 170°]$

Limited angle reconstructions: Experiment 2



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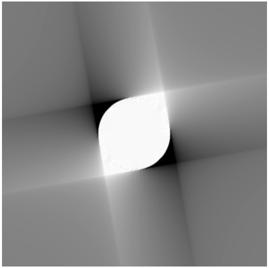
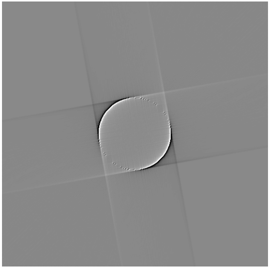


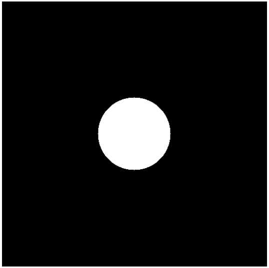
FBP



Lambda

$[100°, 190°]$

LIMITED ANGLE RECONSTRUCTIONS: EXPERIMENT 2



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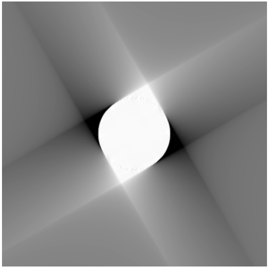
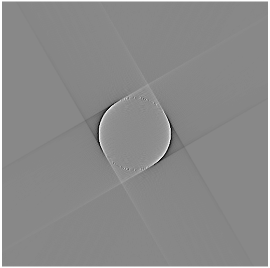
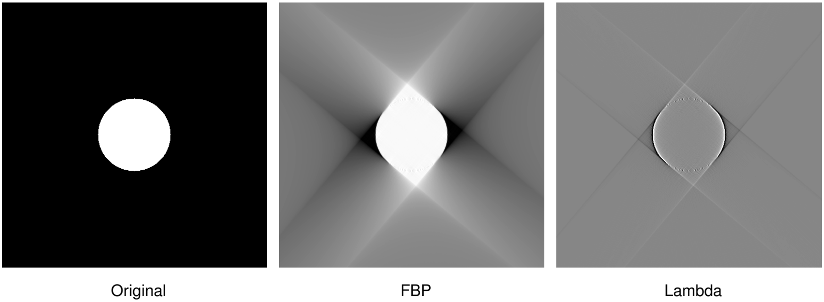


FBP

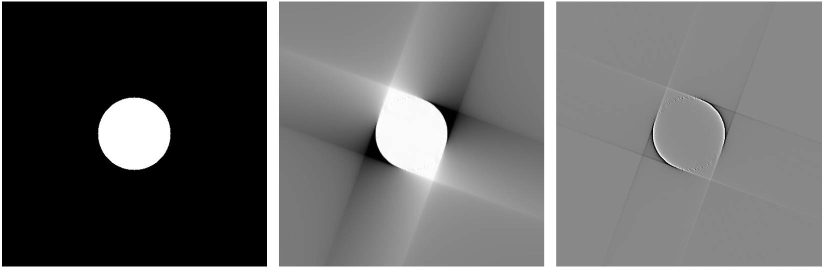


Lambda

$[120°, 210°]$



$[140°, 230°]$



Original FBP Lambda

$[160°, 250°]$

Observations at a first glance:

- Only certain features of the original object can be reconstructed
- Artifacts are generated

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Goal:

- Characterize image features that can be reliably reconstructed?
- Develop strategy to reduce artifacts?

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Observations at a second glance:

- Common characteristic features for both types of reconstructions are edges
- Information about edges is given in terms of projection directions

Practically: Density jumps, boundaries between regions, etc.

f

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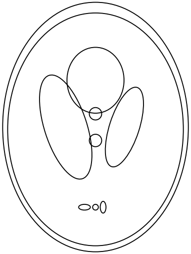
f

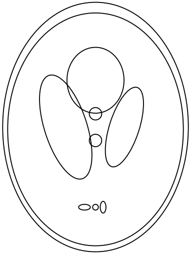


Singularities of f

Practically: Density jumps, boundaries between regions, etc.

Mathematically: Where the function is not smooth → singularities.

f



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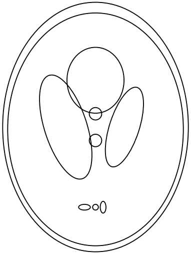
Mathematically: Where the function is not smooth $\rightarrow$ singularities.

Edges are local and oriented: What's an appropriate mathematical framework?

f



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Mathematically: Where the function is not smooth → singularities.

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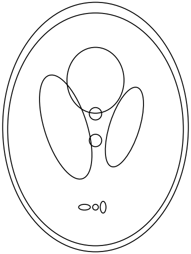
Image processing: Location of singularities = locations of high gradient magnitude;

Direction of singularities = gradient directions

f



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Mathematically: Where the function is not smooth $\rightarrow$ singularities.

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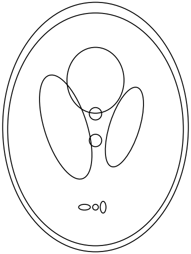
More powerful framework: Microlocal Analysis

f



Singularities of f

Today

- Global smoothness vs. decay of Fourier transform
- Singular locations
- Singular directions
- Wavefront sets: simultaneous description of singular locations and directions
- Examples

Global smoothness vs. decay of Fourier transform

In what follows, we only consider real functions $f : \mathbb{R}^2 \to \mathbb{R}$.

The smoothness of any function f can be charachterized by means of the decay of its Fourier transform $\hat{f}$:

$$f \in C^k \quad \Leftrightarrow \quad \hat{f}(\xi) = \mathcal{O}(|\xi|^{-k}) \text{ for } |\xi| \to \infty.$$

In particular,

$$f \in C^\infty \quad \Leftrightarrow \quad \hat{f}(\xi) = \mathcal{O}(|\xi|^{-k}) \text{ for all } k \in \mathbb{N}.$$

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The smoothness of any function f can be charachterized by means of the decay of its Fourier transform $\hat{f}$:

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Notation: A function which satisfies $f(x) = \mathcal{O}(|x|^{-k})$ for all $k \in \mathbb{N}$ is called **rapidly decreasing**. In all other cases **slowly decreasing**.

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- Above relations can be proven by using the property $\mathcal{F}(D^\alpha f) = i^{|\alpha|}\xi^\alpha \mathcal{F} f$
- Above relations also hold for distributions in $\mathcal{D}'$ and $\mathcal{S}'$
- If the Fourier transform of f has compact support (fastest decay one can imagine), then it can be shown that the function f is analytic (Paley-Wiener)
- One can also use the above relations to define smoothness of order $\alpha \in \mathbb{R}$ (Sobolev-Spaces)

Limitation:

- No information about locations or directions is provided.
- Global smoothness is related to a non-directional decay of the Fourier transform.

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Example: Let f be a function which is smooth except at the point $0 \in \mathbb{R}^2$, i.e. $f \in C^\infty(\mathbb{R}^2 \setminus 0)$. Let $y \in \mathbb{R}^2$ be arbitrary and let $f_y(x) = f(x-y)$. Then

$$\hat{f}_y(\xi) = e^{-i\xi y}\hat{f}(\xi)$$

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That is, we can generate functions f and f_y that have equal decay properties but arbitrarily different locations of jums, discontinuities, etc..

Hence, the decay of the Fourier transform does not provide information about the location of discontinuities.

How can we gain knowledge about locations of jump, discontinuieties, etc.?

What is a proper concept of a singularity that can be used in a distributional setting?

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Idea: Localize and study decay of the Fourier transforms!

Definition

- Let f be a function or distribution. We say that $x_0 \in \mathbb{R}^2$ is a regular point of f, if there exists a (cut-off) function $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi(x_0) \neq 0$ such that the Fourier transform $\mathcal{F}(\varphi f)$ decays rapidly (or in other words s.t. $\varphi f \in C^\infty(\mathbb{R}^2)$).

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- The complement of the (open) set of all regular points is called the singular support of f and is denoted by $\operatorname{sing\,supp}(f)$. It can be explicitly stated as

$$\operatorname{sing\,supp}(f) = \left\{x \in \mathbb{R}^2 : \ \forall \varphi \in C_c^\infty(\mathbb{R}^2), \varphi(x) \neq 0 : \varphi f \notin C^\infty(\mathbb{R}^2)\right\}.$$

- $\operatorname{sing\,supp}(f) \subset \operatorname{supp}(f)$
- $\operatorname{sing\,supp}(f) = \emptyset$ iff $f \in C^\infty(\mathbb{R}^2)$

Singular directions

So far: Locations of singularities are given by the singular support and $x \in \operatorname{sing\,supp} f$ implies that $\mathcal{F}(\varphi f)$ decays slowly for any cut-off function $\varphi(x_0) \neq 0$.

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$$\Sigma(f) = \left\{\xi \in \mathbb{R}^2 \setminus 0 : \ \forall \text{conic neighb. } N \text{ of } \xi : \ \hat{f} \text{ decays slowly in } N\right\}.$$

- $\Sigma(f) = \emptyset$ iff $f \in C^\infty(\mathbb{R}^2)$
- Singular directions only exist if singularities exist and vice versa.
- For any compactly supported distribution f and any $\varphi \in C_c^\infty(\mathbb{R}^2)$ we have

$$\Sigma(\varphi f) \subset \Sigma(f),$$

i.e., multiplication by compactly supported smooth function does not add singular directions.

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- $\Sigma_x(f)$ is the set of singular directions of f at location x

Definition

The wavefront set of a distribution is the set of all tuples (x, ξ) where x is a singular location of f and ξ is a singular direction of f at x. That is,

$$\mathrm{WF}(f) = \left\{(x, \xi) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0) : \ \xi \in \Sigma_x(f)\right\}.$$

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WF simultaneously describes locations and directions of singularities:

$$\pi_1(\mathrm{WF}(f)) = \text{sing supp}(f), \qquad \pi_2(\mathrm{WF}(f)) = \Sigma(f),$$

where $\pi_1(x, \xi) = x$ and $\pi_2(x, \xi) = \xi$.

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A characterization of WF in terms of the complement:

$$(x_0, \xi_0) \in \mathrm{WF}(f) \quad \Leftrightarrow \quad \exists \varphi \in C_c^\infty(\mathbb{R}^2),\ \varphi(x_0) \neq 0 : \; \xi_0 \notin \Sigma(\varphi f)$$

$$\Leftrightarrow \quad \exists \varphi \in C_c^\infty(\mathbb{R}^2),\ \varphi(x_0) \neq 0 : \; \exists \text{ conic neighb. } N(\xi_0) : \; \forall k \in \mathbb{N} : \; \mathcal{F}(\varphi f) = \mathcal{O}(|\xi|^{-k})$$

Examples

Example:

$\Omega \subset \mathbb{R}^2$ such that the boundary $\partial\Omega$ is a smooth manifold:

$$(x, \xi) \in WF(\chi_\Omega) \quad \Leftrightarrow \quad x \in \partial\Omega, \text{ and } \xi \in N_x,$$

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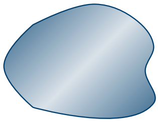
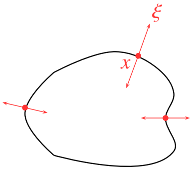


f distribution with

$\operatorname{supp}(f) \subset D$



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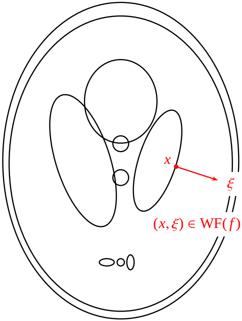

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What happens at crossings and corners?

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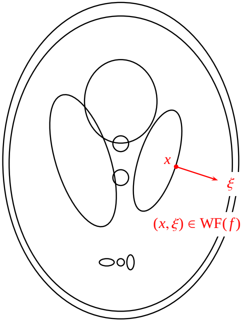

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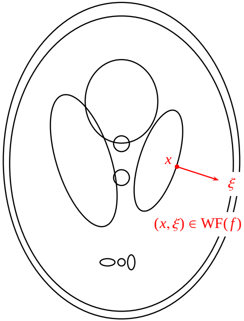
What happens at crossings and corners?

If x is a corner or a crossing singularity (or of similar nature), $\Sigma_x(f) = \mathbb{R}^2 \setminus 0$.

f



Singularities of f

Theorem (Pseudolocal property of ΨDO's)

For any pseudodifferential operator P, we have

$$\mathrm{WF}(Pf) \subset \mathrm{WF}(f).$$

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▹ Pseudodifferential operators do not create new singularities!

Example:

The Riesz-Potentials

$$I^\alpha f = (-\Delta)^{-\alpha/2} f = \mathcal{F}^{-1}(|\xi|^{-\alpha}\,\hat{f}(\xi))$$

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- Pseudodifferential operators do not create new singularities!
- The result is also valid for differential operators P as they are special cases of ΨDo's.
- The result can be also used in order to compute WF, e.g. for Heaviside function H we have $H' = \delta$ so $\mathrm{WF}(H) = \mathrm{WF}(\delta) = \{0\}$.

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  WF does not quantify the strength of singularities. To quantify the strength of singularities, a refined notion of the wavefront set can be defined (Sobolev wavefront set).

- For general FBP type reconstruction operators $\mathcal{R}^* P$ with general pseudodifferential operators P as filters, a minimum requirement on the filter can be formulated as

$$\mathrm{WF}(\mathcal{R}^* P \mathcal{R} f) \stackrel{!}{=} \mathrm{WF}(f).$$

  In general, this is not even if P is elliptic.

Thanks!

Next week: Microlocal analysis in limited angle tomography