Lectures on microlocal characterizations in limited-angle tomography

Jürgen Frikel

DTU Compute
Department of Applied Mathematics and Computer Science

1. **Today:** Introduction to the mathematics of computerized tomography

4 Lectures

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2. **Nov. 18:** Introduction to the basic concepts of microlocal analysis

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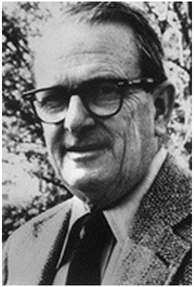
References:

F. Natterer, *The mathematics of computerized tomography*. Stuttgart: B. G. Teubner, 1986.

JF and E. T. Quinto, *Characterization and reduction of artifacts in limited angle tomography*, Inverse Problems 29(12):125007, December 2013.

Image taken from www.wikipedia.org

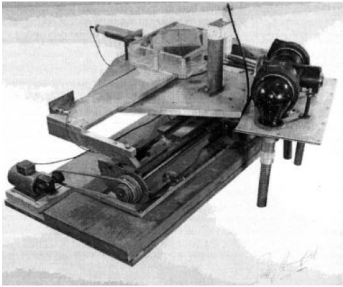
Johan Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte Längs gewisser Manningsfaltigkeiten*, Berichte Sächsische Akadamie der Wissenschaften, Leipzig, Math.-Phys. Kl., 69, pp. 262- 277, 1917



Allan Cormack

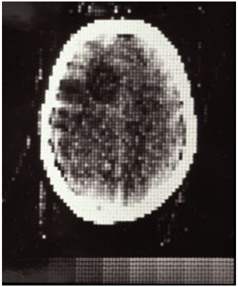
Godfrey Hounsfield

The **Nobel Prize in Physiology or Medicine 1979** was awarded jointly to Allan M. Cormack and Godfrey N. Hounsfield **for the development of computer assisted tomography**



First scanner, ≈ \$300

Modern scanner, > \$1 million



First clinical scan 1971

Modern scan*

*Case courtesy of Dr Maxime St-Amant, Radiopaedia.org

Analytical approach to computerized tomography

- Principle of tomography
- Radon transform - a mathematical model of tomography
- Reconstruction via backprojection
- Fourier slice theorem
- Inversion formulas & Filtered backprojection
- Ill-posedness & regularization

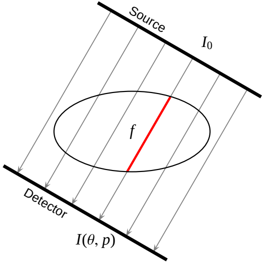
Beer Lambert law

X-rays are attenuated when traveling through object according to

$$\begin{cases} \dfrac{\mathrm{d}I}{\mathrm{d}t} = -f(\gamma(t)) \cdot I(t) \quad \text{for } t \in \mathbb{R} \\ I(0) = I_0 \end{cases}$$

f = **attenuation coefficient**, γ = **x-ray path**

$\gamma(t) = s \cdot \theta + t \cdot \theta^\perp$ = line with direction $\theta^\perp$ starting at $x_{\text{detector}} = s\theta$



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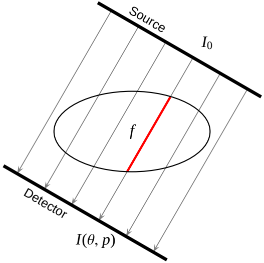

Solution of the initial value problem is given by

$$I(\theta, s) = I_0 \cdot \exp\left\{- \int_0^{t_{\text{detector}}} f(s \cdot \theta + t \cdot \theta^{\perp}) \, dt\right\}$$

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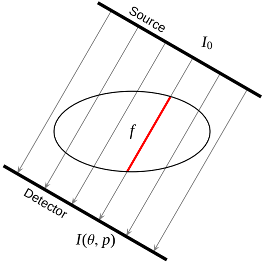

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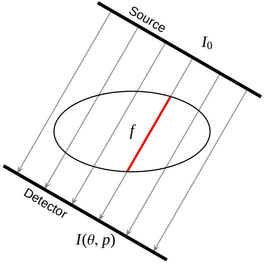

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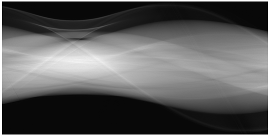
Mathematical model of the measurement process

$$\mathcal{R}f(\theta, s) = \int_{L(\theta,s)} f(x)\ \mathrm{d}x = \ln\left(\frac{I_0}{I(\theta, s)}\right)$$



Data / Sinogram

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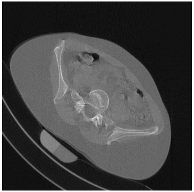
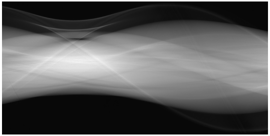


Sought image

Mathematical problem

Solve the integral equation $y = \mathcal{R}f$



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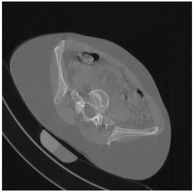

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Solve the integral equation $y = \mathcal{R} f$

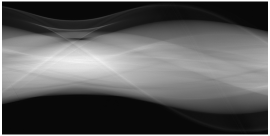
I. Algebraic reconstruction

- Fully discretize formulation of the problem

$\rightsquigarrow$ Linear system of equations $Rx = y$

This is an algebraic problem!

Examples: ART, SART, SIRT, statistical reconstruction methods such as ML-EM, variational methods, etc.



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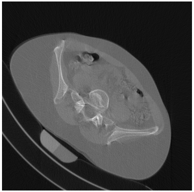

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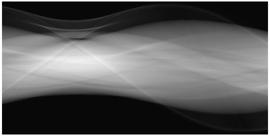
- Study of the continuous problem (above)
- Derivation of inversion formulas
- Discretization of analytic reconstruction formulas

Examples: Filtered backprojection (FBP), Fourier inversion, etc.



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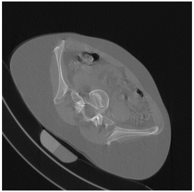

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(1) $$\mathcal{R}f(\theta, s) = \int_{H(\theta,s)} f(x)\,\mathrm{d}\sigma(x), \qquad (\theta, s) \in S^{n-1} \times \mathbb{R},$$

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- Continuity of the Radon transform = Stability of the measurement process

  $\mathcal{R}$ is continuous on many standard function spaces, such as $L^1(\mathbb{R}^n)$, $L^2(\Omega)$, $\mathcal{S}(\mathbb{R}^n)$ and many distributional spaces.

$f : \mathbb{R}^2 \to \mathbb{R}$ is radial if the value $f(x)$ only depends on $\|x\|$, i.e., if there is a function $\varphi : [0, \infty) \to \mathbb{R}$ such that $f(x) = \varphi(\|x\|)$.

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The Radon transform of radial functions is independent of θ!

$\rightsquigarrow$ 1 Projection enough to reconstruct a radial function

Example: $f(x) = \chi_{B(0,1)}(x) = \chi_{[0,1]}(\|x\|)$:

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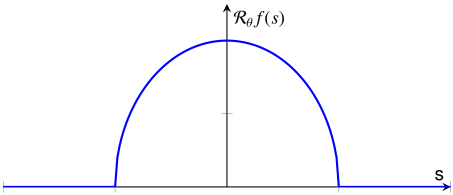

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- The operator $\mathcal{R}^*$ is the L^2 Hilbert space adjoint of the Radon transform $\mathcal{R}$, i.e., for $f \in L^2(\Omega)$ and $g \in L^2(S^{n-1} \times \mathbb{R})$

$$\int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}f(\theta, s) g(\theta, s) \, \mathrm{d}\theta \, \mathrm{d}s = \int_{\mathbb{R}^n} f(x) \mathcal{R}^* g(x) \, \mathrm{d}x.$$

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- As an adjoint operator, $\mathcal{R}^*$ is **continuous** whenever $\mathcal{R}$ is

Theorem

For $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$\mathcal{R}^*\mathcal{R}f(x) = \left|S^{n-2}\right| \cdot \left(f * \frac{1}{\|\cdot\|}\right)(x)$$

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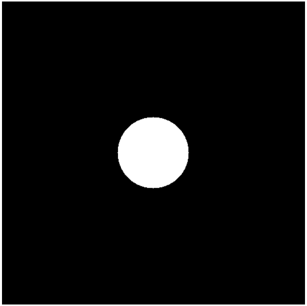

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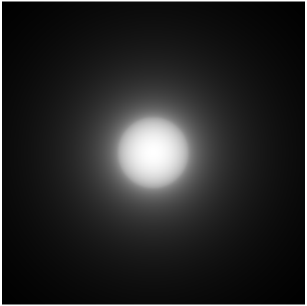

Backprojection reconstruction

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- Backprojection reconstruction produces a function that is smoother than the original function (by 1 order): high frequencies are damped

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- Using the above theorem, the normal equation $\mathcal{R}^*\mathcal{R}f = \mathcal{R}^*g$ (up to a constant) reads

$$f * \frac{1}{\|\cdot\|} = \mathcal{R}^*g, \qquad \text{for data} \quad g = \mathcal{R}f$$

$\rightarrow$ tomographic reconstruction can be interpreted as a deconvolution problem

Fourier transform turns out to be a very useful tool for studying the Radon transform.

Definition (Fourier transform and its inverse)

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Sometimes it's useful to calculate the 1D Fourier transform of the projection function $g(\theta, s) = \mathcal{R}f(\theta, s)$ with respect to the second variable s. To make that clear, we will write

$$\mathcal{F}_s g(\theta, \sigma) = (2\pi)^{-1/2} \int_{\mathbb{R}} g(\theta, s) e^{-is\cdot\sigma} \, \mathrm{d}s$$

for the Fourier transform of $g(\theta, s)$ with respect to the variable s (here θ is considered to be a fixed parameter). Whenever we write $\widehat{\mathcal{R}f}(\theta, \sigma)$ the Fourier transform of $\mathcal{R}f$ has to be understood in that sense. Same holds for the inverse Fourier transform.

Theorem (Fourier slice theorem)

Let $f \in \mathcal{S}(\mathbb{R}^n)$. *Then, for* $\sigma \in \mathbb{R}$,

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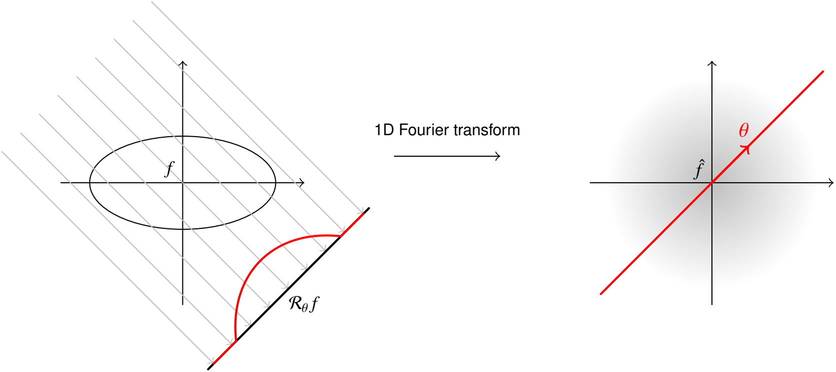

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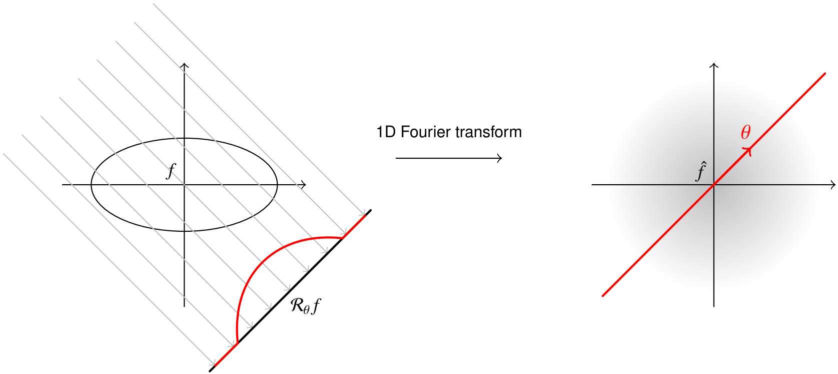

This theorem can be used derive a reconstruction procedure $\rightsquigarrow$ **Fourier reconstructions**

Many proprities of the Radon transform can be derived from the properties of the Fourier transform.

Theorem

The Radon transform $\mathcal{R} : L^1(\mathbb{R}^n) \to L^1(S^{n-1} \times \mathbb{R})$ *is an* ***injective*** *operator.*

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Proof.

Suppose $\mathcal{R}f \equiv 0$ for $f \in L^1(\mathbb{R}^n)$. Then, the Fourier slice theorem implies

$$\widehat{f}(\sigma\theta) = (2\pi)^{(1-n)/2}\mathcal{F}_s\mathcal{R}f(\theta,\sigma) = 0$$

for all $(\theta,\sigma) \in S^{n-1} \times \mathbb{R}$. Hence,

$$\widehat{f} \equiv 0$$

and the injectivity of the Fourier transform implies that $f \equiv 0$. □

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  Unfortuntely, NO. It can be shown that $\mathcal{R}^{-1}$ is not a continuous operator and, hence, that the reconstruction problem $\mathcal{R}f = y$ is ill-posed. However, the ill-posedness is mild (maybe later).

For $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha < n$ we define the **Riesz potential** (which is a linear operator) via

$$I^\alpha f = (-\Delta)^{-\alpha/2} f = \mathcal{F}^{-1}\left(|\xi|^{-\alpha}\,\widehat{f(\xi)}\right).$$

For $g \in \S(S^{n-1} \times \mathbb{R})$ we analogously define the Riesz-potential with respect to the second variable

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Lef $f \in \mathcal{S}(\mathbb{R}^n)$. Then, for any $\alpha < n$, the following inversion formulas hold:

$$f = \frac{1}{2}(2\pi)^{1-n} I^{-\alpha}\mathcal{R}^* I_s^{\alpha-n+1}\mathcal{R}f.$$

$n = 3$ and $\alpha = 2$: $$f = -\frac{1}{8\pi^2} \cdot \Delta \mathcal{R}^* \mathcal{R} f$$

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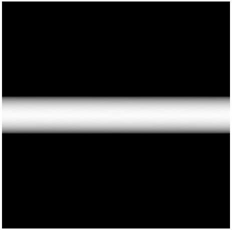
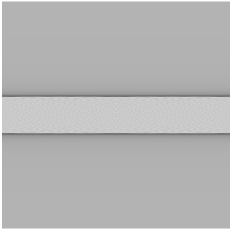


Sinogram



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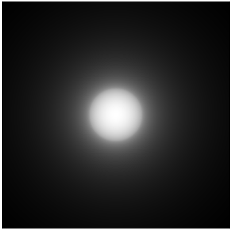
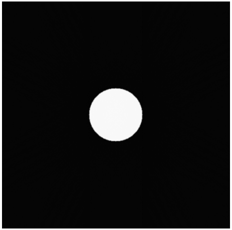


Backprojection



Filtered backprojection

For $\alpha = 0$ we obtain a classical filtered backprojection form

$$f = \frac{1}{2}(2\pi)^{1-n}\mathcal{R}^* I_s^{1-n}\mathcal{R}f,$$

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\end{aligned}$$

where H is defined in the Fourier domain via

$$\widehat{H}g(\sigma) = -i\,\operatorname{sgn}(\sigma)\cdot\widehat{g}(\sigma).$$

INVERSION FORMULAS IN EVEN AND ODD DIMENSIONS

Now observe that for $n \geq 2$ we have

$$(-i\,\mathrm{sgn}(\sigma))^{n-1} = \begin{cases} (-1)^{(n-2)/2} \cdot (-i\,\mathrm{sgn}(\sigma)), & \text{for } n \text{ even} \\ (-1)^{(n-1)/2}, & \text{for } n \text{ odd} \end{cases}$$

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Therefore

$$f = \frac{1}{2}(2\pi)^{1-n}\mathcal{R}^* \cdot \begin{cases} (-1)^{(n-2)/2} \cdot H\,\partial_s^{n-1}\,\mathcal{R}f(\theta, x\cdot\theta), & \text{for } n \text{ even} \\ (-1)^{(n-1)/2} \cdot \partial_s^{n-1}\,\mathcal{R}f(\theta, x\cdot\theta), & \text{for } n \text{ odd} \end{cases}$$

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- If n is odd, the filter is a differential operator and, hence, a local operator. As a result, the inversion formula is local, i.e., the inversion formula can be evaluated at point x if the data $\mathcal{R}f(\theta, y\cdot\theta)$ is known for all $\theta \in S^{n-1}$ and for y in a neighbourhood of x.

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- If n is even, the inversion formula is **not** local, since H is an integral operator

$$Hg(s) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{g(t)}{s-t}\,\mathrm{d}t$$

  H is the so-called **Hilbert transform**.

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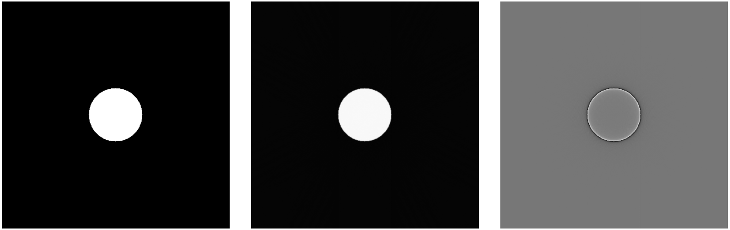
- Instead of reconstructing f we reconstruct $\Lambda f := I^1 f$.
- This formula is local and of filtered backprojection type.
- In the implementation, the standard FBP filter $|\sigma|$ has to be replaces by $|\sigma|^2$ $\rightarrow$ high frequecies are amplified even more, low frequencies are damped.
- Instead of values $f(x)$, it reconstructs the singularities of f.



Original FBP Lambda

Reconstruction is mildly ill-posed (of order (n-1)/2)

$$c\,\|f\|_{H_0^\alpha} \leq \|\mathcal{R}f\|_{H^{\alpha+(n-1)/2}} \leq C\,\|f\|_{H_0^\alpha}$$

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- Regularizationation by replacing the filtered with a band-limited version:

$$|\sigma|^{n-1} \mapsto w(\sigma)\cdot|\sigma|^{n-1}$$

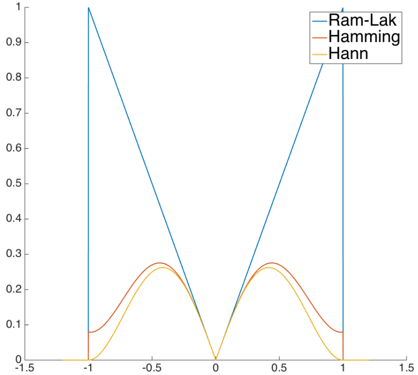
For $\alpha > 0$, let $\omega_\alpha : (-1/\alpha, 1/\alpha) \to [0,\infty)$ be smooth such that $\omega_\alpha(\sigma) \to \sigma$ as $\alpha \to 0$ ($\forall\sigma$), and set $\psi_\alpha(\sigma) := \mathcal{F}^{-1}(\omega_\alpha(\sigma)\cdot|\sigma|)$.

Stabilized FBP inversion

$$R_\alpha(g)(x) := \frac{1}{4\pi}\int_{S^1}(g_\theta * \psi_\alpha)(x\cdot\theta)\,\mathrm{d}\sigma(\theta)$$

Remark

$f_\alpha = R_\alpha(g)$ is a “low-pass filtered version of f”



Plot of $\hat{\psi}_\alpha = \omega_\alpha(\sigma)\cdot|\sigma|$

See you next week!