ON THE USE OF HIGHLY DIRECTIONAL REPRESENTATIONS IN INCOMPLETE DATA TOMOGRAPHY

Jürgen Frikel

Insights and algorithms for incomplete data tomography
DTU Compute

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OSTBAYERISCHE
TECHNISCHE HOCHSCHULE
REGENSBURG

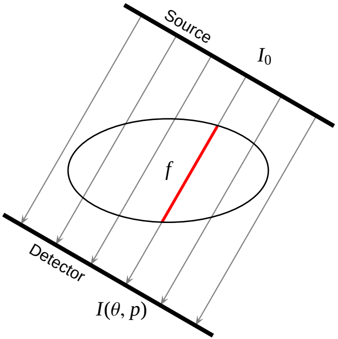
INFORMATIK UND
MATHEMATIK

Outline

- Incomplete data in tomography
- Microlocal Analysis
- Microlocal characterization of incomplete data reconstructions
- Use of directional representations in incomplete data tomography

X-ray tomography: Classical Radon transform

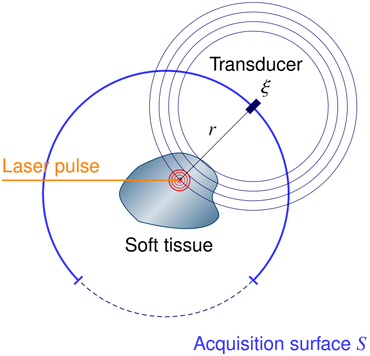
$$\mathcal{R}f(\theta, p) = \int_{\mathbb{R}} f(p\theta + t\theta^{\perp})\,\mathrm{d}t = \ln\left(\frac{I_0}{I(\theta, p)}\right)$$



Notation: $p \in \mathbb{R}$, $\theta = (\theta_1, \theta_2) \in S^1$ and $\theta^{\perp} = (-\theta_2, \theta_1)$

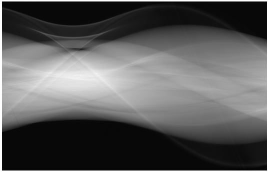
Photoacoustic tomography: Spherical Radon transform

$$\mathcal{M}f(\xi, r) = \int_{S^1} f(\xi + r\zeta)\, \mathrm{d}\zeta$$

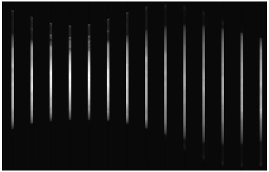


Notation: $r > 0$, $\theta = (\theta_1, \theta_2) \in S^1$

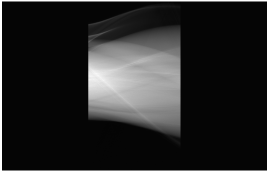
Incomplete data in tomography



full data



sparse angle data



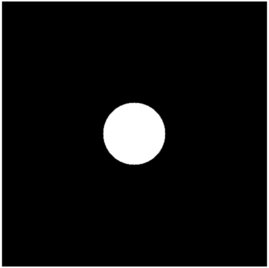
limited angle data

Breast tomosynthesis

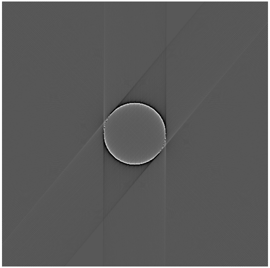
Dental CT

Electron microscopy

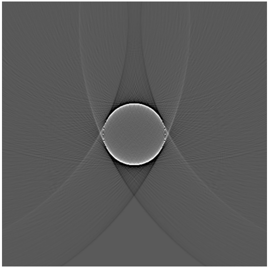
... and many more



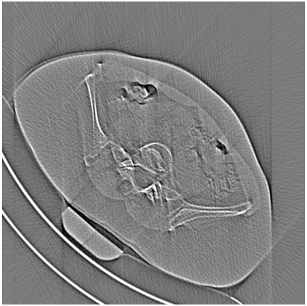
Original



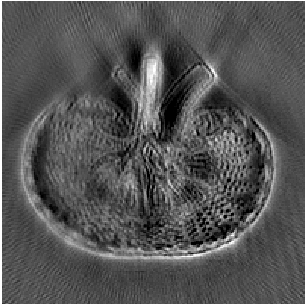
X-ray tomography, $[0°, 140°]$



Photoacoustic tomography, $[-45°, 225°]$



X-ray tomography[1], $[0°, 140°]$



Photoacoustic tomography[2], $[-45°, 225°]$

Data by courtesy of [1]Department of Diagnostic and Interventional Radiology, TUM and [2]Helmholtz Zentrum München

Observations:

- Only certain features of the original object can be reconstructed,
- Artifacts are generated.

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Need mathematical tools to implement these insights into alogrithms

- Applied harmonic analysis provides highly directional and numerically efficient representations.

- JF, E. T. Quinto, *Limited data problems for the generalized Radon transform in* $\mathbb{R}^n$, SIAM J. Math. Anal., 2016.
- JF, E. T. Quinto, *Artifacts in incomplete data tomography – with applications to photoacoustic tomography and sonar*, SIAM J. Appl. Math., 2015.
- JF, E. T. Quinto, *Characterization and reduction of artifacts in limited angle tomography*, Inverse Problems, 2013.
- JF, L. V. Nguyen, L. Barannyk, *On artifacts in limited data spherical Radon transform: curved observation surfaces*, Inverse Problems 2015.
- L. V. Nguyen, *How strong are streak artifacts in limited angle computed tomography?*, Inverse Problems, 2015.
- L. V. Nguyen, *On artifacts in limited data spherical Radon transform: flat observation surfaces*, SIAM J. Math. Anal. (2015)
- A. I. Katsevitch, *Local tomography for the limited-angle problem*, J. Math. Anal. Appl., 1997.
- E. T. Quito, *Singularities of the x-ray transform and limited data tomography in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, SIAM J. Math. Anal., 1993.

- *Microlocal analysis, visible singularities and artifacts in other tomography problems*:
  G. Ambartsoumian, R. Felea, D. V. Finch, A. Greenleaf, V. Guillemin, A. Katsevich, V. P. Krishnan, I.-R. Lan, P. Kuchment, C. Nolan, V. Palamodov, E. T. Quinto, A. Ramm, H. Rullgard, P. Stefanov, G. Uhlmann, . . .

Practically: Density jumps, boundaries between regions

Mathematically: Where the function is not smooth...

Paradigm: Fourier transform of f decays rapidly at ∞ iff f is smooth.

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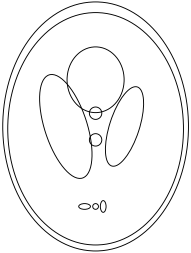

Singularities of f

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Mathematically: Where the function is not smooth...

Paradigm: Fourier transform of f decays rapidly at ∞ iff f is smooth.

Singularities are local and oriented! $\rightsquigarrow$ Wavefront set: localize & microlocalize

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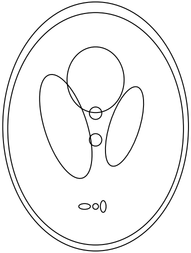

Singularities of f

Definition (Wavefront set)

A tuple $(x_0, \xi_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$ is **not** in the **wavefront set** $\mathrm{WF}(f)$ of $f \in \mathcal{D}'(\mathbb{R}^2)$ iff

- there is a cut-off function $\varphi \in \mathcal{D}(\mathbb{R}^2)$, $\varphi(x_0) \neq 0$, (Localize at x_0)
- there is a conic neighborhood $\mathcal{N}(\xi_0)$, (Microlocalize at ξ_0)

such that $\mathcal{F}(\varphi f)$ **decays rapidly** in $\mathcal{N}(\xi_0)$.

(Hörmander '90)

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WF simultaneously describes locations and directions of a singularity

Example:
$\Omega \subset \mathbb{R}^2$ such that the boundary $\partial\Omega$ is a smooth manifold:

$$(x, \xi) \in WF(\chi_\Omega) \quad \Leftrightarrow \quad x \in \partial\Omega, \ \text{and} \ \xi \in N_x,$$

where N_x is the normal space to $\partial\Omega$ at $x \in \partial\Omega$.

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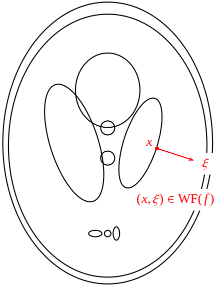

Singularities of f

Forward operator is a Fourier Integral operator (FIO)

$$T : \mathcal{E}'(\Omega) \to \mathcal{E}'(\Xi),$$

where Ω is the object space and Ξ is the data space.

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Reconstruction problem: Recover f (or singularities f) from the data $g = Tf$

- Limited data: $g(y)$ known only for $y \in A \subsetneq \Xi$ (χ_A = characteristic function of A)
- Limited data forward operator: $T_A f = \chi_A T f$

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Reconstruction operators (FBP type):

$$Bg_A = T^* P g_A, \quad g_A = T_A f$$

P is a pseudodifferential operator and T^* dual (or backprojection) operator.

Theorem (Quinto, JF 2013-2015)

Let $T \in \{R, M\}$, $f \in \mathcal{E}'(\Omega)$, and let P be a pseudodifferential operator on $\mathcal{D}'(\Xi)$. Then,

$$\mathrm{WF}(T^* P T_A f) \subset \mathrm{WF}_{[a,b]}(f) \cup \mathcal{A}_{\{a,b\}}(f).$$

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Visible singularities for R:

$$\mathrm{WF}_{[a,b]}(f) := \{(x,\xi) \in \mathrm{WF}(f) : \ \xi = \alpha\theta(\varphi), \alpha \neq 0, \varphi \in [a,b]\}$$

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Added singularities for R:

$$\mathcal{A}_{[a,b]}(f) = \{(x + t\theta^{\perp}(\varphi), \alpha\theta(\varphi)\,\mathrm{d}x) : \ \varphi \in \{a,b\}, \alpha, t \neq 0, x \in L(\varphi, s), \ (x, \alpha\theta(\varphi)) \in \mathrm{WF}(f)\}$$

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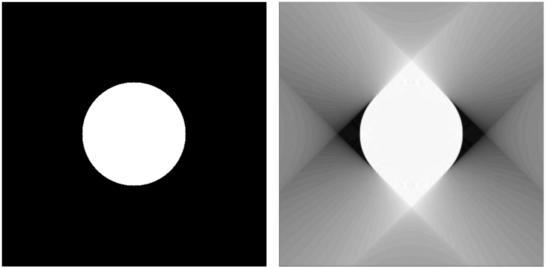
Theorem (Quinto, JF)

Under additional assumptions on P we have

$$\mathrm{WF}_{(a,b)}(f) \subset \mathrm{WF}(T^* P T_A f) \subset \mathrm{WF}_{[a,b]}(f) \cup \mathcal{A}_{\{a,b\}}(f).$$

▹ Visible singularities are characterized in terms of their orientation

Only singularities $(x, \theta(\varphi)) \in \mathrm{WF}(f)$ can be reconstructed for which $\varphi \in [a, b]$



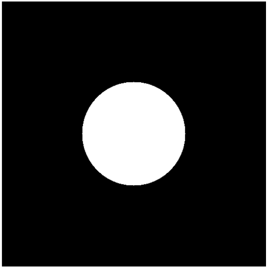
Original Reconstruction

- Visible singularities are characterized in terms of their orientation

  Only singularities $(x, \theta(\varphi)) \in \mathrm{WF}(f)$ can be reconstructed for which $\varphi \in [a, b]$

- Artifacts are spread along lines having orientations corresponding to the boundary of the angular range, $\theta(a)$ or $\theta(b)$, respectively

  Streaks are added at location x whenever $(x, \theta(a)) \in \mathrm{WF}(f)$ or $(x, \theta(b)) \in \mathrm{WF}(f)$



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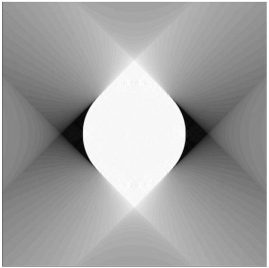
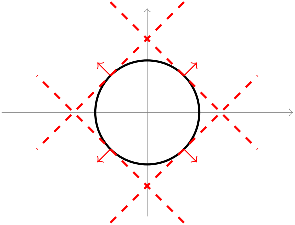


Reconstruction



$\mathcal{A}_{[-45°,45°]}$

- Microlocal characterisations provide insight into the information content of incomplete data.

- X-ray tomography:
  - Reliably reconstructed singularities are $(x, \theta(\varphi)) \in \mathrm{WF}(f_{\mathrm{rec}})$ with $\varphi \in (a, b)$,
  - Any singularity $(x, \theta(\varphi)) \in \mathrm{WF}(f_{\mathrm{rec}})$ with $\varphi \notin \{a, b\}$ can be an added streak artifact.

- To avoid generation of artifacts we need to make sure that

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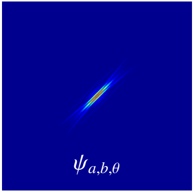
$\rightsquigarrow$ Shearlets, curvelets, or similar transforms

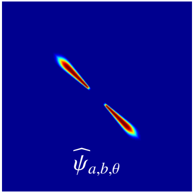
- The shearlet / curvelet dictionary

$$\{\psi_{a,b,\theta}\}_{(a,b,\theta)\in I}$$

simultaneously localize at location a and along direction θ.

$$a = \text{scale}, \quad b = \text{location}, \quad \theta = \text{orientation}.$$





(Candes, Donoho, Kutyniok, Lemvig, Lim, Grohs, Guo, Labate, Easley, . . .)

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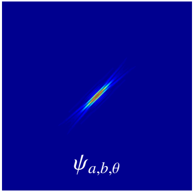

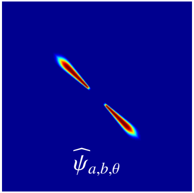

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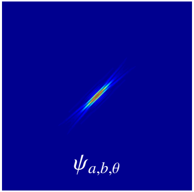

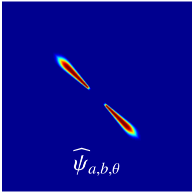

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- Optimally sparse representation of edges (cartoon images)

(Candes, Donoho, Kutyniok, Lemvig, Lim, Grohs, Guo, Labate, Easley, ...)

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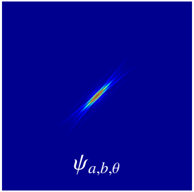

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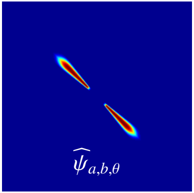

- Optimally sparse representation of edges (cartoon images)

Theorem (Resolution of the wavefront set)

$(b,\theta) \notin \mathrm{WF}(f) \quad \Leftrightarrow \quad \langle f, \psi_{(a,b,\theta)} \rangle$ *decays rapidly as* $a \to 0$

(Candes, Donoho, Kutyniok, Lemvig, Lim, Grohs, Guo, Labate, Easley, . . .)

Definition (Visible coefficients)

We define the index set of visible coefficients at limited angular range $[a, b]$ as

$$I_{[a,b]} = \{(a, b, \theta) \in I : \ \theta \in [a, b]\}.$$

Coefficients with $(a, b, \theta) \in I \setminus I_{[a,b]}$ are called invisible at limited angular range $[a, b]$.

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Coefficients with $(a, b, \theta) \in I \setminus I_{[a,b]}$ are called invisible at limited angular range $[a, b]$.

Decomposition into a visible and an invisible part

$$f = \sum_{(a,b,\theta)\in I_{[a,b]}} \left\langle f, \psi_{(a,b,\theta)} \right\rangle \psi_{(a,b,\theta)} + \sum_{(a,b,\theta)\in I \setminus I_{[a,b]}} \left\langle f, \psi_{(a,b,\theta)} \right\rangle \psi_{(a,b,\theta)}$$

$$= f_{\text{visible}} + f_{\text{invisible}}.$$

OTH IM

Definition (Visible coefficients)

We define the index set of visible coefficients at limited angular range $[a, b]$ as

$$I_{[a,b]} = \{(a, b, \theta) \in I : \ \theta \in [a, b]\}.$$

Coefficients with $(a, b, \theta) \in I \setminus I_{[a,b]}$ are called invisible at limited angular range $[a, b]$.

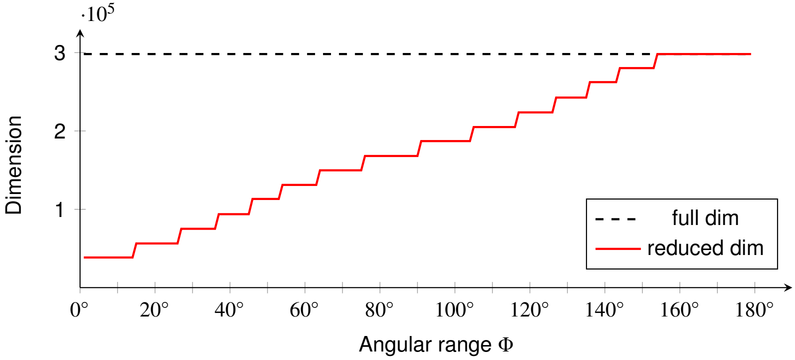
Decomposition into a visible and an invisible part

$$f = \sum_{(a,b,\theta)\in I_{[a,b]}} \left\langle f, \psi_{(a,b,\theta)} \right\rangle \psi_{(a,b,\theta)} + \sum_{(a,b,\theta)\in I\setminus I_{[a,b]}} \left\langle f, \psi_{(a,b,\theta)} \right\rangle \psi_{(a,b,\theta)}$$

$$= f_{\text{visible}} + f_{\text{invisible}}.$$

Dimensionality reduction: reconstruct only the visible part

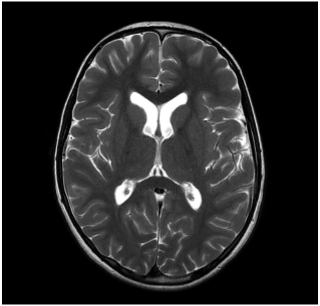
(works with any reconstruction algorithm)



Dimensions of the reconstruction problem in the curvelet domain for an image of size 256×256. The plot shows the dependence of the full dimension – – – and reduced (adapted) dimension —— on the available angular range $[0, \Phi]$.

Sparse regularization

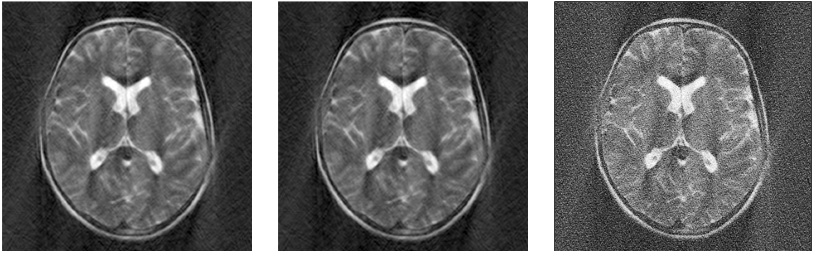
$$\hat{c} = \arg\min_{c} \left\{ \left\| RT^* c - y^\delta \right\|_2^2 + \|c\|_{\ell^1_w} \right\}, \qquad \hat{f} = T^* \hat{c} = \sum_{\gamma} \hat{c}_\gamma \psi_\gamma.$$



Testimage

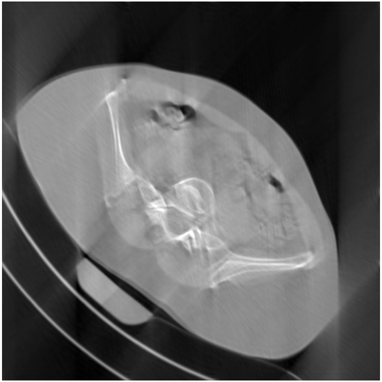
(JF, 2013; Vandeghinste et al., 2013; Wieczorek et al., 2015)

CSR	ACSR	FBP

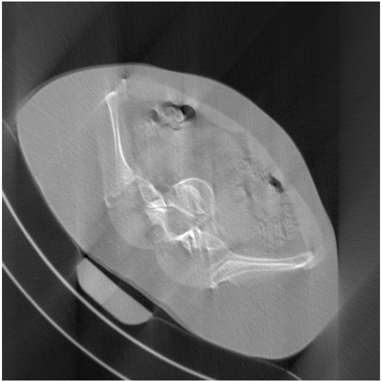


Reconstruction of the Brainstem image of size 300×300 using curvelet sparse regularisation (CSR) and adapted curvelet sparse regularisation (ACSR):

Angular range $[0^\circ, 160^\circ]$, $\Delta\theta = 1^\circ$, Noiselevel 2%.



ACSR

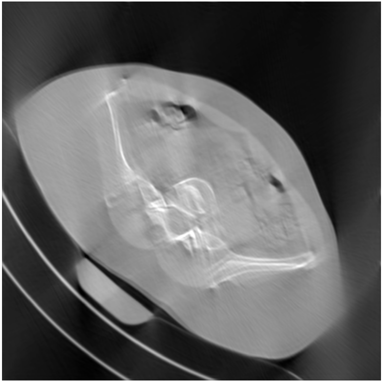


FBP

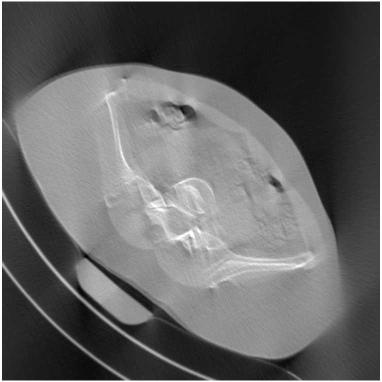
No artifact reduction

CT data[1] of an abdomen examination; limited angular range $\sim 140°$.

[1] Data by courtesy of Dr. Peter Noël (Department of Diagnostic and Interventional Radiology, TUM).



ACSR



FBP

With artifact reduction

CT data[1] of an abdomen examination; limited angular range $\sim 140^\circ$.

[1] Data by courtesy of Dr. Peter Noël (Department of Diagnostic and Interventional Radiology, TUM).

- Microlocal is a powerful framework for characterisation of incomplete data reconstructions in tomography
  - Visible singularities
  - Added artifacts
- Harmonic analysis provides tools and makes microlocal insights accessible algorithmically
  - Shearlets, curvelets or similar dictionaries
  - Dimensionality reduction and artifact reduction in limited angle x-ray tomography

Thank you!