02157 Functional Programming

Lecture 2: Functions, Types and Lists

Michael R. Hansen

DTU Compute

Department of Applied Mathematics and Computer Science

Outline

- Functional decomposition

short break

- Functions as ”first-class citizens”
  - Anonymous functions

short break

- Types, type inference and overloading
  - Tuples and equality and comparison constraints

short break

- Let expressions and lists

On functional decomposition

Functional decomposition

A simple technique when solving a complex problem is

- to partition it into smaller well-defined parts and, thereafter,
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- small comprehensible programs

Notice: there are better sorting algorithms than insertion sort

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Consider multiplication of polynomials:

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\begin{aligned}
0 \cdot Q(x) \quad &= 0 \\
(a_0 + a_1 \cdot x + ... + a_n \cdot x^n) \cdot Q(x) & \\
&= a_0 \cdot Q(x) + x \cdot \big((a_1 + a_2 \cdot x + ... + a_n \cdot x^{n-1}) \cdot Q(x)\big)
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That makes the task of declaring multiplication easier

Invent helper function(s): Blackboard exercise

Declare a function `sumProd: int list -> int*int`:

$$\texttt{sumProd } [x_0; x_1; \ldots; x_{n-1}] = (x_0 + x_1 + \ldots + x_{n-1}\,,\; x_0 * x_1 * \ldots * x_{n-1})$$

$$\texttt{sumProd } [\,] = (0,1)$$

On functions as first-class citizens

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Higher-order functions are useful

- succinct code
- highly parameterized programs
- Program libraries typically contain many such functions

An example

Suppose that we have a cube with side length s, containing a liquid with density ρ. The weight of the liquid is then given by $\rho \cdot s^3$:

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val weight : float -> float -> float
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The formula $\rho \cdot s^3$ is represented just once in the program

Currying and Uncurrying

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Consider declarations:

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let wC ro s = ro * s ** 3.0;;
val wC : float -> float -> float

let wUC(ro, s) = ro * s ** 3.0;;
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- `wC` is the curried version of `wUC`
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Have a look at exercise HR 2.13:

- declare functions for curring and uncurring.

```
curry:   ('a * 'b -> 'c) -> 'a -> 'b -> 'c
uncurry: ('a -> 'b -> 'c) -> 'a * 'b -> 'c
```

A well-known example: function composition

Function composition: $(f \circ g)(x) = f(g(x))$

For example, if $f(y) = y + 3$ and $g(x) = x^2$, then $(f \circ g)(z) = z^2 + 3$.

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The infix operator $<<$ in F# denotes function composition:

```
let f y = y+3;;

let g x = x*x;;

let h = f << g;;          // h = (f o g)
val h : int -> int

h 4;;                     // h(4) = (f o g)(4)
val it : int = 19
```

- An infix operator appears between the arguments

Infix functions

The prefix version ($(\oplus)$) of an infix operator $\oplus$ is a curried function, that is, higher-order function where argument are supplied one by one

For example:

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(<<);;
val it : (('a -> 'b) -> ('c -> 'a) -> 'c -> 'b)
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Declaration

`<<` is a built-in function

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Infix operators are written as strings of special characters including

```
!  %  &  *  +  -  /  <  =  >  ?  @  ^  \  ~
```

Consult F# specification for complete rules.

The built-in infix function `@`

List.append is a higher-order function from the `List` library:

- `List.append` $[x_0; \ldots; x_{n-1}]\,[y_0; \ldots; x_{m-1}] =$
  $[x_0; \ldots; x_{n-1}; y_0; \ldots; x_{m-1}]$

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The declaration of `(@)` *xs ys* follows the structure of *xs*:

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```

```
[["a"]; ["ab";"abc"; ""]; []] @ [["x"]; ["xy"; "xyz"]];;
val it : string list list =
    [["a"]; ["ab"; "abc"; ""]; []; ["x"]; ["xy"; "xyz"]]
```

On anonymous functions

Function expressions

There are two kinds of expression for anonymous functions

- One originates from abstraction $\lambda x.e$ in the lambda calculus:

  `fun x` $\rightarrow$ ***e***

  reads: “the function of *x* given by *e*”.

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- The other support pattern matching:

  function
  | $pat_1 \rightarrow e_1$
  $\vdots$
  | $pat_n \rightarrow e_n$

You can write functions without naming them

Anonymous functions: Example 1

An expressions denoting the circle-area function

```
fun r -> System.Math.PI * r * r ;;
val it : float -> float = <fun:clo@10-1>

it 2.0 ;;
val it : float = 12.56637061
```

Anonymous functions: Example 2

An anonymous function computing the number of days in a month:

```
function
| 2  -> 28   // February
| 4  -> 30   // April
| 6  -> 30   // June
| 9  -> 30   // September
| 11 -> 30   // November
| _  -> 31;; // All other months
 val it : int -> int = <fun:clo@17-2>
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it 2;;
val it : int = 28
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Anonymous functions: Example 3

Function expressions with general patterns, e.g.

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| 2          -> 28  // February
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Anonymous functions: Example 3

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Exploits an or pattern in the second clause

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$$\texttt{fun}\ x\ y\ \cdots\ z\ \rightarrow e$$

has the same meaning as

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For example: The function below takes an integer as argument and returns a function of type `int -> int` as value:

```
fun x y -> x + x*y;;
val it : int -> int -> int = <fun:clo@2-1>
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f 3;;
val it : int = 8
```

On types, type inference and overloading

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Purposes:

- Modelling, readability: types are used to indicate the intention behind a program

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Type inference is an algorithm to automatically calculate types of expressions without use of explicit type annotations.

Fundamental type-checking problem

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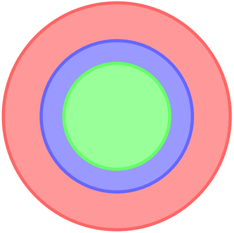
Consequence: A type-checking algorithm provides an approximation:

ill-typed, bad programs

ill-typed, good programs

well-typed, good programs



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Examples:

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Remark: identity function id is a built-in function

Polymorphic type inference – informally

Given a declaration, for example,

```
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   match xs with
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Every sub-expression is now consistently typed.

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'a list -> 'a list -> 'a list
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- First inference algorithm for ML DamasMilner82
- A nice introduction and F# implementation: Sestoft12

Parametric and ad-hoc polymorphism - Strachey 1967

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  – a function can be written so that it handles values identically independent on their types

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- Ad-hoc polymorphism/overloading:
  – a function has different implementations depending on the type of arguments

For example, + can be used on integers, floating-points values, strings, ...

Overloaded Operators and Type inference

A squaring function on integers:

Declaration	Type	
`let square x = x * x`	`int -> int`	Default

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`let square x = x * x:  float`	Type expression for the result
`let square x = x:float * x`	Type a variable

You can mix these possibilities

On tuples and equality and comparison constraints

Basic types: equality and comparison

Equality and comparison are defined for the basic types of F#, including integers, floats, booleans, characters and strings.

Examples:

```
true < false;;
val it : bool = false

'a' < 'A';;
val it : bool = false

"a" < "ab";;
val it : bool = true
```

Composite Types: equality and comparison

Equality and comparison carry over to composite types
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Equality is defined structurally on values with the same type:

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Equality is defined structurally on values with the same type:

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[[1;2]; [3;4;5]] = [[1..2]; [3..5]];;
val it : bool = true
```

Comparison is typically defined using lexicographical ordering:

```
[1; 2; 3] < [1; 4];;
val it : bool = true

(2, [1; 2; 3]) > (2, [1;4]);;
val it : bool = false
```

Tuples

An ordered collection of n values $(v_1, v_2, \ldots, v_n)$ is called an n-tuple

Examples

`(3, false);` `val it = (3, false) : int * bool`	2-tuples (pairs)
`(1, 2, ("ab",true));` `val it = (1, 2, ("ab", true)) :` ?	3-tuples (triples)

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`(1, 2, ("ab",true));` `val it = (1, 2, ("ab", true)) :` ?	3-tuples (triples)

Equality defined componentwise, ordering lexicographically

```
(1, 2.0, true) = (2-1, 2.0*1.0, 1<2);;
val it = true : bool
```

provided = is defined on components

Tuple patterns

Extract components of tuples

```
let ((x,_),(_,y,_)) = ((1,true),("a","b",false));;
val x :  int = 1
val y :  string = "b"
```

Pattern matching yields bindings

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Restriction

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let (x,x) = (1,1);;
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 ...  ERROR ...  'x' is bound twice in this pattern
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Restriction

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let (x,x) = (1,1);;
...
 ...  ERROR ...  'x' is bound twice in this pattern
```

Restriction can be circumvented using `when` clauses, for example:

```
let f = function
        | (x,y) when x=y -> x
        | (x,y)          -> x+y
```

Polymorphic types: equality and comparison constraints (I)

Polymorphic types may be accompanied with equality and comparison constraints like:

- when 'a : comparison
- when 'b : equality

For example, there is a built-in function:

$$\texttt{compare}\; x\; y \;=\; \begin{cases} > 0 & \text{if } x > y \\ 0 & \text{if } x = y \\ < 0 & \text{if } x < y \end{cases}$$

with the type:

```
'a -> 'a -> int  when 'a : comparison
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with the type:

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```

For example:

```
compare (2, [1; 2; 3]) (2, [1;4]);;
val it : int = -1
```

Polymorphic types: equality and comparison constraints (II)

The built-in function List.contains can be declared as follows:

```
let rec contains x =
   function
   | []    -> false
   | y::ys -> x=y || contains x ys
contains: 'a -> 'a list -> bool when 'a : equality

contains [3;4] [[1..2]; [3..5]];;
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contains [3;4] [[1..2]; [3..5]];;
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```

Notice:

- The equality constraint in the type
- Lazy (short-circuit) evaluation of $e_1 || e_2$ causes termination as soon as an element y equal to x is found
- Yet a recursion following the structure of lists

On let-expressions and lists

Let-expressions

A let-expression e_l has the (verbose) form

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let x = e1 in e2
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2. env' is obtained by adding binding $x \mapsto v_1$ to env and
3. v_2 is the value obtained by evaluating e_2 in env'

then

$$(\texttt{let}\ x = e_1\ \texttt{in}\ e_2, env) \leadsto (v_2, env)$$

Let-expression – an example

```
let g y = let a = 6
          let b = y + a
          y + b;;
val g : int -> int

g 1;;
val it : int = 8
```

Note: a and b are not visible outside of g

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Evaluation ?

Pattern matching on results of recursive calls

sumProd [x_0;x_1; ...;x_{n-1}]
= ($x_0 + x_1 + \ldots + x_{n-1}$, $x_0 \star x_1 \star \ldots \star x_{n-1}$)
sumProd [] = (0,1)

Pattern matching on results of recursive calls

$$\texttt{sumProd}\ [x_0; x_1; \ldots; x_{n-1}] = (x_0 + x_1 + \ldots + x_{n-1},\ x_0 * x_1 * \ldots * x_{n-1})$$

$$\texttt{sumProd}\ [\,] = (0,1)$$

The declaration is based on the recursion formula:

$$\texttt{sumProd}\ [x_0; x_1; \ldots; x_{n-1}] = (x_0 + \texttt{rSum}, x_0 * \texttt{rProd})$$

where $(\texttt{rSum}, \texttt{rProd}) = \texttt{sumProd}\ [x_1; \ldots; x_{n-1}]$

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where $(\texttt{rSum},\texttt{rProd}) = \texttt{sumProd}\ [x_1;\ldots;x_{n-1}]$

This gives the declaration:

```
let rec sumProd =
   function
   | []       -> (0,1)
   | x::rest -> let (rSum,rProd) = sumProd rest
                (x+rSum,x*rProd);;
val sumProd : int list -> int * int

sumProd [2;5];;
val it : int * int = (7, 10)
```

A blackboard exercise

A function from the `List` library:

- `List.unzip([(`x_0, y_0`);(`x_1, y_1`);...;(`x_{n-1}, y_{n-1}`)]`
  `= ([`x_0`;`x_1`;...;`x_{n-1}`],[`y_0`;`y_1`;...;`y_{n-1}`])`

Function expressions and match expressions

Consider

$$\text{Let } e_m \text{ be } \left\{ \begin{array}{lcl} \texttt{match}\ e\ \texttt{with} & & \\ \mid pat_1 & \to & e_1 \\ & \vdots & \\ \mid pat_n & \to & e_n \end{array} \right.$$

and

$$\text{Let } e_f \text{ be } \left\{ \begin{array}{lcl} \texttt{function} & & \\ \mid pat_1 & \to & e_1 \\ & \vdots & \\ \mid pat_n & \to & e_n \end{array} \right.$$

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- Can you express e_f using e_m and ... ?
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Overview: Syntactical constructs in “our part of” F#

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- Expressions: x $(e_1, \ldots, e_n)$ $e_1 :: e_2$ $[p_1; \ldots; p_n]$

  $e_1\, e_2$ $e_1 \oplus e_2$ $(\oplus)$ `let` $p_1 = e_1$ `in` e_2

  $e : t$ `if` e `then` e_1 `then` e_2 `match` e `with` *clauses*

  `fun` $p_1 \cdots p_n$`->`e `function` *clauses* ...

where the construct *clauses* has the form:

$|\ p_1\ \texttt{->}\ e_1\ |\ \ldots\ |\ p_n\ \texttt{->}\ e_n$

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- Declarations `let` $f\ p_1 \ldots p_n = e$ `let rec` $f\ p_1 \ldots p_n = e$, $n \geq 0$

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  `int` `float` `bool` `string` $'a \ldots$
  $t_1 * t_2 * \cdots * t_n$ t `list` t_1`->`$t_2 \ldots$

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where the construct *clauses* has the form:

| p_1 -> e_1 | ... | p_n -> e_n

In addition to that

- type declarations, precedence and associativity rules, parenthesis around p and e and type correctness

Summary

- Functional decomposition
- Functions as ”first-class citizens”
- Anonymous functions
- Types, type inference and overloading
- Tuples and equality and comparison constraints
- Let expressions and lists