Mathematical morphology

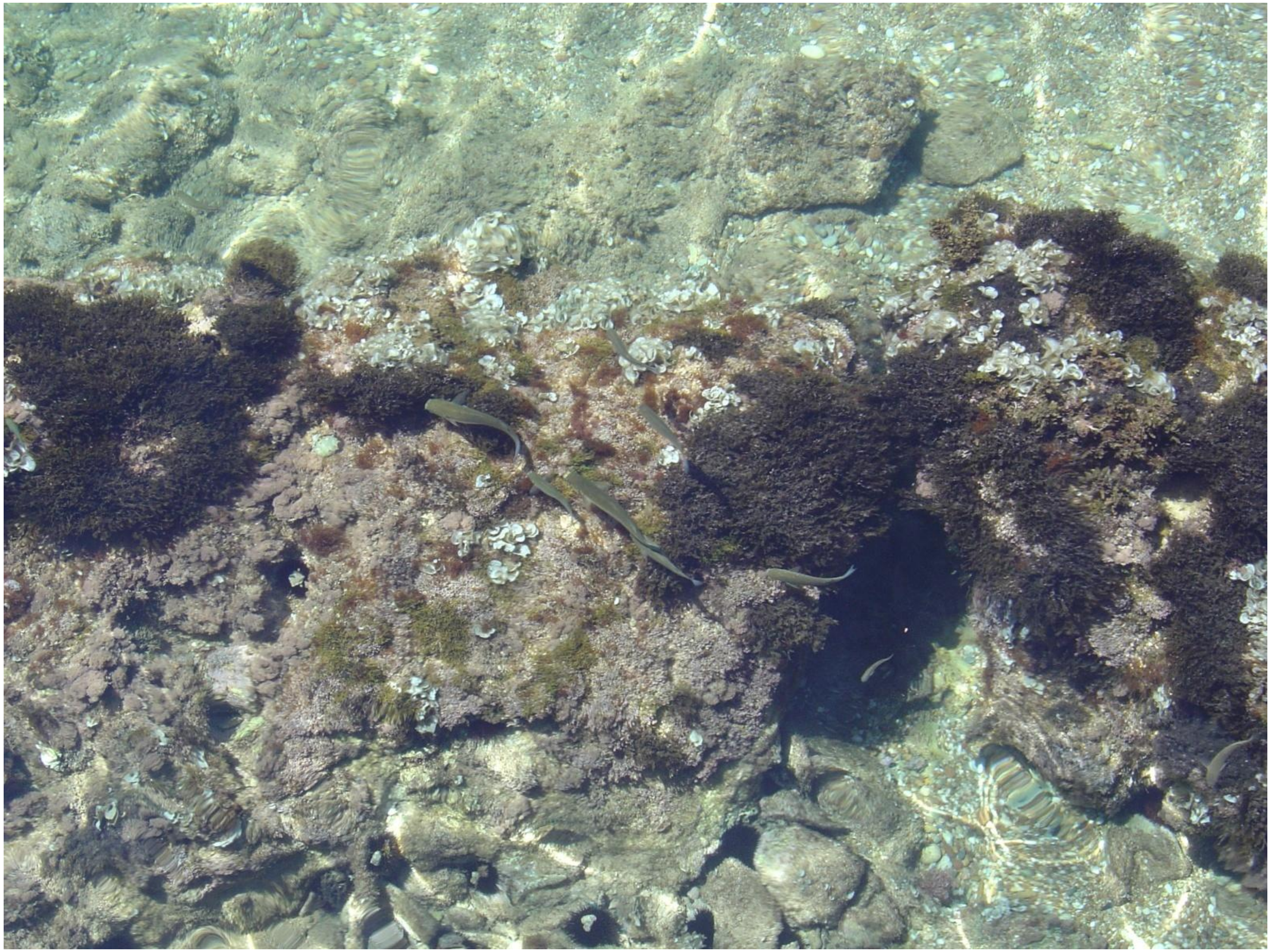
- École Nationale Supériure des Mines de Paris, Fontainebleau
- Matheron and Serra

A quantification consists of a transformation followed by a measurement

Source: www.carmanahdesign.com

DTU

Filters that relate to shape



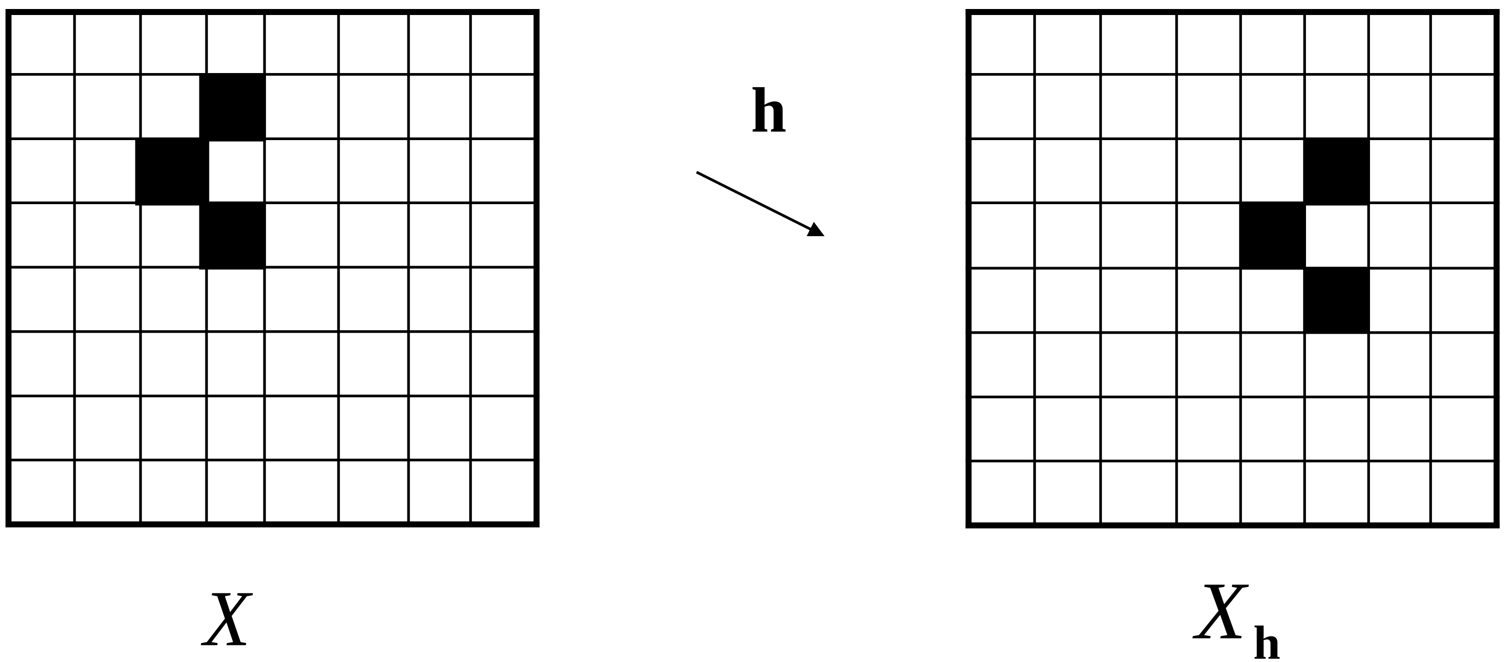
Set theory

- Empty set: $\emptyset$
- Universal set: S
- Set union: $X \cup Y$
- Set intersection: $X \cap Y$
- Subset: $X \subset Y, X \subseteq Y$
- Set complement: $X^c = S \setminus X$
- Set difference: $X \setminus Y = X \cap Y^c$

Set theory on binary images

- A binary image is represented by the set of foreground pixels. The universal set is the set of all pixels in the image.

Translation



$$X_{\mathbf{h}} = \{ z \in S \mid \exists x \in X : z = x + \mathbf{h} \}$$

Morphological operation

- A morphological operation φ is specified with respect to a structuring element *B*.

$$\varphi(X) = X \oplus B$$

- The dual operation φ^* is defined as

$$\varphi^*(X) = \varphi(X^c)^c$$

Basic properties

A morphological operation can be characterized as to which of the following four basic properties it has.

- Anti-extensive/extensixe $\varphi(X) \subseteq X \;/\; \varphi(X) \supseteq X$
- Increasing $X \subseteq Y \Rightarrow \varphi(X) \subseteq \varphi(Y)$
- Idempotent $\varphi(\varphi(X)) = \varphi(X)$
- Homotopic
  - Preserves the homotopic tree

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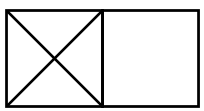
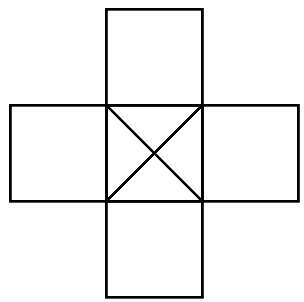
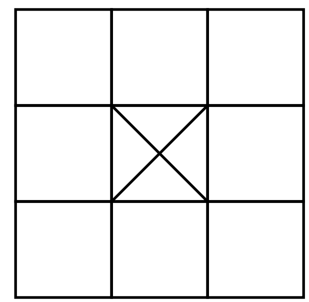
Simple structuring elements



Right-neighbor correlation

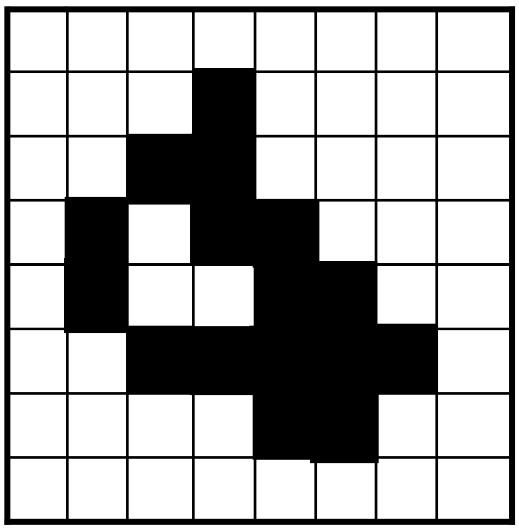


5-cross



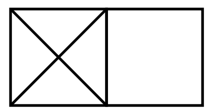
3x3

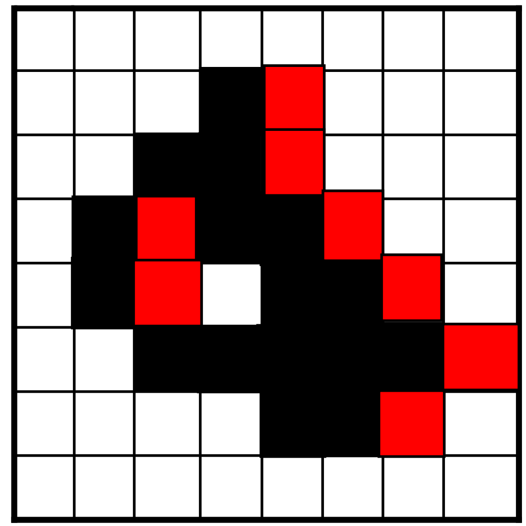
Dilation



X

B





$X \oplus B$

$$X \oplus B = \bigcup_{\mathbf{b} \in B} X_{\mathbf{b}}$$

Dilation

- Definition

$$X \oplus B = \bigcup_{\mathbf{b} \in B} X_{\mathbf{b}}$$

- Properties
  - Commutative
  - Associative
  - Increasing
  - Extensive if origin belongs to B

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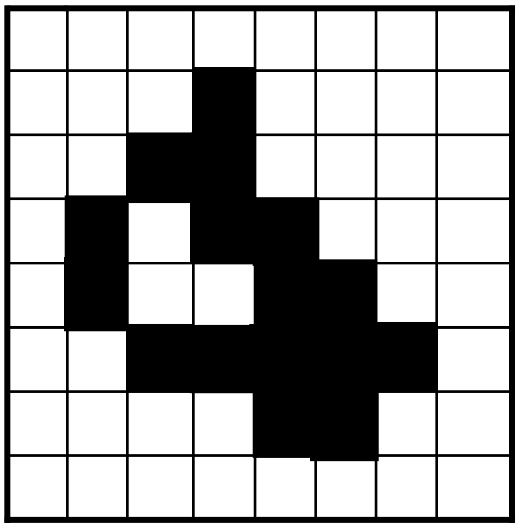
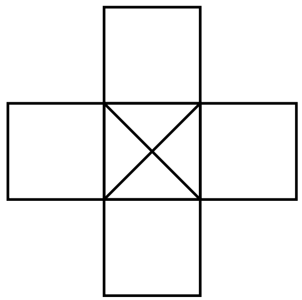
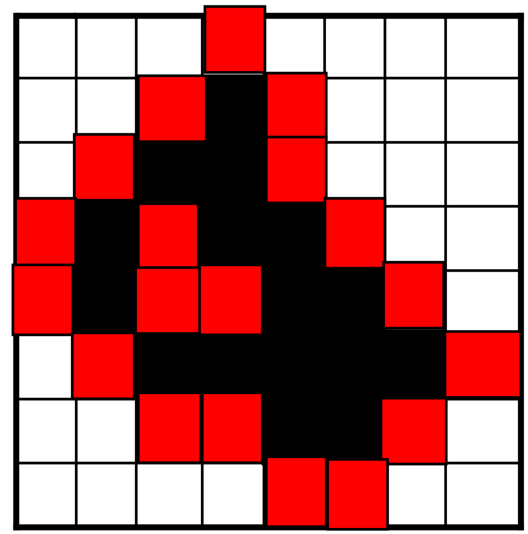
Dilation



X

B

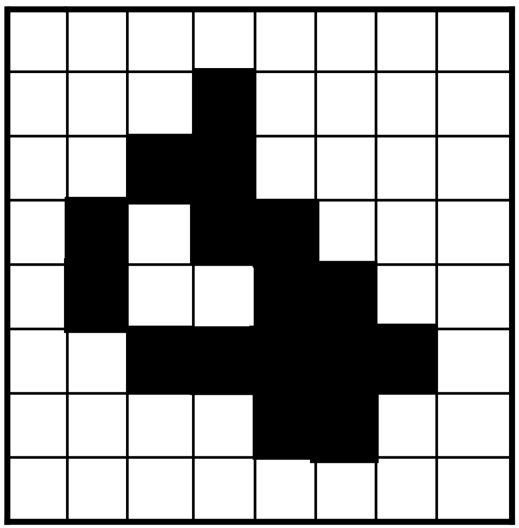




$X \oplus B$

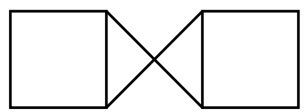
$$X \oplus B = \bigcup_{\mathbf{b} \in B} X_{\mathbf{b}}$$

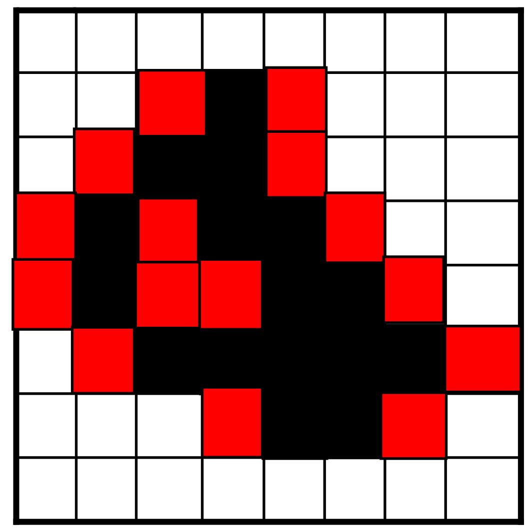
Dilation



X

B

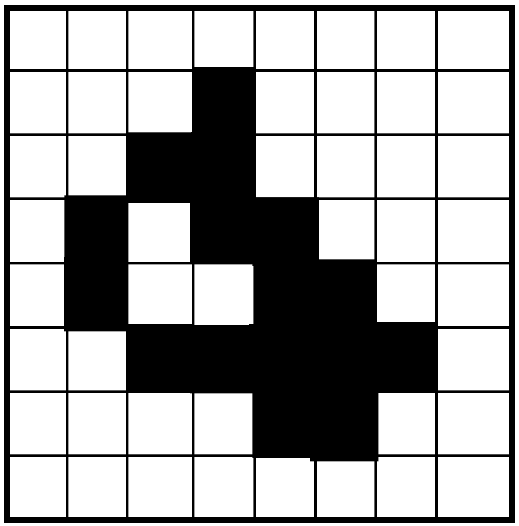




$X \oplus B$

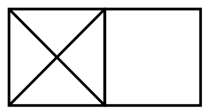
$$X \oplus B = \bigcup_{\mathbf{b} \in B} X_{\mathbf{b}}$$

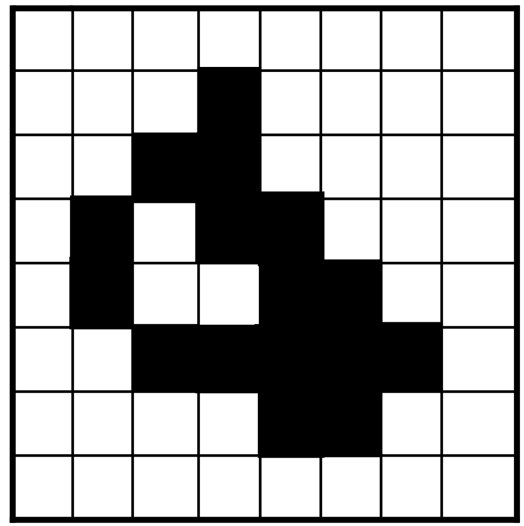
Erosion



X

B





$X \ominus B$

$$X \ominus B = \bigcap_{\mathbf{b} \in B} X_{-\mathbf{b}}$$

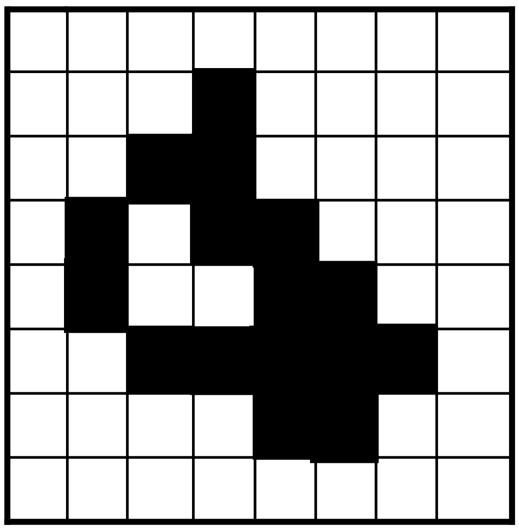
Erosion

- Definition

$$X \ominus B = \bigcap_{\mathbf{b} \in B} X_{-\mathbf{b}}$$

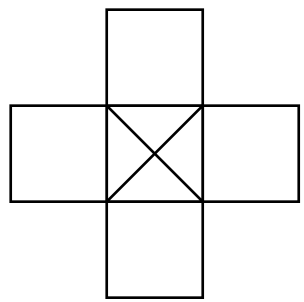
- Properties
  - Not commutative
  - Not associative
  - Increasing
  - Anti-extensive

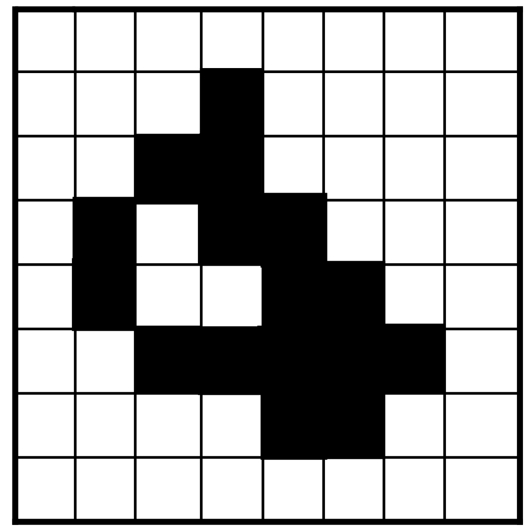
Erosion



X

B

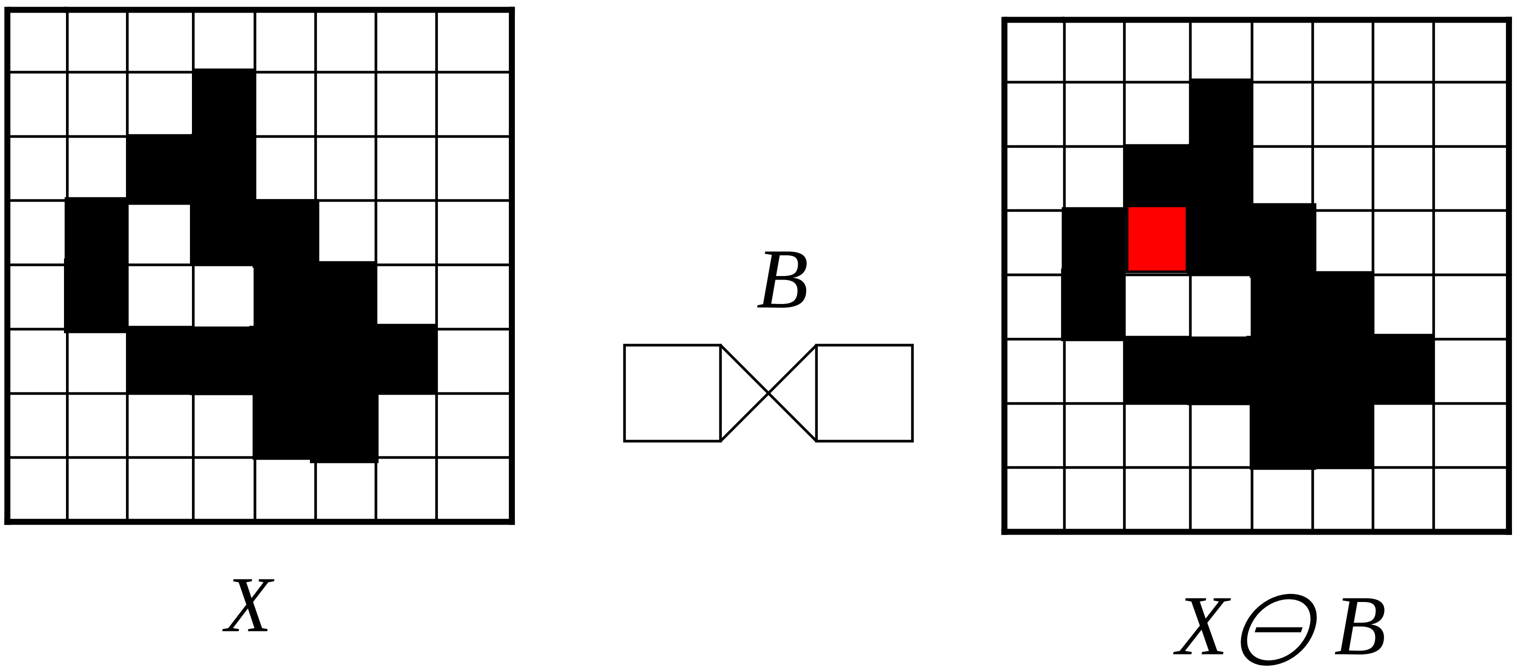




$X \ominus B$

$$X \ominus B = \bigcap_{\mathbf{b} \in B} X_{-\mathbf{b}}$$

Erosion



$$X \ominus B = \bigcap_{\mathbf{b} \in B} X_{-\mathbf{b}}$$

Erosion/dilation duality

- Duality

$$X \ominus B = (X^c \oplus \breve{B})^c$$

where we use the transpose of B

$$\breve{B} = \{-\mathbf{b} \mid \mathbf{b} \in B\}$$

  - *An erosion of the foreground with B corresponds to a dilation of the background with* $\breve{B}$.

DTU

Structuring element decomposition

- $(X \oplus B) \oplus C = X \oplus (B \oplus C)$
- $(X \ominus B) \ominus C = X \ominus (B \oplus C)$

Erosion in Matlab

IM2 = IMERODE(IM,SE) erodes the grayscale, binary, or packed binary image IM, returning the eroded image, IM2.

SE is a structuring element object, or array of structuring element objects, returned by the STREL function.

IM2 = IMERODE(IM,NHOOD) erodes the image IM, where NHOOD is an array of 0s and 1s that specifies the structuring element. This is equivalent to the syntax MERODE(IM,STREL(NHOOD)).

Structuring elements - strel

```
% 11-by-11 square
se1 = strel('square',11)
% line, length 10, angle 45 degrees
se2 = strel('line',10,45)
% disk, radius 15
se3 = strel('disk',15)
% ball, radius 15, height 5
se4 = strel('ball',15,5)
```

Opening and closing

- Opening

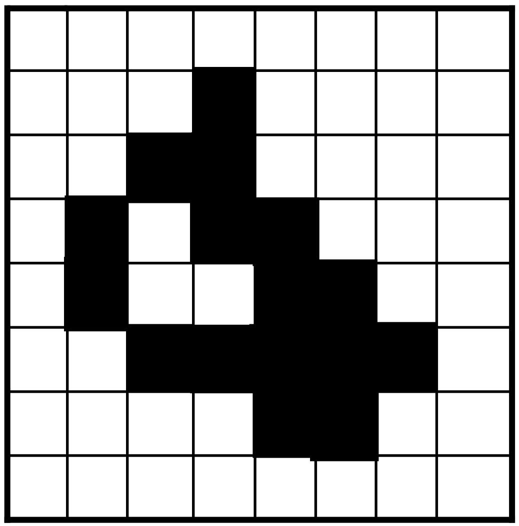
  $X \circ B = (X \ominus B) \oplus B$

- Closing

  $X \bullet B = (X \oplus B) \ominus B$

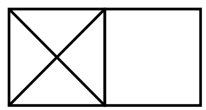
DTU

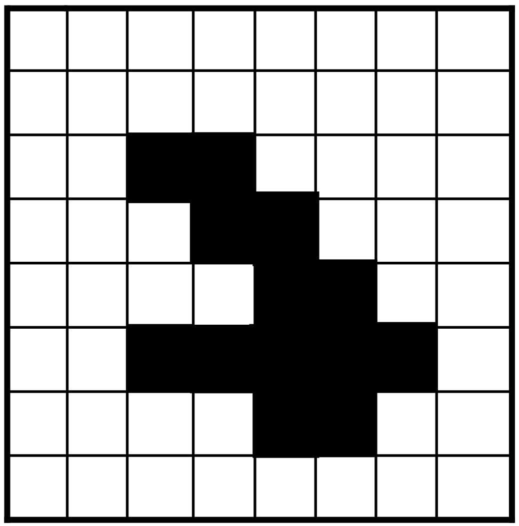
Opening



X

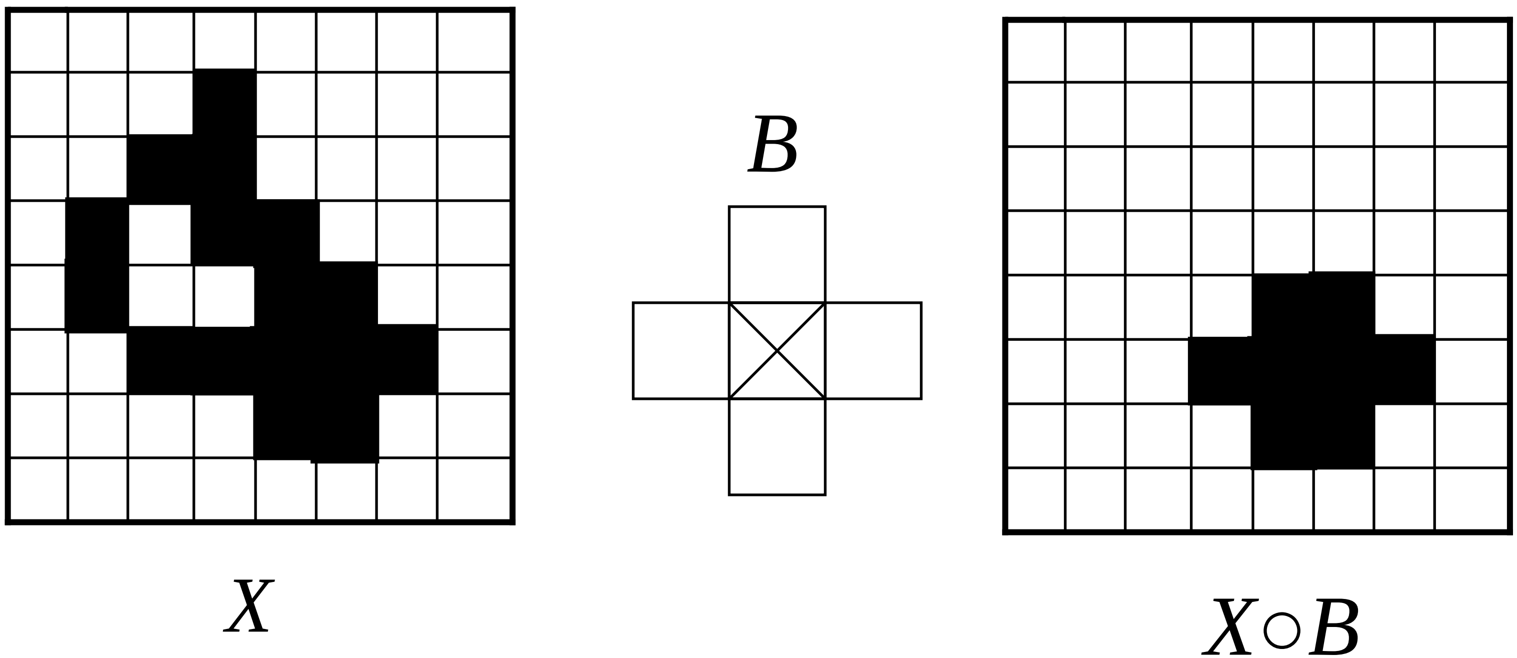
B



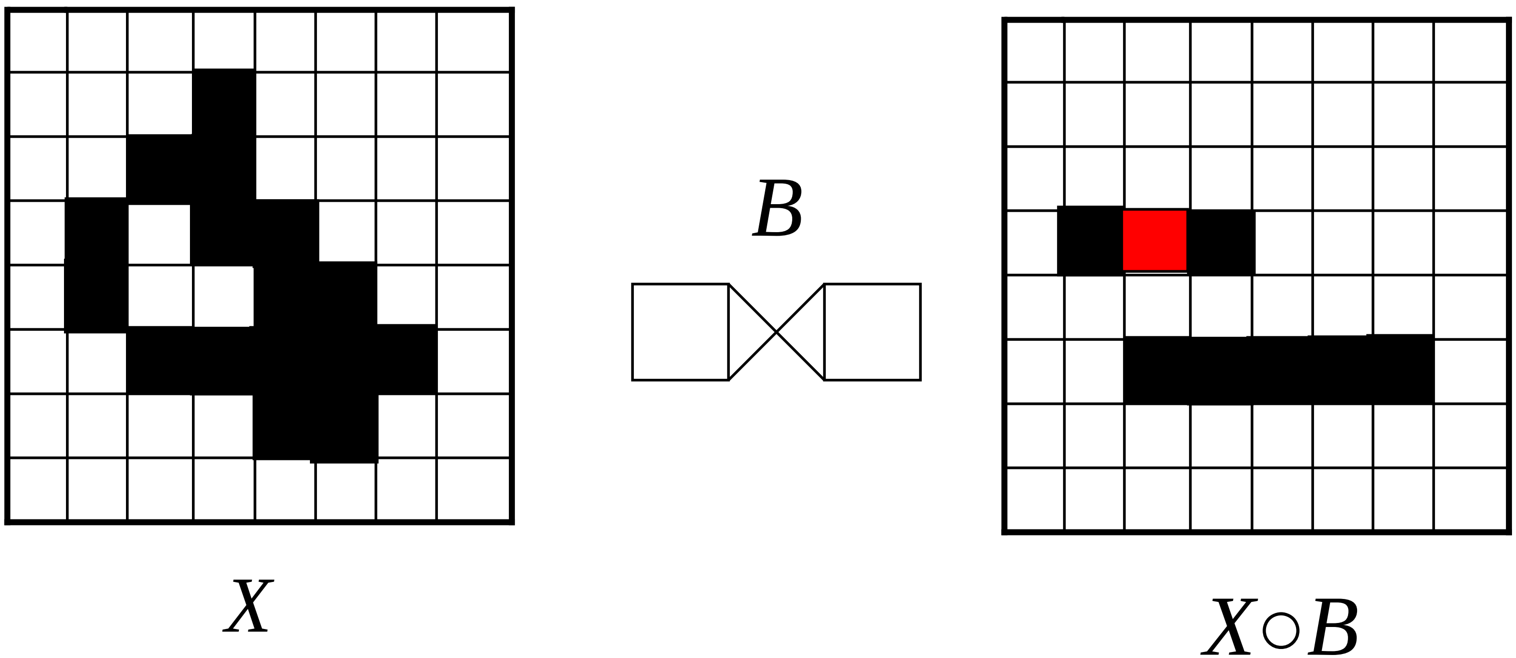


$X \circ B$

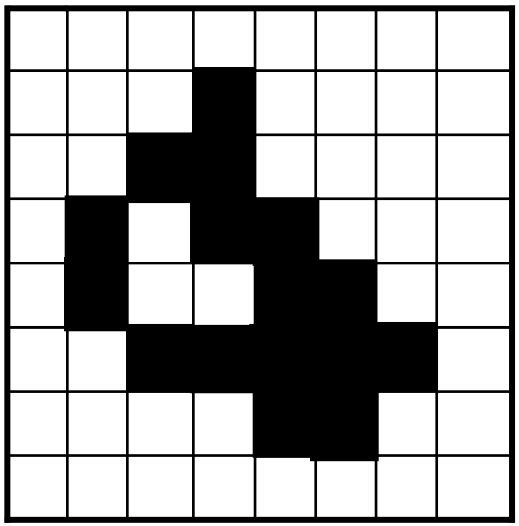
Opening



Opening

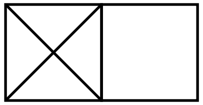


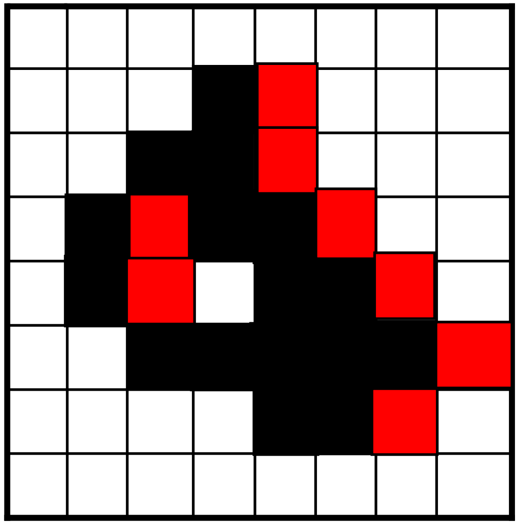
Closing



X

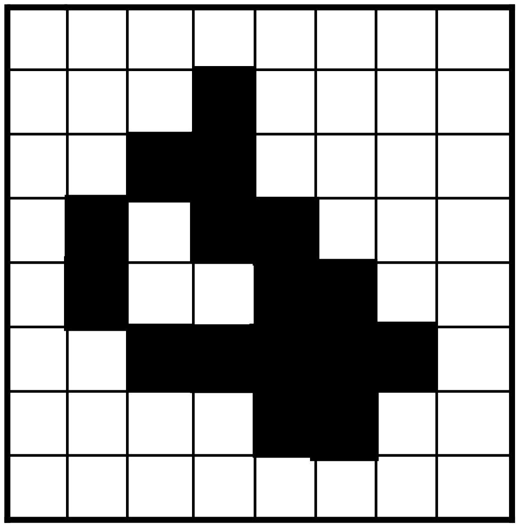
B

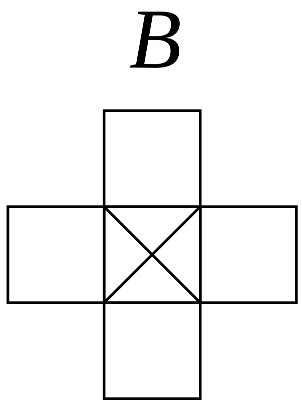


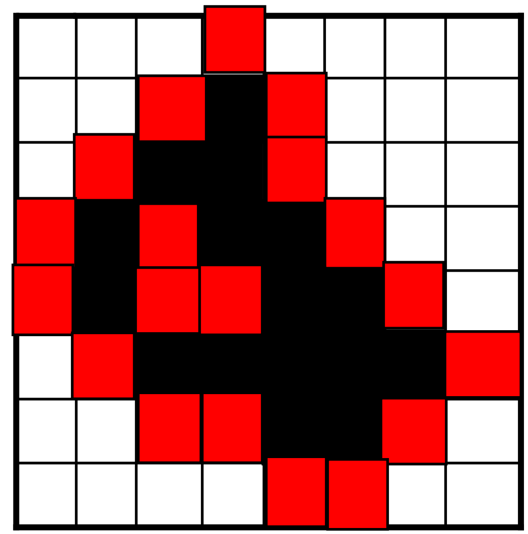


$X \bullet B$

Closing

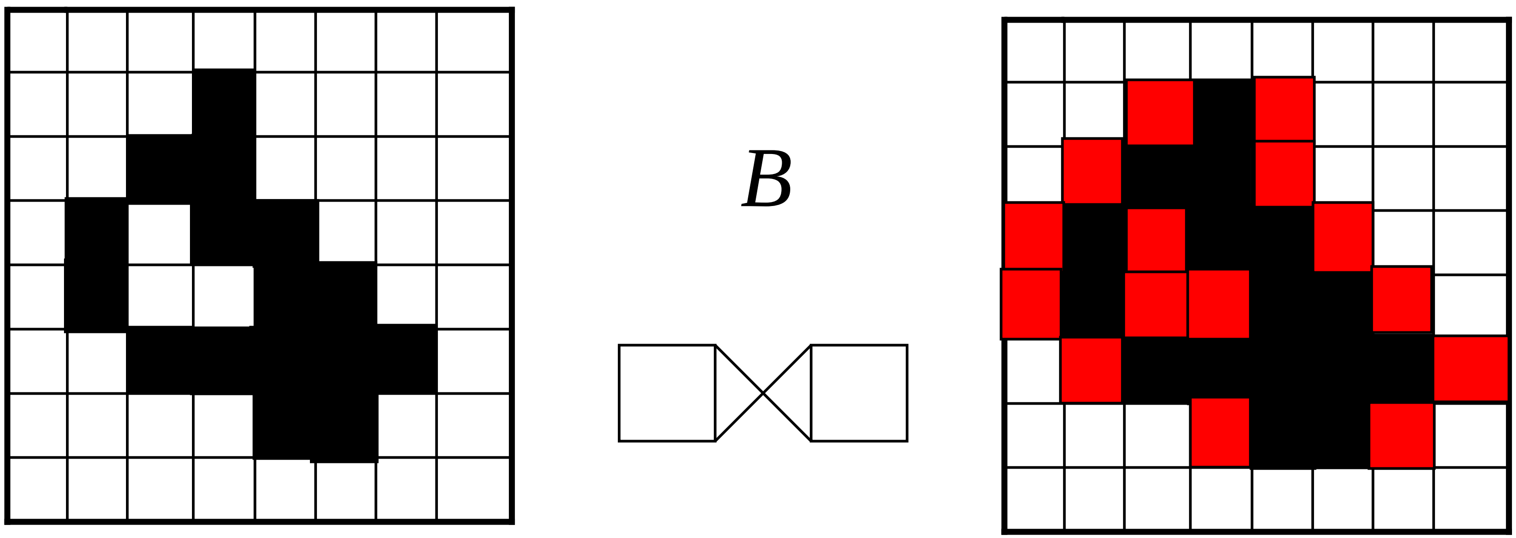






Closing

B



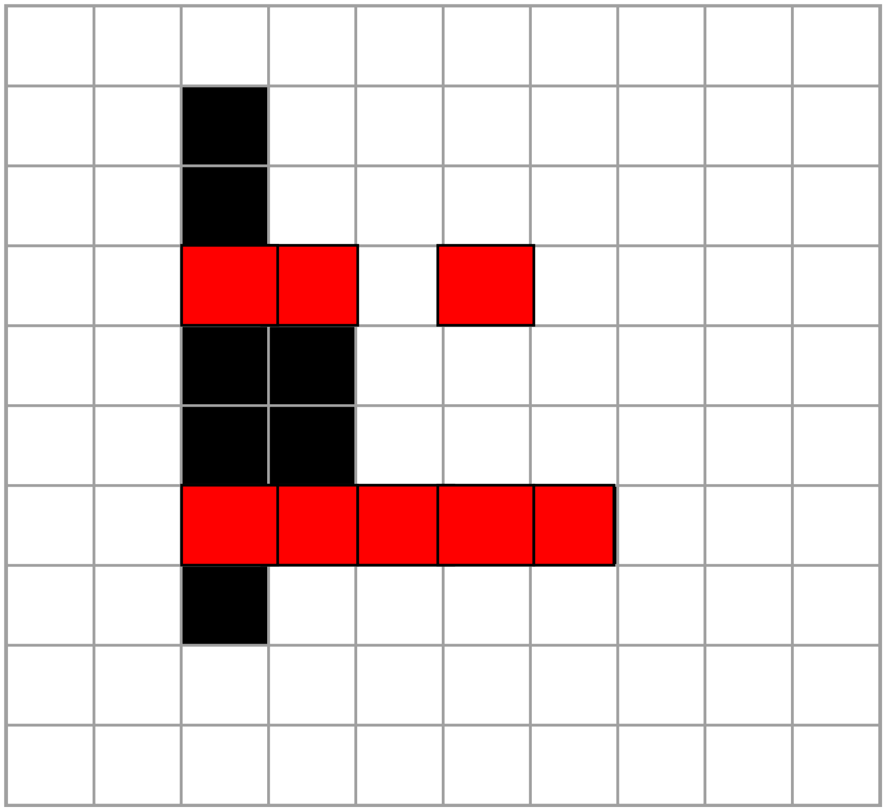
X

$X \bullet B$

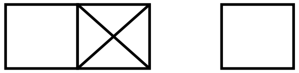
Properties of opening and closing

- Not commutative
- Not associative
- Anti-extensive/extensive
- Idempotent
- Translation invariant wrt. structuring element
- Dual

Exercise



X is the set of black pixels in the binary image. X is opened with the structure element



What is the number of black pixels in the result?

Result: 8 pixels

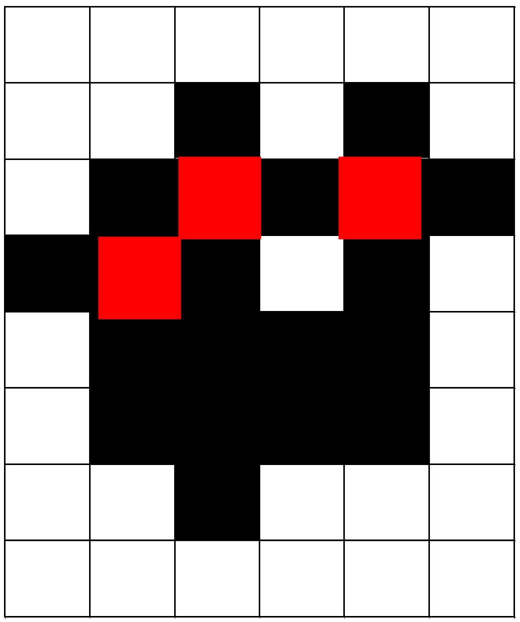
Hit-or-miss transformation

- Definition

$$X \otimes B = (X \ominus B_1) \setminus (X \oplus \breve{B}_2)$$

where $B = (B_1, B_2)$, $B_1 \cap B_2 = \varnothing$ is a composite structuring element.

Exercise

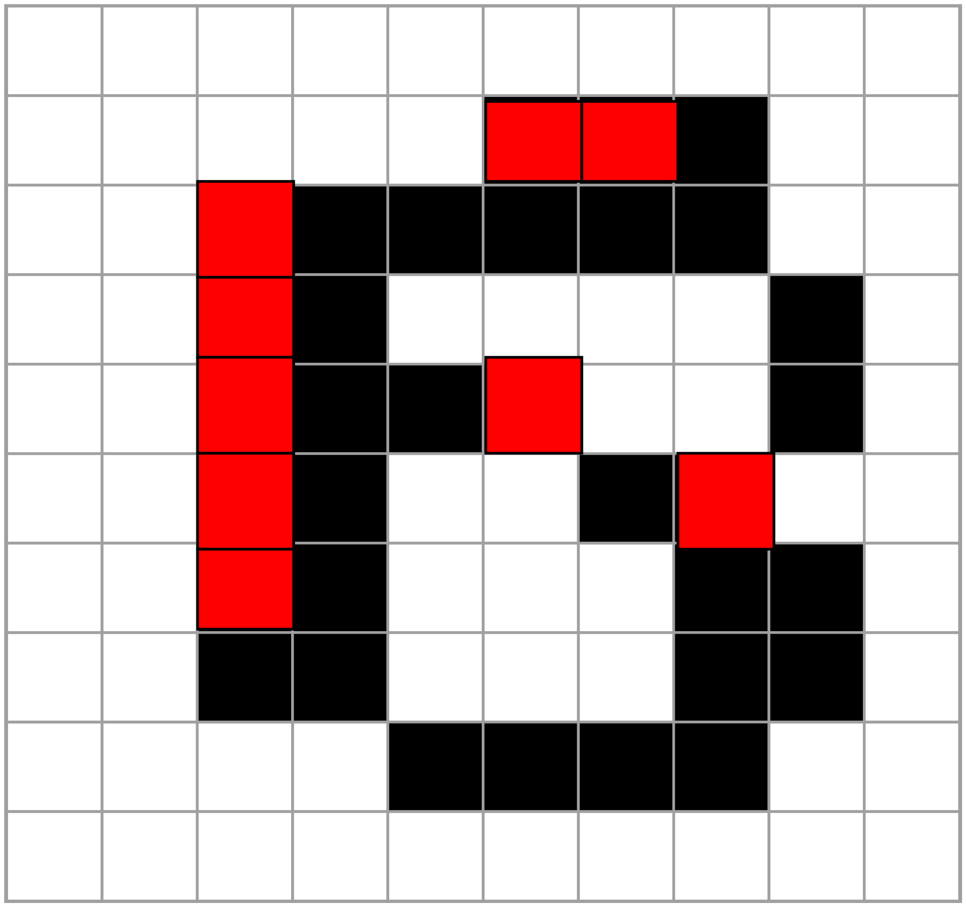


We perform a hit-or-miss-transformation on the black pixels in the image with the composite structuring element

0	1	*
1	1	1
*	1	*

How many black pixels are there in the resulting image?

Exercise



A hit-or-miss transformation is performed on the set of black pixels using the following hit-or-miss structure element

0	*	*
*	1	*
*	*	1

How many black pixels are left in the resulting image?

DTU

Thinning

- Definition

  $X \oslash B = X \setminus (X \otimes B)$

- Often sequence

  $X \oslash \{B(i)\} = (\ldots((X \oslash B(1)) \oslash B(2)) \ldots \oslash B(n))$

- Thinning until convergence

  Skeletonizing

L-thinning

0	0	0
*	1	*
1	1	1

*	0	*
1	1	0
1	1	*

1	*	0
1	1	0
1	*	0

1	1	*
1	1	0
*	0	*

1	1	1
*	1	*
0	0	0

*	1	1
0	1	1
*	0	*

0	*	1
0	1	1
0	*	1

*	0	*
0	1	1
*	1	1

Result: Homotopic skeleton

M-thinning

*	0	*
*	1	*
1	1	1

*	0	*
1	1	0
1	1	*

1	*	*
1	1	0
1	*	*

1	1	*
1	1	0
*	0	*

1	1	1
*	1	*
*	0	*

*	1	1
0	1	1
*	0	*

*	*	1
0	1	1
*	*	1

*	0	*
0	1	1
*	1	1

Result: Homotopic (jagged) skeleton

DTU

E-thinning

*	*	*
0	1	0
*	0	*

*	0	*
0	1	*
*	0	*

*	0	*
0	1	0
*	*	*

*	0	*
*	1	0
*	0	*

Result: Pruning

D-thinning

*	0	*
0	1	0
*	1	*

*	0	0
1	1	0
1	1	*

*	0	*
1	1	0
*	0	*

1	1	*
1	1	0
*	0	0

*	1	*
0	1	0
*	0	*

*	1	1
0	1	1
0	0	*

*	0	*
0	1	1
*	0	*

0	0	*
0	1	1
*	1	1

Result: Homotopic marking

Thickening

- Definition

  $X \odot B = X \cup (X \otimes B)$

- Often sequence

  $X \odot \{B(i)\} = (\ldots((X \odot B(1)) \odot B(2))\ldots \odot B(n))$

- Thickening until convergence

  Skiz (skeleton by influence zone)

Conditional dilation

- Definition

  $(M \oplus Q) \cap X$

  i.e. dilation of marker image *M* under condition *X*.

Geodesic dilation (FeatureAND)