Geometric transformations

Applications

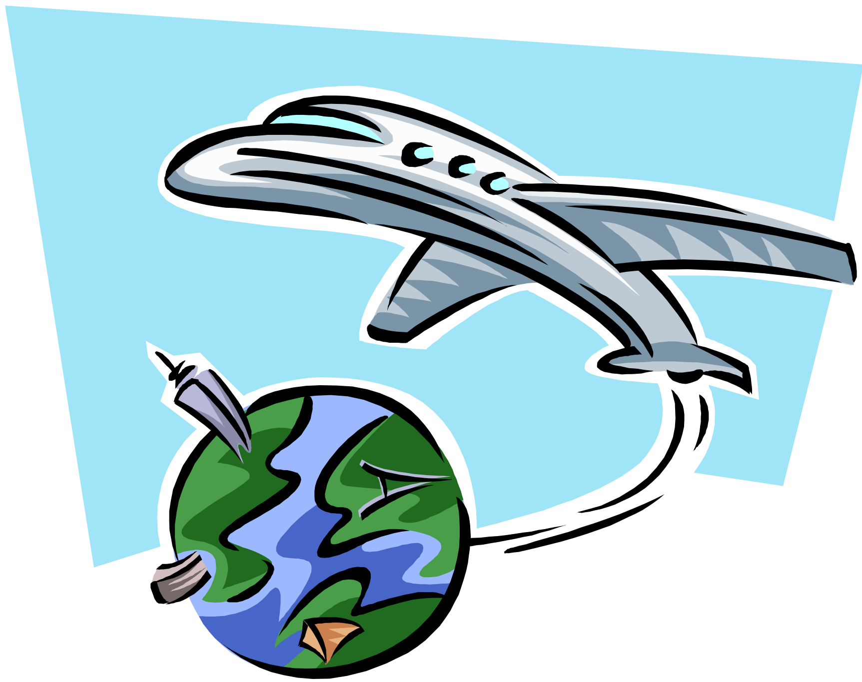
- Registration
  - To geometrically adjust images to each other or to a reference coordinate system
- Morphing
  - In combination with pixel value interpolation any image can fade into another in a convincing fashion (mainly for amusement)

Image registration

Two different principles

1. Physical (geometrical) model of rays in the scene
2. Statistical model based on control points

Physical model



- Kamera model
- Flight path
- Atmospheric distortion
- Earth rotation
- Earth shape and curvature
- E.g. Camera calibration

Statistical model

- Data fitting
- Control points with known reference coordinates (ground control points)
- Model for the type of mapping
  - Polynomial
  - Spline
- This is the approach of this lecture

Input-to-output transformation

Moving pixel values

$$(x, y) \rightarrow (x', y')$$

Position in input image — Position in output image

These transformations will in general either

1. leave holes in the output image, or
2. require interpolation based on an irregularly sampled grid

Output-to-input transformation

Retrieving pixel values

$$(x, y) \leftarrow (x', y')$$

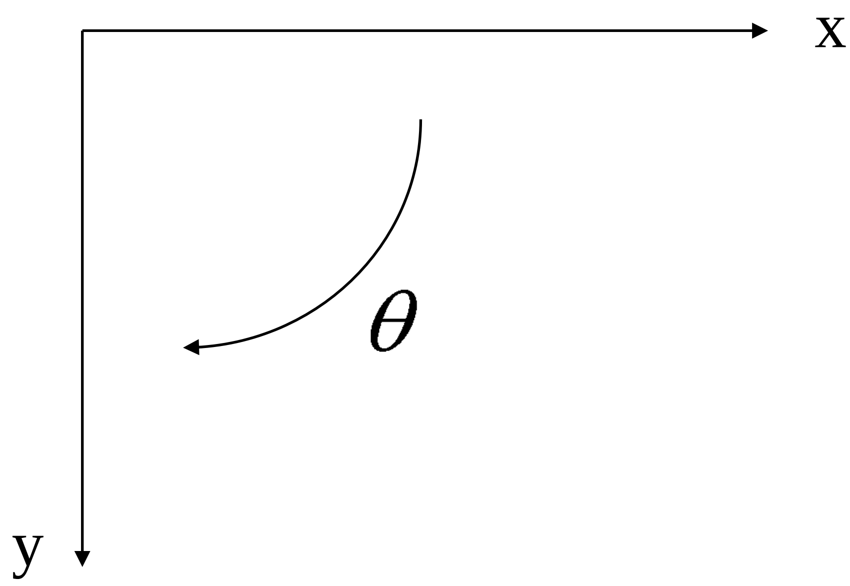
Position in input image

Position in output image

These transformations will in general only require interpolation based on a regularly sampled grid.

Used subsequently.

Coordinate system



Linear geometric transformations

Affine transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{t} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} - \mathbf{A}^{-1} \cdot \mathbf{t}$$

Examples

- Translation
- Rotation
- Scaling
- Affine transformation

Translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

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Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

DTU

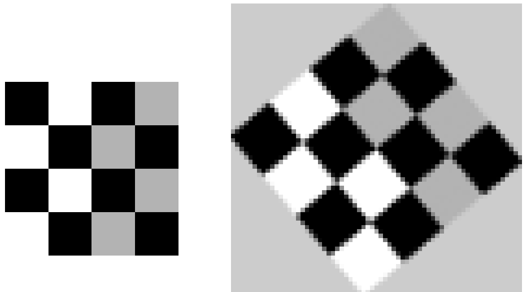
Scaling

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Linear conformal transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s \cdot \cos\theta & s \cdot \sin\theta \\ -s \cdot \sin\theta & s \cdot \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

i.e. translation, rotation and isotropic scaling.



Linear geometric transformations in homogeneous coordinates

The affine transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{t} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} - \mathbf{A}^{-1} \cdot \mathbf{t}$$

can be expressed in homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} & t_1 \\ A_{21} & A_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{T}} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Linear geometric transformations in homogeneous coordinates

The affine transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{t} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} - \mathbf{A}^{-1} \cdot \mathbf{t}$$

can be expressed in homogeneous coordinates

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}}_{\mathbf{T}} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}}_{\mathbf{T}^{-1}} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

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Control points

Given N points with known coordinates in both input and output image we can estimate the parameters of a chosen model.

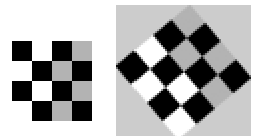
$$\{(x_i, y_i, x'_i, y'_i) \mid i = 1,..N\}$$

These points are called

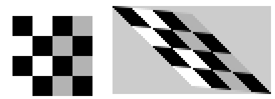
- control points,
- landmarks,
- or fiducial markers.

Types of transformations

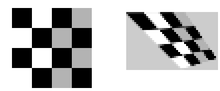
- Linear conformal, 2 cp pairs



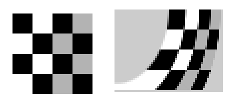
- Affine, 3 cp pairs



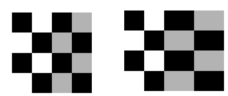
- Projective, 4 cp pairs



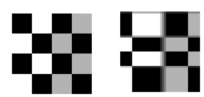
- Polynomial, order 2, 3, or 4, (6, 10, 16 cp pairs)



- Piecewise linear, 4 cp pairs



- Local weighted mean (Lwm), 6 cp pairs



Polynomial transformation

First-order output-to-input transformation

$$x = a_0 + a_1 x' + a_2 y'$$

$$y = b_0 + b_1 x' + b_2 y'$$

Second-order output-to-input transformation

$$x = a_0 + a_1 x' + a_2 y' + a_3 x'^2 + a_4 y'^2 + a_5 x' y'$$

$$y = b_0 + b_1 x' + b_2 y' + b_3 x'^2 + b_4 y'^2 + b_5 x' y'$$

Fitting a first-order transformation

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} 1 & x'_1 & y'_1 \\ \vdots & \vdots & \vdots \\ 1 & x'_N & y'_N \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{pmatrix} \Rightarrow \mathbf{x} = \mathbf{A} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \boldsymbol{\varepsilon}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x'_1 & y'_1 \\ \vdots & \vdots & \vdots \\ 1 & x'_N & y'_N \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \Rightarrow \mathbf{y} = \mathbf{A} \cdot \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} + \boldsymbol{\eta}$$

Solution

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{x}$$

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}$$

Warping

- Rubber-sheet transformation

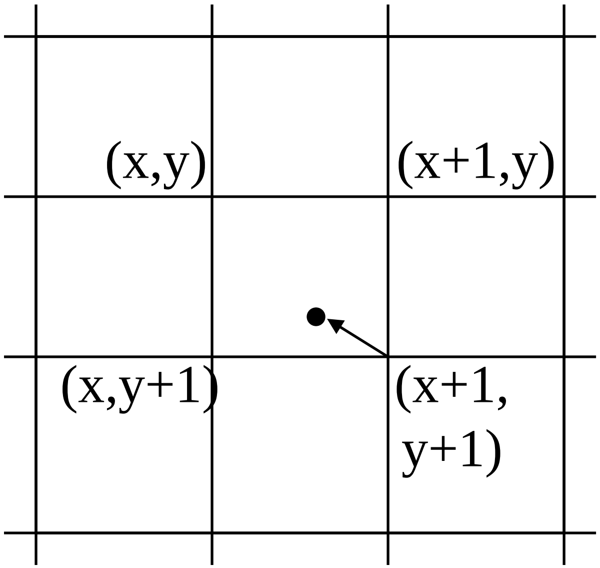
Resampling

- Upsampling
  – Pixel interpolation
- Downsampling
  – Low-pass filtering

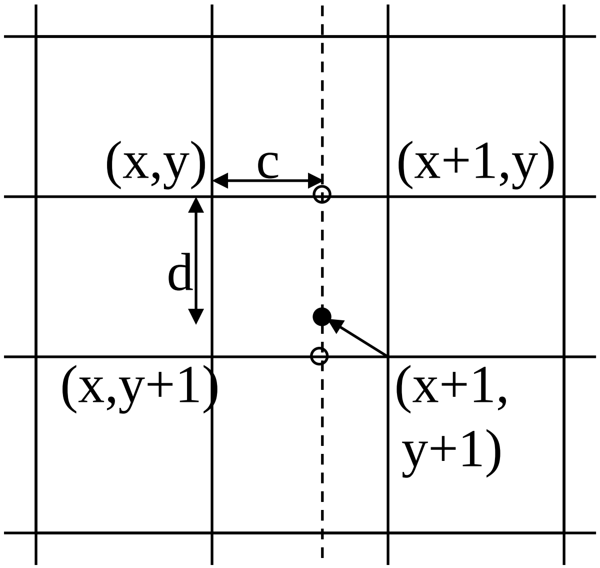
Interpolation methods

- Nearest neighbor
- Bilinear
- Bicubic

Nearest neighbor interpolation

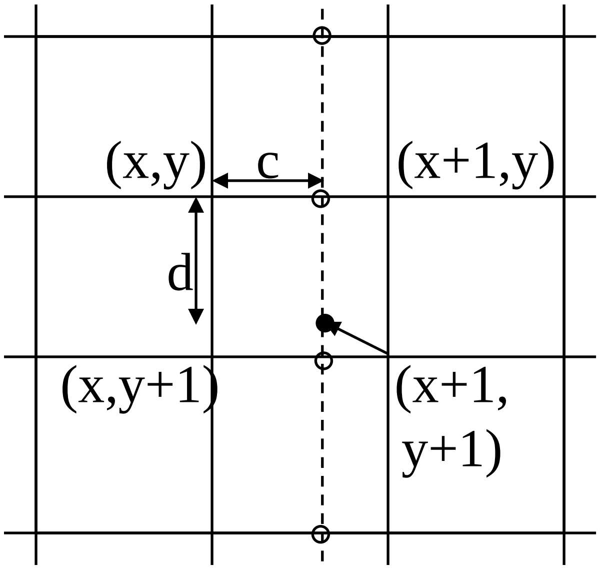


Bilinear interpolation



$$p'(x',y') = (1-c)\cdot(1-d)\cdot p(x,y) + c\cdot(1-d)\cdot p(x+1,y)$$
$$+ (1-c)\cdot d\cdot p(x,y+1) + c\cdot d\cdot p(x+1,y+1)$$

Bicubic interpolation



OPGAVE 05.18

2	2	3	5	2	5
2	4	1	3	3	2
3	5	1	2	5	3
4	3	2	1	4	2
2	2	2	2	5	2
2	1	2	4	5	4

The image on the left is translated with the vector (xt,yt)=(2.5 , 1.4). What is the value of pixel position (x,y)=(7 , 4) in the translated image, when bilinear resampling is used in the input image?

1. 3.4
2. 3.5
3. 3.6
4. 4
5. 3.2

OPGAVE 03.16

2	2	3	5	2	5
2	4	1	3	3	2
3	5	1	2	5	3
4	3	2	1	4	2
2	2	2	2	5	2
2	1	2	4	5	4

We perform a geometric alignment of the image on the left. The alignment is given by the output-to-input transform

$$x = 3.5 - 0.7 \cdot x' + 1.1 \cdot y' - 0.2 \cdot x' \cdot y'$$
$$y = 1.0 - 0.9 \cdot x' + 0.8 \cdot y' + 0.1 \cdot x' \cdot y'$$

What is the value of pixel (3 , 4) in the output image when nearest neighbor resampling is used in the input image?

1. 1
2. 2
3. 3
4. 4
5. 5