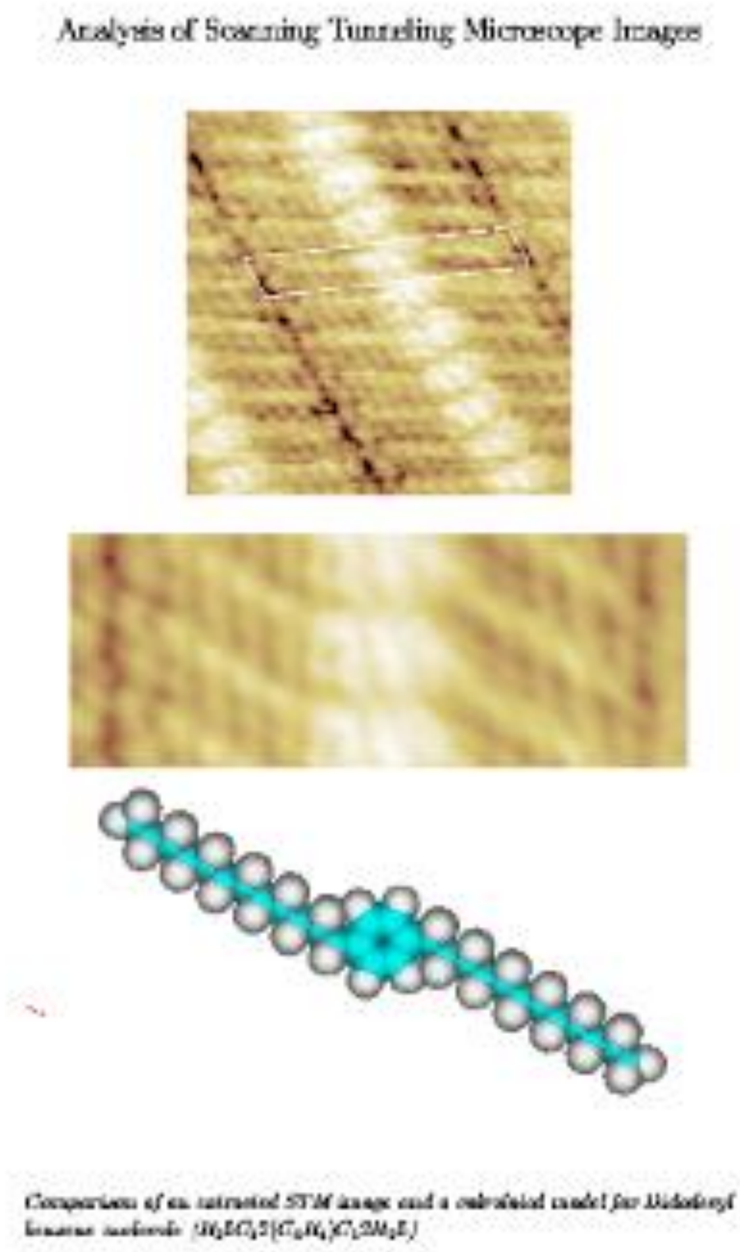
Fourier transform

Related concepts

- Spatial frequency
- Periodicity
- Complex images
- Convolution

Example: STM image



Fourier Transform

- A Fourier Transform of a function is a decomposition of the function in sinusoidals, which - when added together - reproduce the function.

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1D Fourier Transform

Fourier Transform,

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi \cdot u \cdot x} dx$$

Inverse Fourier Transform,

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi \cdot u \cdot x} du$$

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2D Fourier Transform

Fourier Transform,

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(u\cdot x+v\cdot y)}dxdy$$

Inverse Fourier Transform,

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i2\pi(u\cdot x+v\cdot y)}dudv$$

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2D Digital Fourier Transform

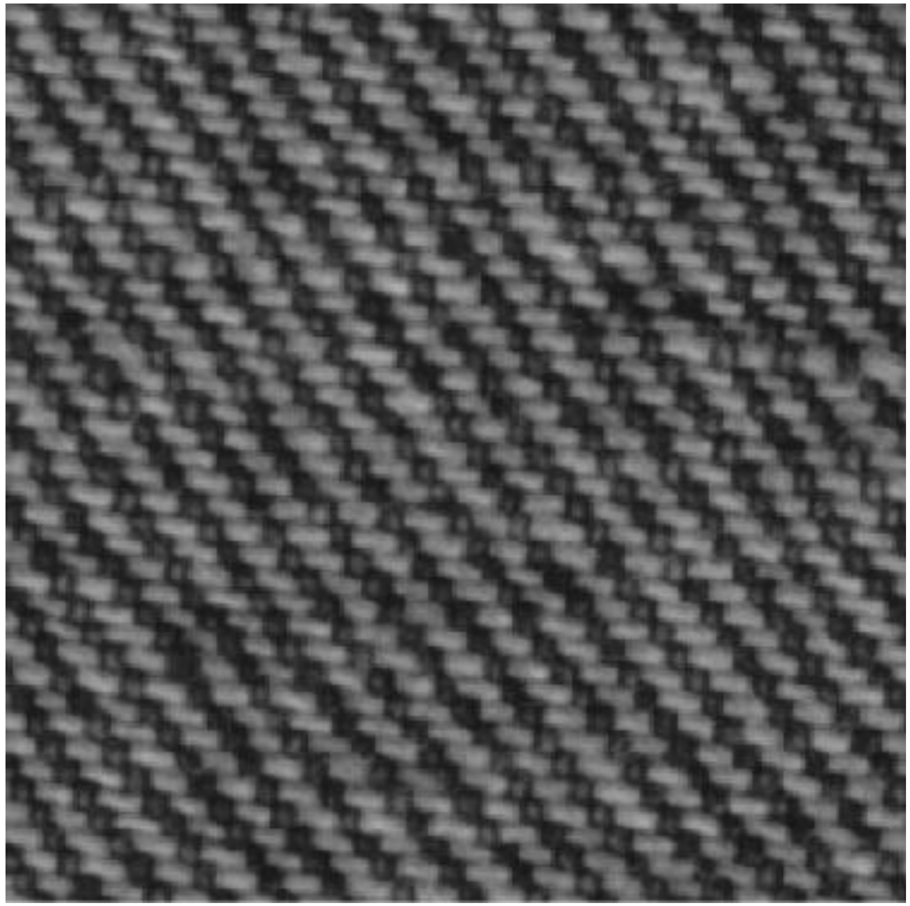
Fourier Transform,

$$F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{n-1} f(m,n) e^{-i2\pi(m \cdot u/M + n \cdot v/N)}$$

Inverse Fourier Transform,

$$f(m,n) = \sum_{u=0}^{M-1} \sum_{v=0}^{n-1} F(u,v) e^{i2\pi(m \cdot u/M + n \cdot v/N)}$$

Textile



Power and phase spectrum

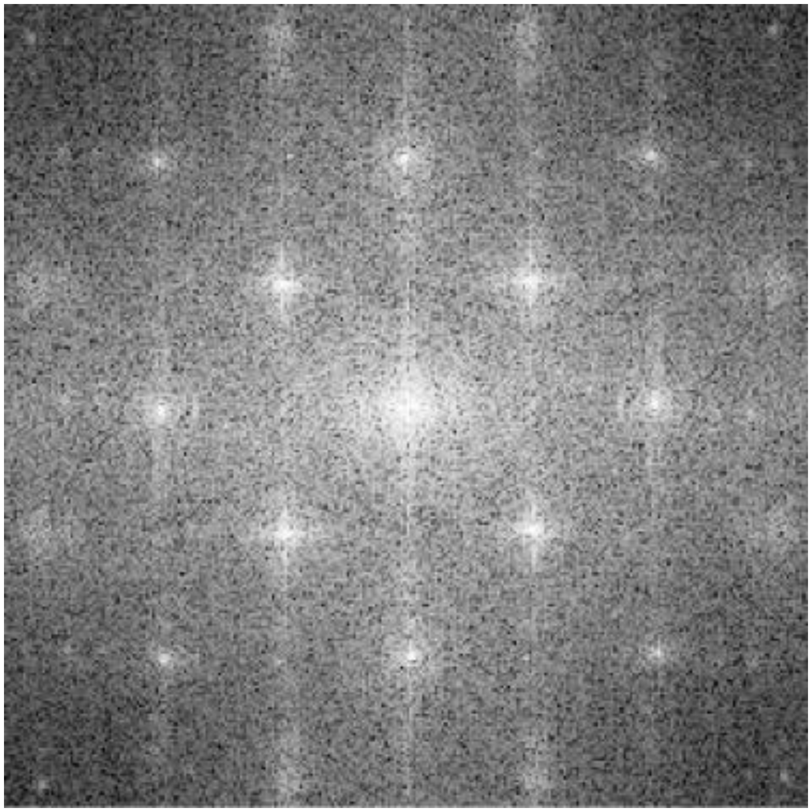
- F(u,v) is a complex number: $F(u,v) = a + i \cdot b$
- Power spectrum consists of squared magnitudes:

$$\left|F(u,v)\right|^2 = a^2 + b^2$$

- Phase spectrum

$$P(u,v) = \tan^{-1}\frac{b}{a}$$

Power spectrum display



Min	0.00032
5% quantile	0.01261
Median	0.07678
95% quantile	0.36108
Max	116.771

$|F(u,v)|^2$ displayed as

$$\log\left(\alpha + |F(u,v)|^2\right)$$

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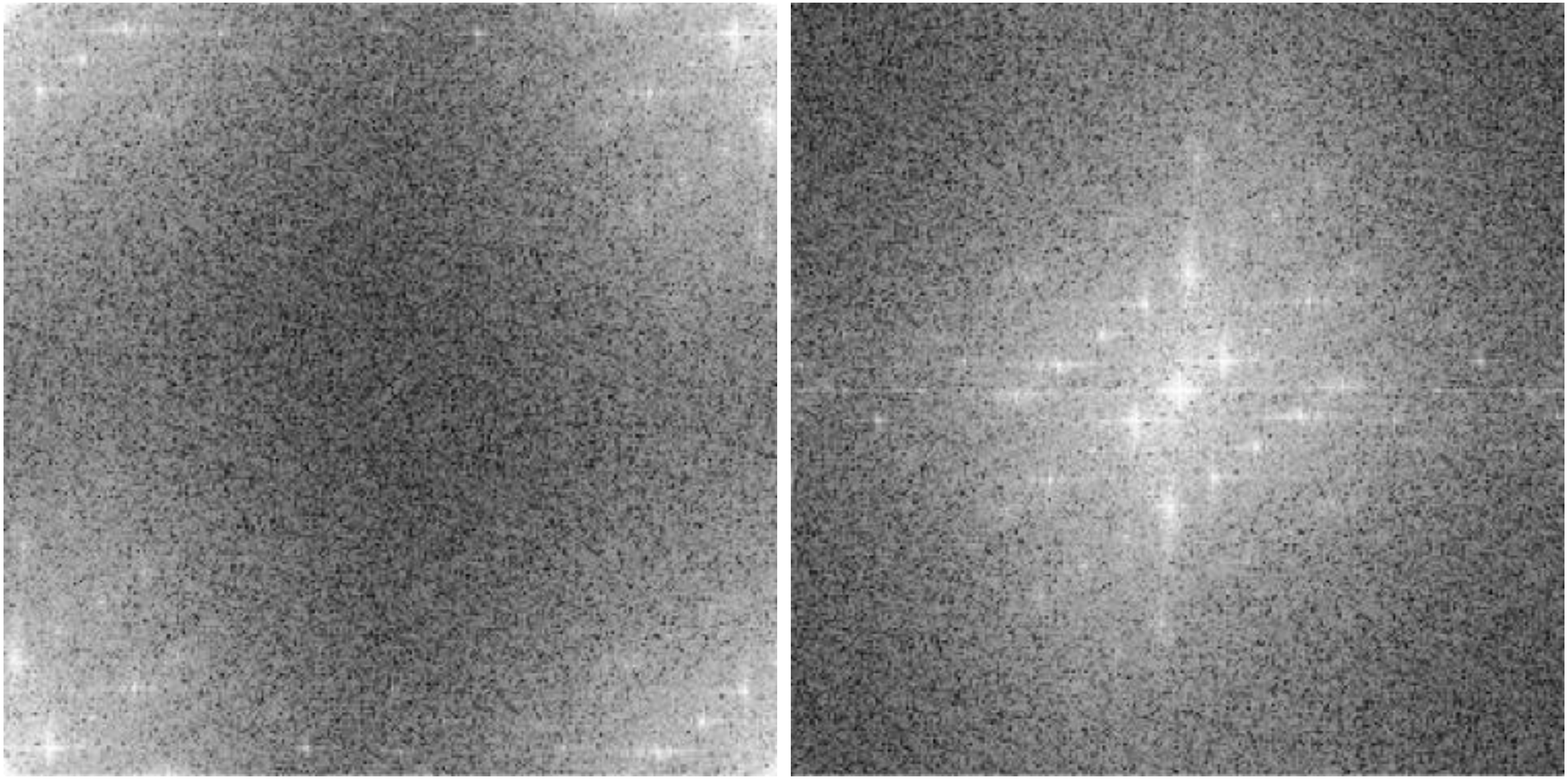
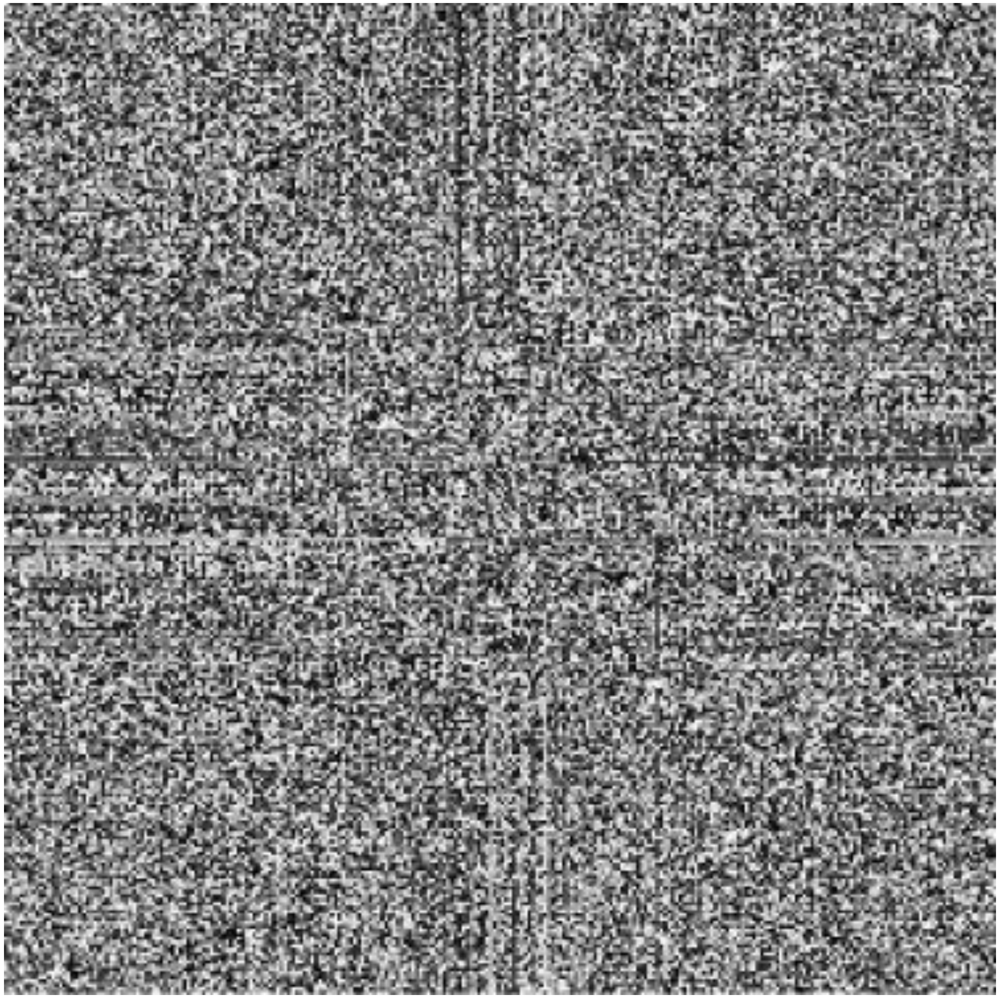
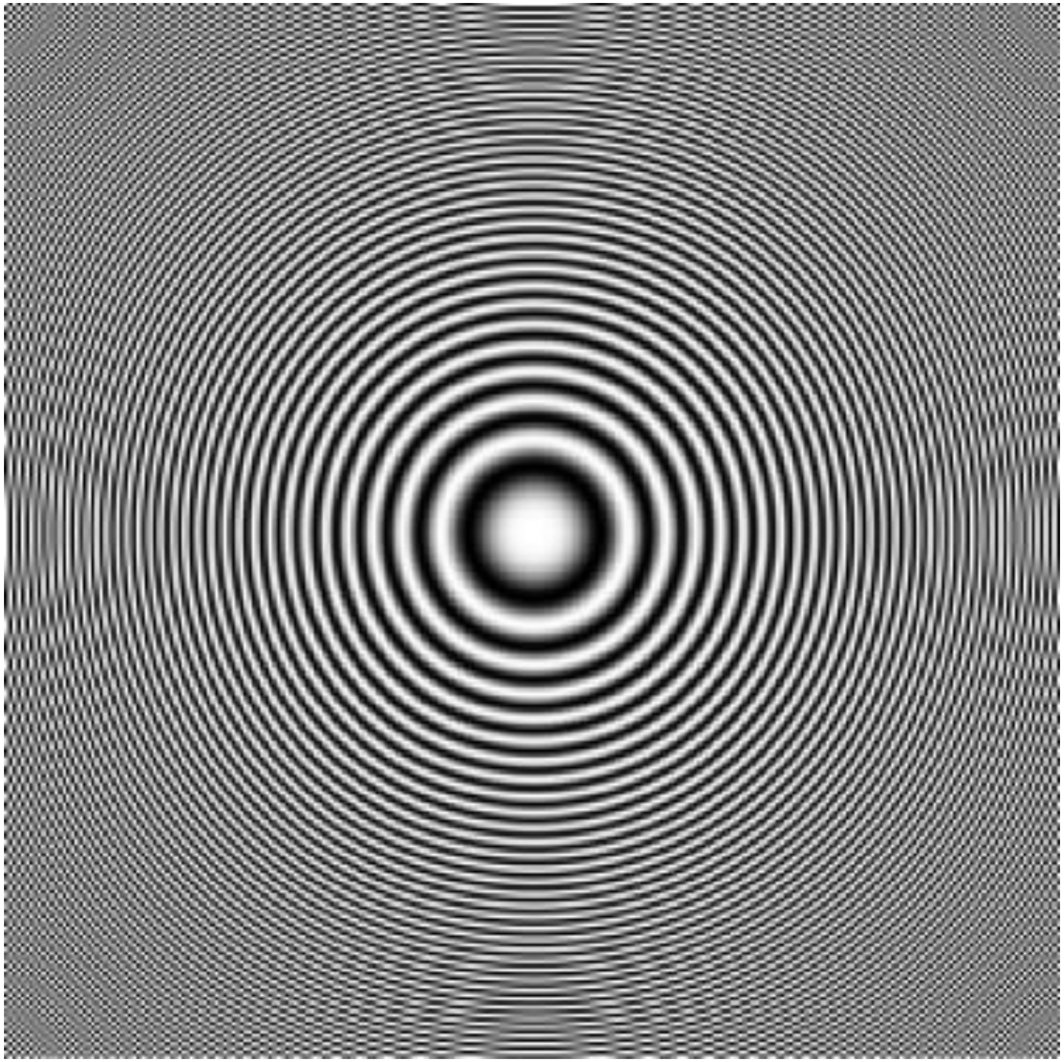
Standard and optical power spectrum



Fourier phase

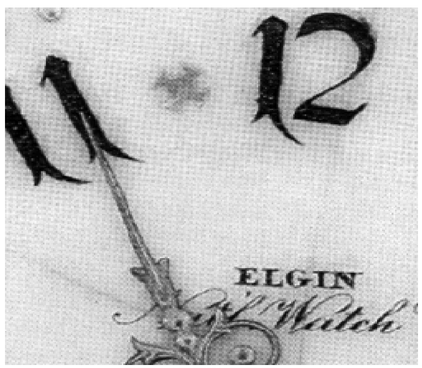


Interpretation of optical DFT

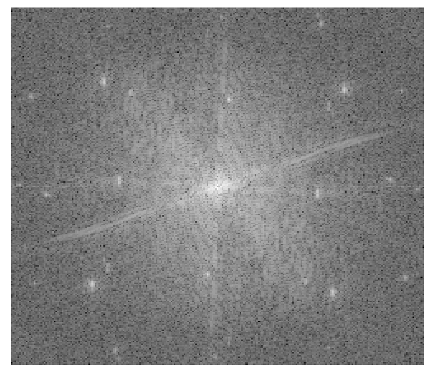


Example (Two Dimension)

f(x,y)



FT(f(x,y))

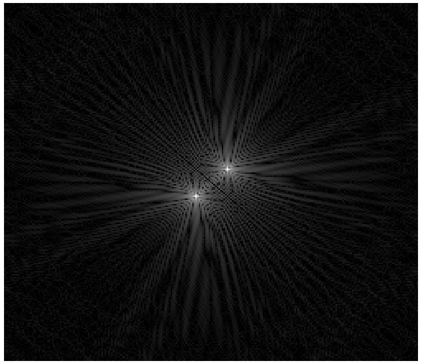


Clock

Example (Two Dimension)

f(x,y)

FT(f(x,y))

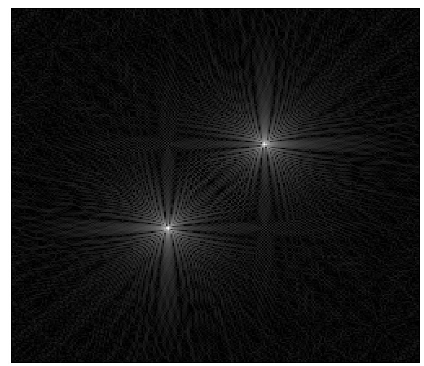


2D - sinewave

Example (Two Dimension)

f(x,y)

FT(f(x,y))

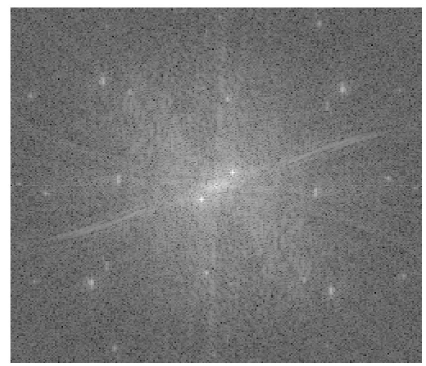


2D - sinewave

Property

f(x,y)

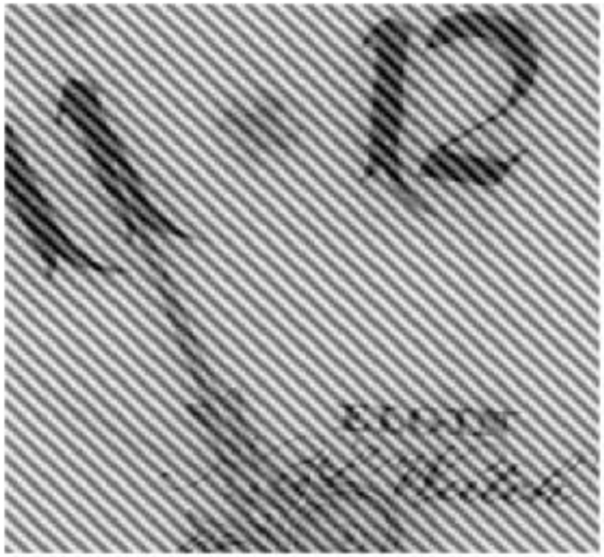
FT(f(x,y))



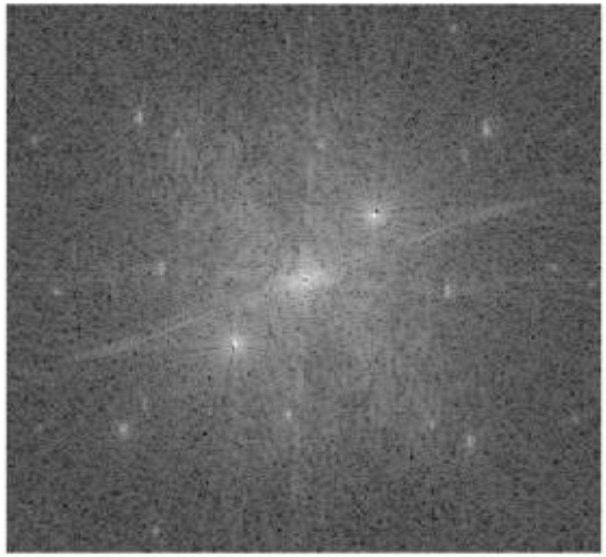
Clock and 2D - sinewave

Example 1

f(x,y)

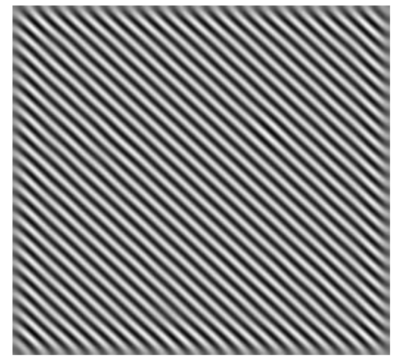


FT(f(x,y))

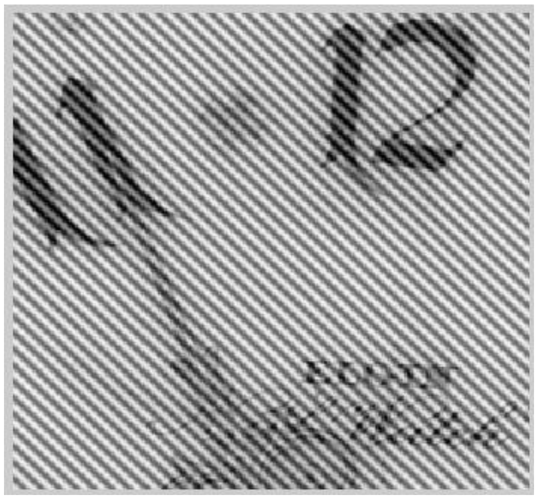


Example 2

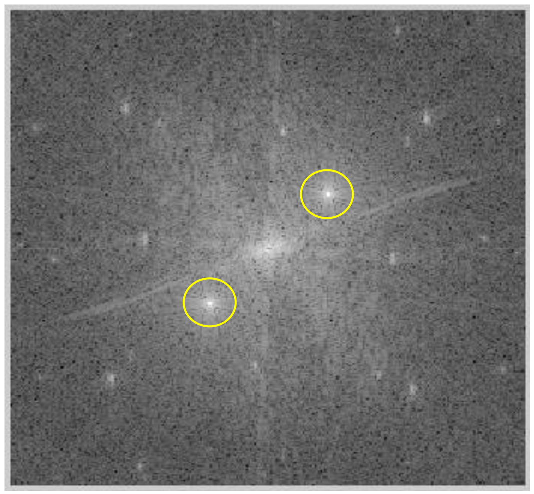
FT^{-1}

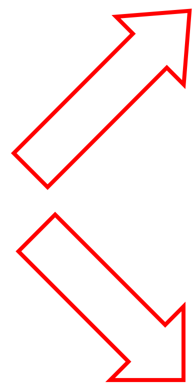


f(x,y)

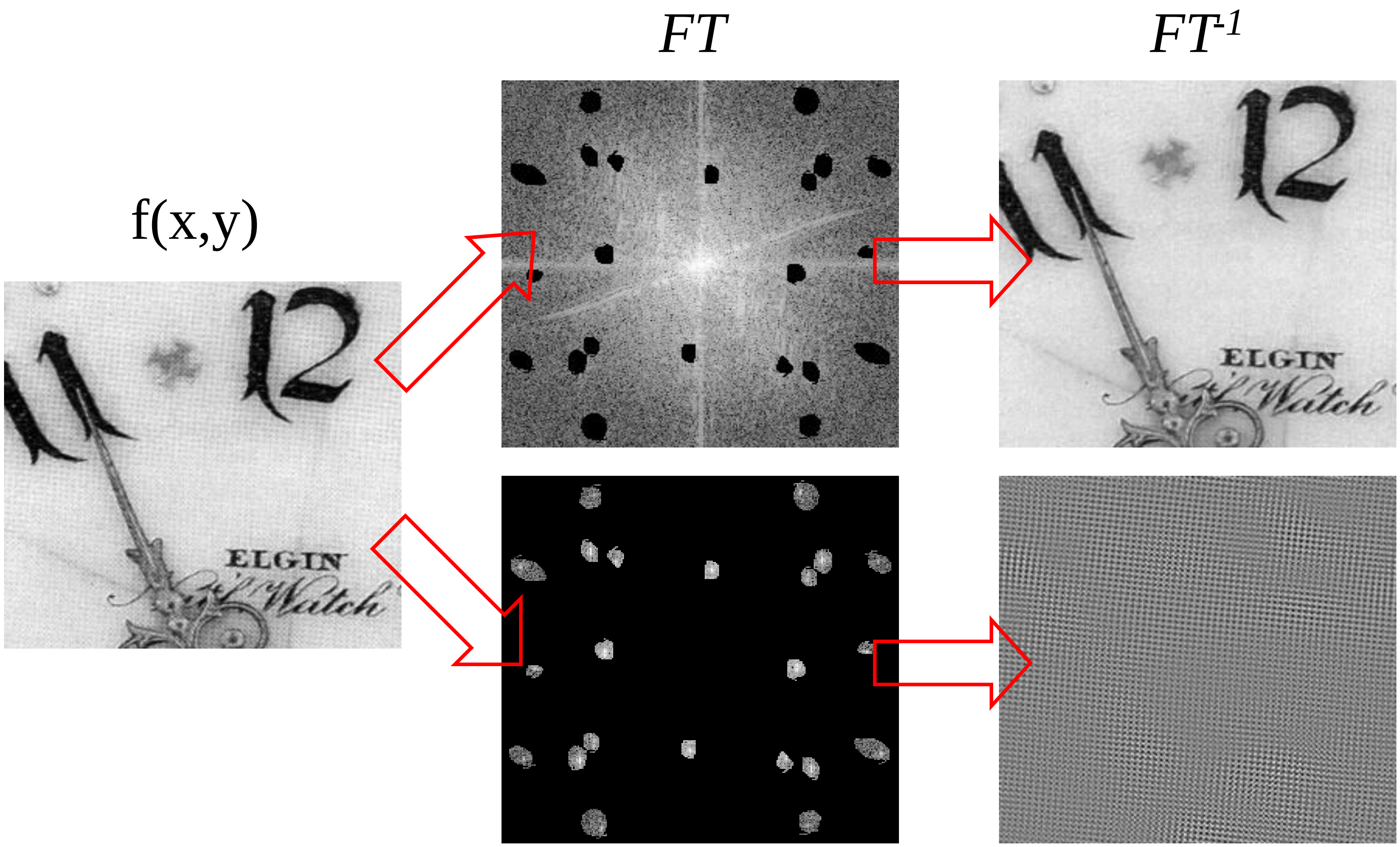


FT

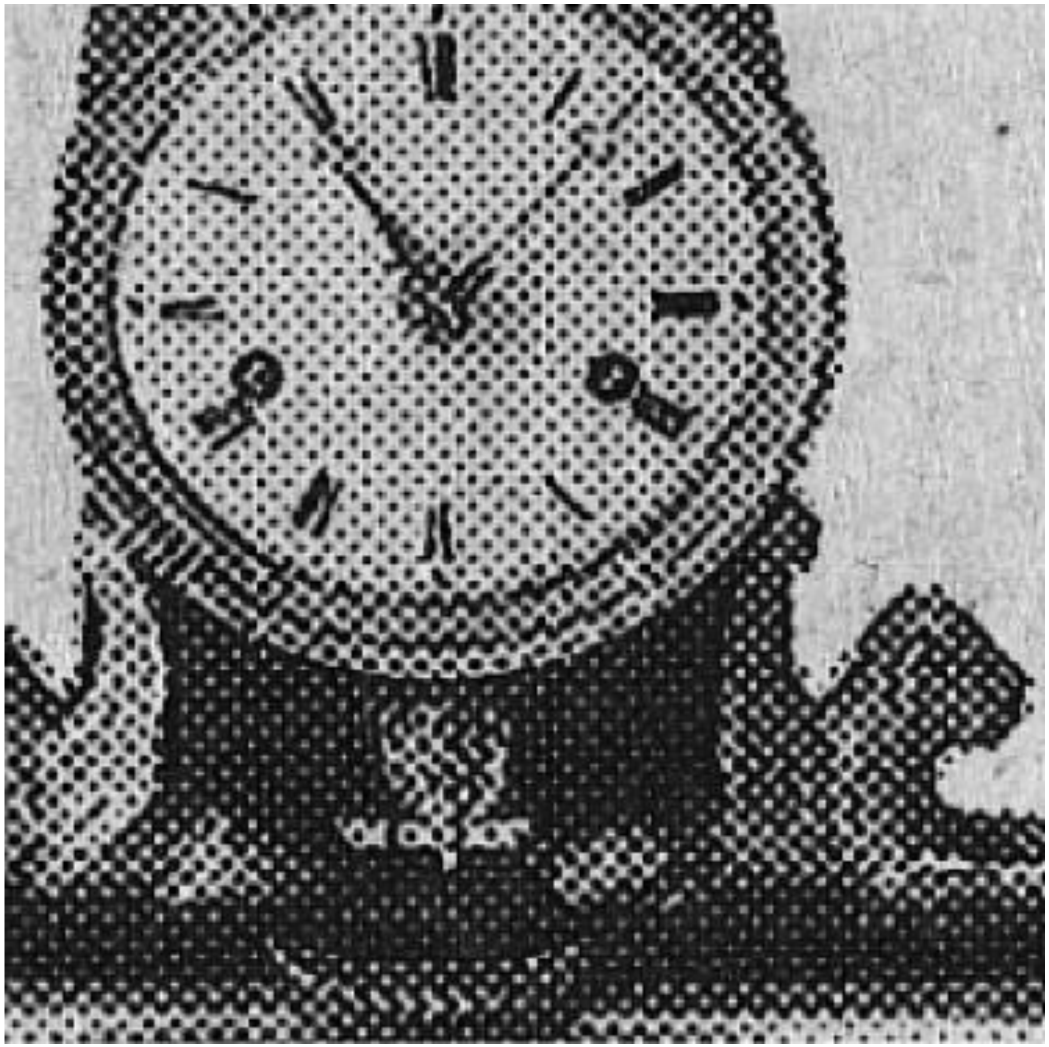




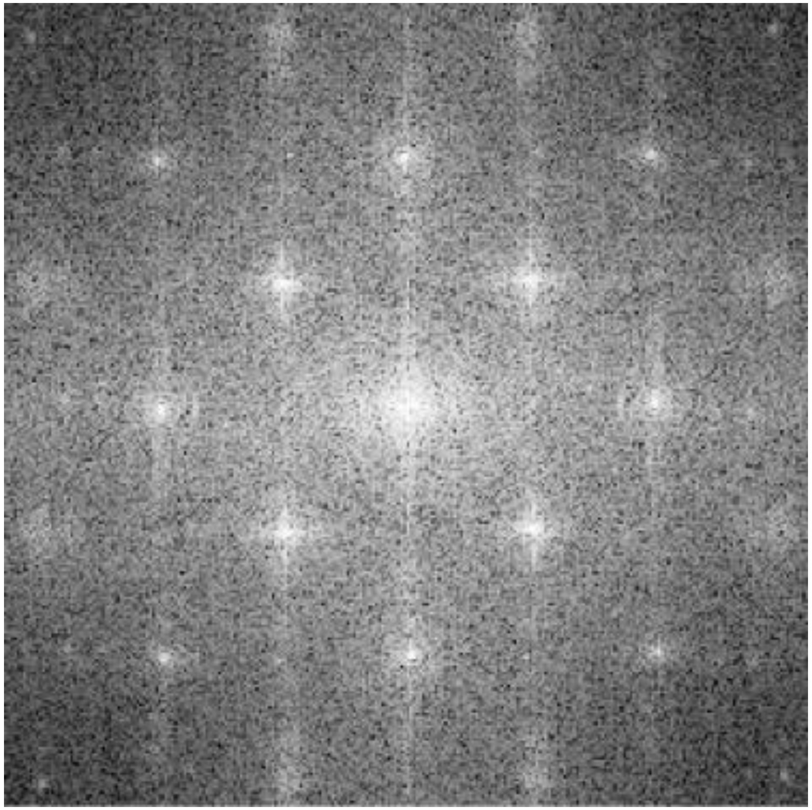
Example 3

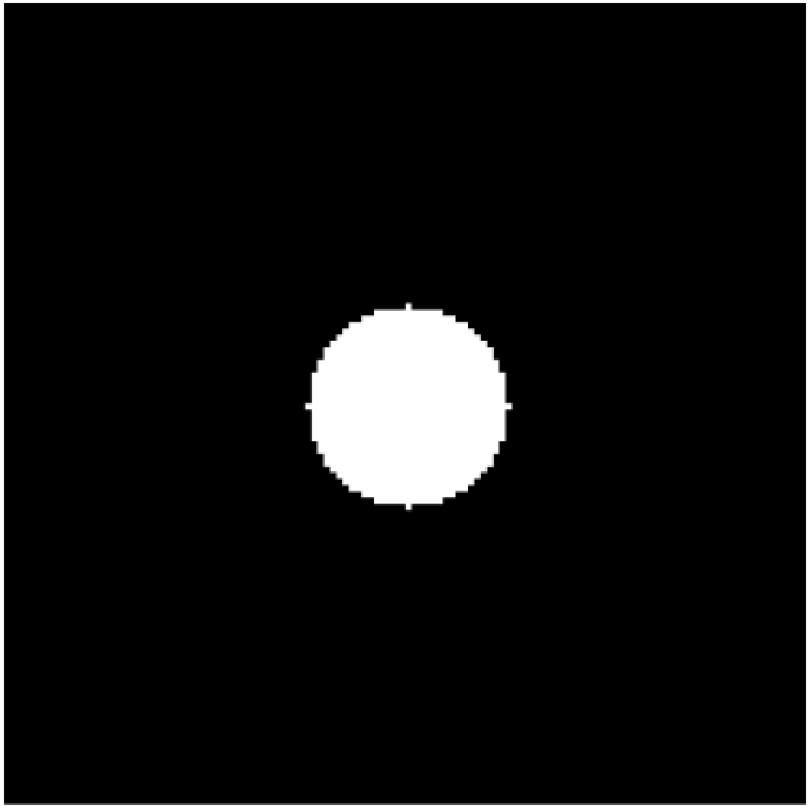


Raster image



Power spectrum and mask





Derasterized image

Scale determination

Convolution theorem

$$\mathbf{F}(f * h) = \mathbf{F}(f) \cdot \mathbf{F}(h)$$

or

$$f * h = \mathbf{F}^{-1}(\mathbf{F}(f) \cdot \mathbf{F}(h))$$

Exercise 00.20

We consider a digital image that has a peak in the Fourier power spectrum for the frequency (a,b) = (27,3). The value of the Fourier transform in the peak is 6+2i. The image is convolved with an unknown lowpass convolution filter, whereby the value of the Fourier transform in the peak is changed to 4-2i.

What is the value of the Fourier transform of the filter in the point (27,3)?

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Exercise 99.14

The discrete Fourier transform is performed on a 480 x 640 (rows x columns) image. A peak is found at frequency (u,v) = (27,0). What is the length in pixels of one period of this frequency?

Exercise 99.14 solution (1)

Looking at the Fourier transform

$$F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{n-1} f(m,n) e^{-i2\pi(m \cdot u/M + n \cdot v/N)}$$

we see that to go from one top to the next we need to increment

$$m \cdot u/M + n \cdot v/N = \begin{pmatrix} u/M \\ v/N \end{pmatrix} \cdot \begin{pmatrix} m \\ n \end{pmatrix}$$

by one through the right choice of m and n.

Exercise 99.14 solution (2)

Setting

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = k \begin{pmatrix} u/M \\ v/N \end{pmatrix}$$

gives

$$k \begin{pmatrix} u/M \\ v/N \end{pmatrix} \cdot \begin{pmatrix} u/M \\ v/N \end{pmatrix} = 1 \Rightarrow k = \frac{1}{(u/M)^2 + (v/N)^2}$$

Thus

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \frac{1}{(u/M)^2 + (v/N)^2} \begin{pmatrix} u/M \\ v/N \end{pmatrix}$$

E.g. $\begin{pmatrix} m' \\ n' \end{pmatrix} = \frac{1}{(27/480)^2 + (0/640)^2} \begin{pmatrix} 27/480 \\ 0/640 \end{pmatrix} = \begin{pmatrix} 480/27 \\ 0 \end{pmatrix}$

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