Paraconsistent Computational Logic

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Abstract. In classical logic everything follows from inconsistency and this makes classical logic problematic in areas of computer science where contradictions seem unavoidable. We describe a many-valued paraconsistent logic, discuss the truth tables and include a small case study.

A Paraconsistent Logic

We consider the propositional fragment of a higher-order paraconsistent logic.

$\Delta = \{\bullet, \circ\}$, the two classical determinate truth values for truth and falsity, respectively.

$\nabla = \{\text{I}, \text{II}, \text{III}, \ldots\}$, a countably infinite set of indeterminate truth values.

The only designated truth value $\bullet$ yields the logical truths.

None of the indeterminate truth values imply the others and there is no specific ordering of the indeterminate truth values.

Definitions I

$$\llbracket \neg\varphi \rrbracket = \begin{cases} \bullet & \text{if } \llbracket \varphi \rrbracket = \circ \\ \circ & \text{if } \llbracket \varphi \rrbracket = \bullet \\ \llbracket \varphi \rrbracket & \text{otherwise} \end{cases} \qquad \begin{array}{ccc} \top & \Leftrightarrow & \neg\bot \\ \bot & \Leftrightarrow & \neg\top \end{array}$$

$$\llbracket \varphi \wedge \psi \rrbracket = \begin{cases} \llbracket \varphi \rrbracket & \text{if } \llbracket \varphi \rrbracket = \llbracket \psi \rrbracket \\ \llbracket \psi \rrbracket & \text{if } \llbracket \varphi \rrbracket = \bullet \\ \llbracket \varphi \rrbracket & \text{if } \llbracket \psi \rrbracket = \bullet \\ \circ & \text{otherwise} \end{cases} \qquad \begin{array}{ccc} \varphi & \Leftrightarrow & \varphi \wedge \varphi \\ \psi & \Leftrightarrow & \top \wedge \psi \\ \varphi & \Leftrightarrow & \varphi \wedge \top \end{array}$$

Abbreviations:

$$\bot \equiv \neg\top \qquad \varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$$

Definitions II

$$[\![\varphi \Leftrightarrow \psi]\!] \;=\; \begin{cases} \bullet & \text{if } [\![\varphi]\!] = [\![\psi]\!] \\ \circ & \text{otherwise} \end{cases}$$

$$[\![\varphi \leftrightarrow \psi]\!] \;=\; \begin{cases} \bullet & \text{if } [\![\varphi]\!] = [\![\psi]\!] \\ [\![\psi]\!] & \text{if } [\![\varphi]\!] = \bullet \\ [\![\varphi]\!] & \text{if } [\![\psi]\!] = \bullet \\ [\![\neg\psi]\!] & \text{if } [\![\varphi]\!] = \circ \\ [\![\neg\varphi]\!] & \text{if } [\![\psi]\!] = \circ \\ \circ & \text{otherwise} \end{cases} \qquad \begin{array}{rcl} \top & \Leftrightarrow & \varphi \leftrightarrow \varphi \\ \psi & \Leftrightarrow & \top \leftrightarrow \psi \\ \varphi & \Leftrightarrow & \varphi \leftrightarrow \top \\ \neg\psi & \Leftrightarrow & \bot \leftrightarrow \psi \\ \neg\varphi & \Leftrightarrow & \varphi \leftrightarrow \bot \end{array}$$

Abbreviations:

$$\varphi \Rightarrow \psi \;\equiv\; \varphi \Leftrightarrow \varphi \wedge \psi \qquad\qquad \varphi \rightarrow \psi \;\equiv\; \varphi \leftrightarrow \varphi \wedge \psi$$

$$\Box\varphi \;\equiv\; \varphi = \top \qquad\qquad {\sim}\varphi \;\equiv\; \neg\Box\varphi$$

Truth Tables I

Although we have a countably infinite set of truth value we can investigate the logic by truth tables since the indeterminate truth values are not ordered with respect to truth content.

$\wedge$	$\bullet$	$\circ$	$\mid$	$\mid\mid$
$\bullet$	$\bullet$	$\circ$	$\mid$	$\mid\mid$
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\mid$	$\mid$	$\circ$	$\mid$	$\circ$
$\mid\mid$	$\mid\mid$	$\circ$	$\circ$	$\mid\mid$

$\vee$	$\bullet$	$\circ$	$\mid$	$\mid\mid$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\circ$	$\bullet$	$\circ$	$\mid$	$\mid\mid$
$\mid$	$\bullet$	$\mid$	$\mid$	$\bullet$
$\mid\mid$	$\bullet$	$\mid\mid$	$\bullet$	$\mid\mid$

$\neg$	
$\bullet$	$\circ$
$\circ$	$\bullet$
$\mid$	$\mid$
$\mid\mid$	$\mid\mid$

Truth Tables II

$\Leftrightarrow$	●	○	$\mid$	$\mid\mid$
●	●	○	○	○
○	○	●	○	○
$\mid$	○	○	●	○
$\mid\mid$	○	○	○	●

$\Rightarrow$	●	○	$\mid$	$\mid\mid$
●	●	○	○	○
○	●	●	●	●
$\mid$	●	○	●	○
$\mid\mid$	●	○	○	●

$\Box$	
●	●
○	○
$\mid$	○
$\mid\mid$	○

$\leftrightarrow$	●	○	$\mid$	$\mid\mid$
●	●	○	$\mid$	$\mid\mid$
○	○	●	$\mid$	$\mid\mid$
$\mid$	$\mid$	$\mid$	●	○
$\mid\mid$	$\mid\mid$	$\mid\mid$	○	●

$\rightarrow$	●	○	$\mid$	$\mid\mid$
●	●	○	$\mid$	$\mid\mid$
○	●	●	●	●
$\mid$	●	$\mid$	●	$\mid$
$\mid\mid$	●	$\mid\mid$	$\mid\mid$	●

$\sim$	
●	○
○	●
$\mid$	●
$\mid\mid$	●

Indeterminate Truth Values I

The required number of indeterminacies corresponds to the number of propositions in a given formula.

A larger number of indeterminacies weakens the logic.

For an atomic formula P, $\nabla = \{\shortmid\}$ suffices, because $[\![P]\!] = \shortmid\shortmid$ would not be different from $[\![P]\!] = \shortmid$ when we consider logical truths.

Indeterminate Truth Values II

Contraposition: $P \to Q \;\leftrightarrow\; \neg Q \to \neg P$

The formula holds in a logic with a single indeterminacy.

Counter-example for two indeterminacies:

$$\begin{array}{ccccccccc} P & \to & Q & \leftrightarrow & \neg & Q & \to & \neg & P \\ \mathsf{I} & \mathsf{I} & \mathsf{II} & \circ & \mathsf{II} & \mathsf{II} & \mathsf{II} & \mathsf{I} & \mathsf{I} \end{array}$$

Having $\nabla = \{\mathsf{I}, \mathsf{II}, \mathsf{III}\}$ does not weaken the logic further.

Case Study I

Consider an agent with a set of beliefs (0) and rules (1-2):

0. $P \wedge Q \wedge \neg R$
1. $P \wedge Q \rightarrow R$
2. $R \rightarrow S$

$$(\; P \wedge Q \wedge \neg \; R \;) \wedge (\; P \wedge Q \rightarrow R \;) \Rightarrow \ldots$$

● ● ● ● ● ○ ○ ● ● ● ○ ○ ●

$$(\; P \wedge Q \wedge \neg \; R \;) \wedge (\; P \wedge Q \rightarrow R \;) \Rightarrow R$$

● | | | ● ○ | ● | | | ○ | ○

$$(\; P \wedge Q \wedge \neg \; R \;) \wedge \Box (\; P \wedge Q \rightarrow R \;) \Rightarrow R$$

● | | | ● ○ ○ ○ ● | | | ○ ● ○

Case Study II

We let $\rhd_{XYZ}\ P$ mean that P follows from the agents beliefs and rules X, Y and Z, where rules are boxed, so $\rhd_{012}\ Q \wedge R$ considers the logical truth of the formula:

$$\underbrace{(P \wedge Q \wedge \neg R)}_{0} \wedge \Box\underbrace{(P \wedge Q \rightarrow R)}_{1} \wedge \Box\underbrace{(R \rightarrow S)}_{2} \Rightarrow Q \wedge R$$

In particular:

$$\begin{array}{lll} \not\rhd_{012}\ \neg P & \not\rhd_{012}\ \neg Q & \not\rhd_{012}\ \neg S \\ \rhd_{012}\ R & \rhd_{012}\ \neg R & \rhd_{012}\ S \end{array}$$

Conclusions

We have defined an infinite-valued paraconsistent logic using semantic clauses and motivated by key equalities.

Only a finite number of truth values need to be considered for a given formula.

The logic allows agents to reason using inconsistent beliefs and rules without entailing everything.

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