Problem Set 3

Ph.D. Course 2009:
An Introduction to DG-FEM for solving partial differential equations

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If you have not already done so, download all the Matlab codes from the book from

http://www.nudg.org/

and store and unpack them in a directory you can use with Matlab.

Let us go back to the simple problem

$$\partial_t u + a(x)\partial_x u = g(x,t), \quad x \in [-2,2].$$

You can used the codes from Exercise 1 (Advecxxx.m) for this with the boundary condition at $x = -2$ being given by the exact solution.

First assume that $a(x) = 1$ and $g(x,t) = 0$.

- We first consider the case with the exact solution being based on the initial condition
$$u(x,0) = -(\mathrm{sign}(x) - 1)/2.$$
Run the code until T=1 and evaluate the hp-convergence in the L^2-norm. Remember to look at the solution !

- Repeat the exercise but for the initial condition
$$u(x,0) = |x|.$$
Do the results in these two exercises agree with what you know about the error estimates of the method ? – what about the smooth example you considered in Exercise 1 ?

Let us now assume that

$$a(x) = \begin{cases} 1.5 & |x| \leq 0.5 \\ 1 & \text{otherwise} \end{cases}$$

and that the exact solution is assumed to $u(x,t) = \sin(\pi(x - a(x)t))$.

- Derive $g(x,t)$ so the provided $u(x,t)$ is the exact solution.

- Run the code until T=1 and evaluate the hp-convergence in the L^2-norm. What kind of accuracy do you obtain – are the results in agreement with your expectations?

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- Does it matter whether you have an element interface to co-inside with the position of the discontinuities of $a(x)$?

We next consider the 1D model for acoustic waves in an mean flow, given as

$$\begin{aligned} u_t + M(x)u_x &= -p_x \\ p_t + M(x)p_x &= -u_x \end{aligned} \tag{1}$$

where the unknowns are the velocity, $u(x,t)$, and the pressure, $p(x,t)$. The Machnumber, $M(x) = u_0(x)/c_0(x)$, reflects the velocity of the steady mean flow. Note that this is a slight generalization of the problem discussed in Exercise 2.

The Mach-number profile can be both constant, variable, and even piecewise smooth only, i.e., $M(x)$ can be discontinuous when an acoustic signal propagates through a shock.

- Use an energy method to show that the system is hyperbolic (diagonizable and with real eigenvalues) and conserves energy if (u,p) are assumed periodic.
- Using an energy technique, discuss how many boundary conditions are needed in a finite domain at each end – note that this depends on $M(x)$!

At this point, we have established wellposedness and understand what kind of boundary conditions are needed. We shall now seek the development of a numerical solver of this problem using DG-FEM.

- Assume that $M(x)$ varies smoothly and write it on the form

  $$\mathbf{q}_t + \mathcal{A}(q)_x = 0,$$

  where

  $$\mathbf{q} = \begin{bmatrix} u \\ p \end{bmatrix}, \; \mathcal{A} = \begin{bmatrix} M & 1 \\ 1 & M \end{bmatrix}.$$

  One can show that the upwind flux in this case takes the form

  $$(\mathcal{A}\mathbf{q})^* = \mathcal{A}\left(\begin{bmatrix} \{\{u\}\} \\ \{\{p\}\} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} [[p]] \\ [[u]] \end{bmatrix} \right).$$

  Is it reasonable that it takes the form ? – can you identify the different terms ?

  Discuss why we only really need to consider the case where $|M(x)| \leq 1$.

- Modify your codes (Waves1Dxxx.m) to implement a DG-FEM solver for the advective acoustic equations with flux given above. Writing out the numerical flux gives

$$\hat{\mathbf{n}} \cdot (\mathcal{A}\mathbf{q})^* = \frac{\hat{n}_x}{2}\mathcal{A}\left[\begin{array}{c} u^+ - u^- \\ p^+ + p^- \end{array}\right] + \frac{1}{2}\mathcal{A}\left[\begin{array}{c} p^- - p^+ \\ u^- - u^+ \end{array}\right].$$

- Extend the code to also have a simple central flux.
- How do you determine the stable timestep ?

To test the accuracy of the code, we shall use the old trick of a constructed solution – simply choose a solution (u, p), insert it into the equation and find the remainder. Adding this to the equation then guarantees that you know the exact solution

- Validate the accuracy of the code, i.e., show that $\|\varepsilon\|_{\Omega,h} \leq Ch^s$ - what is s?
- Is the accuracy impacted by the choice of flux ?

If you have time, you can consider the simple local time-stepping scheme based on two Adam-Bashforth schemes, one for 1/2 timestep that was discussed in class.

- Implement the local time-stepping scheme for the linear advection problems with just two levels – this requires changes in Advec1D.m and Advec1DRHS.m.
- Construct a simple 1D grid with very different cell-sizes to test the scheme. Do you see any accuracy problems with this – is the code faster?

Enjoy!