Problem Set 2

Ph.D. Course 2009:
An Introduction to DG-FEM for solving partial differential equations

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If you have not already done so, please download all the Matlab codes from the book from

http://www.nudg.org/

and store and unpack them in a directory you can use with Matlab.

To familarize ourselves with the setup for problems with one spatial direction (1D), we consider a coupled system of two linear advection equations, namely

$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + \partial_x u &= 0 \end{aligned} \quad , \quad x \in [-\pi, \pi] \tag{1}$$

These equations can be reduced to the classical wave equation

$$\partial_{tt} u - \partial_{xx} u = 0 \tag{2}$$

which can describe various linear wave phenomena, e.g. in acoustics, electromagnetics and free surface hydraulics.

Exact solutions to the wave equation can be shown to be of the form

$$\begin{aligned} u(x,t) &= f(x+t) + g(x-t) \\ v(x,t) &= -f(x+t) + g(x-t) \end{aligned}$$

where $f(\cdot)$ and $g(\cdot)$ are arbitrary functions representing respectively left and right moving wave forms. The exact solution can be used for defining both initial and boundary conditions.

Make a copy of the three Matlab scripts for solving the linear advection equation (Advec1Dxxx) and rename them to Wave1Dxxx. Your task will be to make appropriate changes to the previous setup to solve the wave equation in 1D.

- Verify that the coupled system of equations is equivalent to the classical wave equation.
- Show that the exact solution satisfies the wave equations.
- Show, using an Energy Method, that solutions to the wave equation conserves energy if u and v are assumed periodic.

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- Derive a DG-FEM scheme for solving the coupled set of first-order equations.
- Implement and solve the wave equation on a periodic domain. This is possible by using the indexmaps mapI and mapO as demonstrated in AdvecRHS1D.m.
- Discuss how many boundary conditions are needed and where in order to solve the wave equation in a finite domain.
- Implement and solve the wave equation on a finite domain.
- Carry out a *hp*-convergence test and determine a global error estimate for convergence.

If time permits it

- Derive and implement an upwind flux scheme to solve the two coupled equations.

To derive an upwind scheme one could use a flux decomposition method based on the characteristics of the system. Consider the general system

$$\partial_t \mathbf{q} + \partial_x F(\mathbf{q}) = \partial_t \mathbf{q} + \partial_x(\mathcal{A}\mathbf{q}) = 0, \quad \mathbf{q} = (u, v)^T \tag{3}$$

Let us first assume that $\mathcal{A}$ is a constant matrix and put the system on the form

$$\partial_t \mathbf{q} + \mathcal{A}\partial_x \mathbf{q} = 0, \quad \mathbf{q} = (u, v)^T \tag{4}$$

Then, determine the eigenvalues and eigenvectors of the flux jacobian $\partial F/\partial \mathbf{q} = \mathcal{A}$. If the eigenvalues are distinct and real then the problem is strictly hyperbolic and, thus, wellposed. It is then possible to diagonalize $\mathcal{A} = \mathcal{R}\mathcal{D}\mathcal{R}^{-1}$, where $\mathcal{R}$ is a matrix with right vectors and $\mathcal{D}$ is diagonal matrix which holds the eigenvalues. With this diagonalization we can determine the characteristic variables as $\mathbf{w} = \mathcal{R}^{-1}\mathbf{q}$ and the characteristic equations

$$\partial_t \mathbf{w} + \mathcal{D}\partial_x \mathbf{w} = 0, \quad \mathbf{w} = (w_1, w_2)^T \tag{5}$$

which consists of two decoupled linear advection equations.

- Would there be any changes in this scheme if $\mathcal{A}$ would depend smoothly on x.
- What if $\mathcal{A}$ would vary in a nonsmooth fashion ?

Enjoy!