DG-FEM for PDE's
Lecture 4

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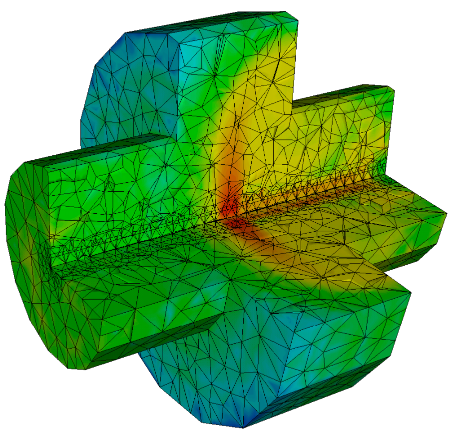
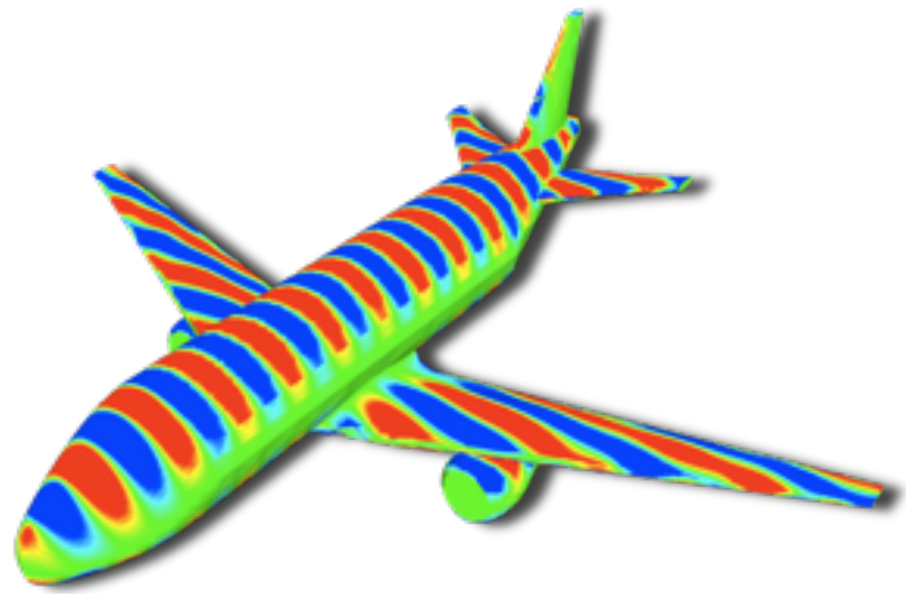




A brief overview of what's to come

- Lecture 1: Introduction and DG-FEM in 1D
- Lecture 2: Implementation and numerical aspects
- Lecture 3: Insight through theory
- Lecture 4: Nonlinear problems
- Lecture 5: Extension to two spatial dimensions
- Lecture 6: Introduction to mesh generation
- Lecture 7: Higher order/Global problems
- Lecture 8: 3D and advanced topics

Lecture 4

- ✓ Let's briefly recall what we know
- ✓ Part I: Smooth problems
  - ✓ Conservations laws and DG properties
  - ✓ Filtering, aliasing, and error estimates
- ✓ Part II: Nonsmooth problems
  - ✓ Shocks and Gibbs phenomena
  - ✓ Filtering and limiting
  - ✓ TVD-RK and error estimates

A brief summary

We now have a good understanding all key aspects of the DG-FEM scheme for linear first order problems

- We understand both accuracy and stability and what we can expect.
- The dispersive properties are excellent.
- The discrete stability is a little less encouraging. A scaling like

$$\Delta t \leq C \frac{h}{aN^2}$$

is the Achilles Heel -- but there are ways!

... but what about nonlinear problems ?

Conservation laws

Let us first consider the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in [L, R] = \Omega,$$
$$u(x,0) = u_0(x),$$

with boundary conditions specified at inflow

$$\hat{\boldsymbol{n}} \cdot \frac{\partial f}{\partial u} = \hat{\boldsymbol{n}} \cdot f_u < 0.$$

The equation has a fundamental property

$$\frac{d}{dt} \int_a^b u(x) dx = f(u(a)) - f(u(b));$$

Changes by inflow-outflow differences only

Conservation laws

Importance ?

This is perhaps most basic physical model
in continuum mechanics:

- ✓ Maxwell's equations for EM
- ✓ Euler and Navier-Stokes equations of fluid/gas
- ✓ MHD for plasma physics
- ✓ Navier's equations for elasticity
- ✓ General relativity
- ✓ Traffic modeling

Conservation laws are fundamental

Conservation laws

One major problem with them:

Discontinuous solutions can form spontaneously even for smooth initial conditions

... and how do we compute a derivate of a step ?

Conservation laws

One major problem with them:

Discontinuous solutions can form spontaneously even for smooth initial conditions

... and how do we compute a derivate of a step ?

Introduce weak solutions satisfying

$$\int_0^\infty \int_{-\infty}^\infty \left(u(x,t)\frac{\partial \phi}{\partial t} + f(u)\frac{\partial \phi}{\partial x} \right) \, dx \, dt = 0,$$

$$\int_{-\infty}^\infty \left(u(x,0) - u_0(x) \right) \phi(x,0) dx = 0.$$

where $\phi(x,t)$ is a smooth compact testfunction

Conservation laws

Now, we can deal with discontinuous solutions

... but we have lost uniqueness!

To recover this, we define a convex entropy

$$\eta(u), \quad \eta''(u) > 0$$

and an entropy flux

$$F(u) = \int_u \eta'(v) f'(v) \, dv,$$

If one can prove that

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} F(u) \leq 0,$$

uniqueness is restored (for *f* convex)

Back to the scheme

Recall the two DG formulations

$$\int_{\mathsf{D}^k} \left(\frac{\partial u_h^k}{\partial t} \ell_i^k(x) - f_h^k(u_h^k) \frac{d\ell_i^k}{dx} \right) dx = - \int_{\partial \mathsf{D}^k} \hat{\boldsymbol{n}} \cdot f^* \ell_i^k(x) \, dx,$$

$$\int_{\mathsf{D}^k} \left(\frac{\partial u_h^k}{\partial t} + \frac{\partial f_h^k(u_h^k)}{\partial x} \right) \ell_i^k(x) \, dx = \int_{\partial \mathsf{D}^k} \hat{\boldsymbol{n}} \cdot \left(f_h^k(u_h^k) - f^* \right) \ell_i^k(x) \, dx.$$

We shall be using a monotone flux, e.g., the LF flux

$$f^*(u_h^-, u_h^+) = \{\{f_h(u_h)\}\} + \frac{C}{2} [\![u_h]\!],$$

Recall also the assumption on the local solution

$$x \in \mathsf{D}^k \colon u_h^k(x,t) = \sum_{i=1}^{N_p} u^k(x_i,t) \ell_i^k(x), \; f_h^k(u_h(x,t)) = \sum_{i=1}^{N_p} f^k(x_i,t) \ell_i^k(x),$$

Note: $f^k(x_i,t) = \mathcal{P}_N(f^k)(x_i,t)$

Properties of the scheme

Using our common matrix notation we have

$$\mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k - \mathcal{S}^T \boldsymbol{f}_h^k = -\left[\boldsymbol{\ell}^k(x) f^*\right]_{x_l^k}^{x_r^k},$$

$$\mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k + \mathcal{S} \boldsymbol{f}_h^k = \left[\boldsymbol{\ell}^k(x)(f_h^k - f^*)\right]_{x_l^k}^{x_r^k},$$

$$\boldsymbol{u}_h^k = [u_h^k(x_1^k), \dots, u_h^k(x_{N_p}^k)]^T, \quad \boldsymbol{f}_h^k = [f_h^k(x_1^k), \dots, f_h^k(x_{N_p}^k)]^T.$$

Multiply with a smooth testfunction from the left

$$\boldsymbol{\phi}_h^T \mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k - \boldsymbol{\phi}_h^T \mathcal{S}^T \boldsymbol{f}_h^k = -\boldsymbol{\phi}_h^T \left[\boldsymbol{\ell}^k(x) f^*\right]_{x_l^k}^{x_r^k}$$

$\phi = 1$

$$\frac{d}{dt}\int_{x_l^k}^{x_r^k} u_h \, dx = f^*(x_l^k) - f^*(x_r^k).$$

Local/elementwise conservation

Properties of the scheme

Summing over all elements we have

$$\sum_{k=1}^{K} \frac{d}{dt} \int_{x_l^k}^{x_r^k} u_h \, dx = \sum_{k_e} \hat{\boldsymbol{n}}_e \cdot [\![f^*(x_e^k)]\!],$$

but the numerical flux is single valued, i.e.,

Global conservation

Let us now assume a general smooth test function

$$x \in \mathsf{D}^k : \quad \phi_h(x,t) = \sum_{i=1}^{N_p} \phi(x_i^k, t) \ell_i^k(x),$$

so we obtain

$$\left(\phi_h, \frac{\partial}{\partial t} u_h \right)_{\mathsf{D}^k} - \left(\frac{\partial \phi_h}{\partial x}, f_h \right)_{\mathsf{D}^k} = - \left[\phi_h f^* \right]_{x_l^k}^{x_r^k} .$$

Properties of the scheme

Integration by parts in time yields

$$\int_0^\infty \left[\left(\frac{\partial}{\partial t}\phi_h, u_h \right)_{\mathsf{D}^k} + \left(\frac{\partial \phi_h}{\partial x}, f_h \right)_{\mathsf{D}^k} - [\phi_h f^*]_{x_l^k}^{x_r^k} \right] dt + (\phi_h(0), u_h(0))_{\mathsf{D}^k} = 0.$$

Summing over all elements yields

$$\int_0^\infty \left[\left(\frac{\partial}{\partial t}\phi_h, u_h \right)_{\Omega,h} + \left(\frac{\partial \phi_h}{\partial x}, f_h \right)_{\Omega,h} \right] dt$$

$$+ (\phi_h(0), u_h(0))_{\Omega,h} = \int_0^\infty \sum_{k_e} \hat{\boldsymbol{n}}_e \cdot [\![\phi_h(x_e^k) f^*(x_e^k)]\!] \, dt.$$

Since the test function is smooth, RHS vanishes

⇒ Solution is a weak solution

⇒ Shocks propagate a correct speed

Properties of the scheme

Consider again

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0,$$

Define the convex entropy

$$\eta(u) = \frac{u^2}{2}, \quad F'(u) = \eta' f'.$$

and note that

$$F(u) = \int_u f' u \, du = f(u)u - \int_u f \, du = f(u)u - g(u),$$

$$g(u) = \int_u f(u) \, du.$$

Properties of the scheme

Consider the scheme

$$\mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k + \mathcal{S}\boldsymbol{f}_h^k = \left[\boldsymbol{\ell}^k(x)(f_h^k - f^*)\right]_{x_l^k}^{x_r^k}.$$

multiply with u from the left to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_h^k\|_{\mathsf{D}^k}^2 + \int_{\mathsf{D}^k} u_h^k \frac{\partial}{\partial x} f_h^k \, dx = \left[u_h^k(x)(f_h^k - f^*)\right]_{x_l^k}^{x_r^k}.$$

Realize now that

$$\begin{aligned}\int_{\mathsf{D}^k} u_h^k \frac{\partial}{\partial x} f_h^k \, dx &= \int_{\mathsf{D}^k} \eta'(u_h^k) f'(u_h^k) \frac{\partial}{\partial x} u_h^k \, dx \\ &= \int_{\mathsf{D}^k} F'(u_h^k) \frac{\partial}{\partial x} u_h^k \, dx = \int_{\mathsf{D}^k} \frac{\partial}{\partial x} F(u_h^k) \, dx,\end{aligned}$$

Properties of the scheme

This yields

$$\frac{1}{2}\frac{d}{dt}\|u_h^k\|^2_{\mathsf{D}^k} + \left[F(u_h^k)\right]_{x_l^k}^{x_r^k} = \left[u_h^k(x)(f_h^k - f^*)\right]_{x_l^k}^{x_r^k}.$$

At each interface we have a term like

$$F(u_h^-) - F(u_h^+) - u_h^-(f_h^- - f^*) + u_h^+(f_h^+ - f^*) \geq 0,$$

$$\Rightarrow \quad -g(u_h^-) + g(u_h^+) - f^*(u_h^+ - u_h^-) \geq 0.$$

Use the mean value theorem to obtain

$$g(u_h^+) - g(u_h^-) = g'(\xi)(u_h^+ - u_h^-) = f(\xi)(u_h^+ - u_h^-),$$

$$g(u) = \int_u f(u)\,du.$$

Properties of the scheme

Combining everything yields the condition

$$(f(\xi) - f^*)(u_h^+ - u_h^-) \geq 0,$$

This is an E-flux -- and all monotone fluxes satisfy this!

We have just proven that

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{\Omega,h} \leq 0.$$

Nonlinear stability -- just by the monotone flux

✓ No limiting

✓ No artificial dissipation

This is a very strong result!

Properties of the scheme

It gets better -- define the flux

$$\hat{F}(x) = f^*(x)u(x) - g(x),$$

Using similar arguments as above, one obtains

$$\frac{d}{dt}\int_{\mathsf{D}^k} \eta(u_h^k)\,dx + \hat{F}(x_r^k) - \hat{F}(x_r^{k-1}) \leq 0,$$

A cell entropy condition

If the flux is convex and the solution bounded

⇒ Convergence to the unique entropy solution

Properties of the scheme

We have managed to prove

- ✓ Local conservation
- ✓ Global conservation
- ✓ Solution is a weak solution
- ✓ Nonlinear stability
- ✓ A cell entropy condition

No other known method can match this!

Note: Most of these results are only valid for scalar convex problems -- but this is due to an incomplete theory for conservation laws and not DG

Consider an example

Consider

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad x \in [-1,1], \qquad f(u) = a(x)u(x,t), \quad a(x) = (1-x^2)^5 + 1.$$

- Scheme I

$$\mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k + \mathcal{S}\boldsymbol{f}_h^k = \frac{1}{2}\oint_{x_l^k}^{x_r^k} \hat{\boldsymbol{n}} \cdot [\![f_h^k]\!] \boldsymbol{\ell}^k(x)\,dx,$$

$$f_h^k(x) = \mathcal{P}_N(a(x)u_h^k(x)) \qquad f_h^k(x,t) = \sum_{i=1}^{N_p} f_h^k(x_i^k)\ell_i^k(x),$$

- Scheme II

$$\mathcal{S}_{ij}^{k,a} = \int_{x_l^k}^{x_r^k} \ell_i^k \frac{d}{dx} a(x)\ell_j^k\,dx, \qquad \mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k + \mathcal{S}^{k,a}\boldsymbol{u}_h^k = \frac{1}{2}\oint_{x_l^k}^{x_r^k} \hat{\boldsymbol{n}} \cdot [\![a(x)u_h^k]\!] \boldsymbol{\ell}^k(x)\,dx.$$

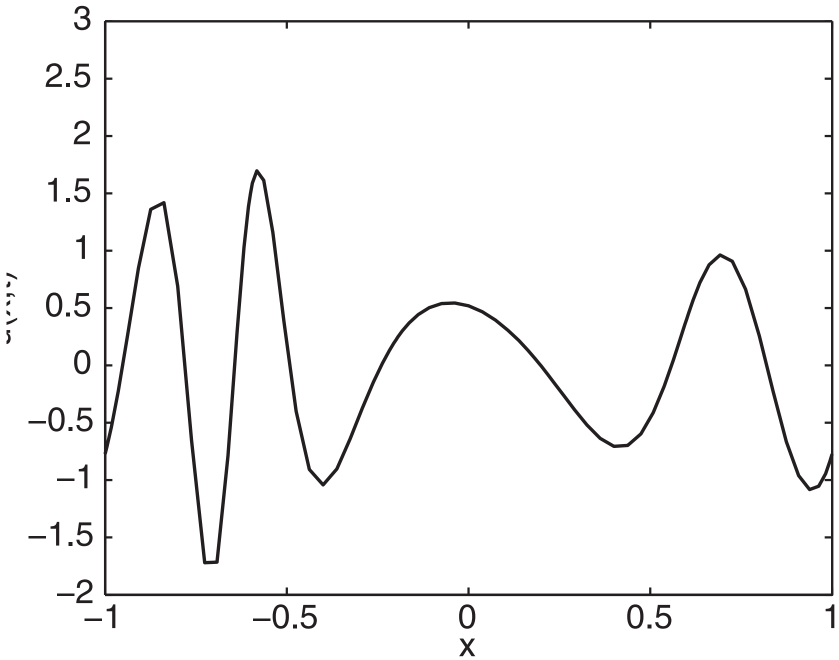
- Scheme III

$$\mathcal{M}^k \frac{d}{dt}\boldsymbol{u}_h^k + \mathcal{S}\boldsymbol{f}_h^k = \frac{1}{2}\oint_{x_l^k}^{x_r^k} \hat{\boldsymbol{n}} \cdot [\![f_h^k]\!] \boldsymbol{\ell}^k(x)\,dx,$$

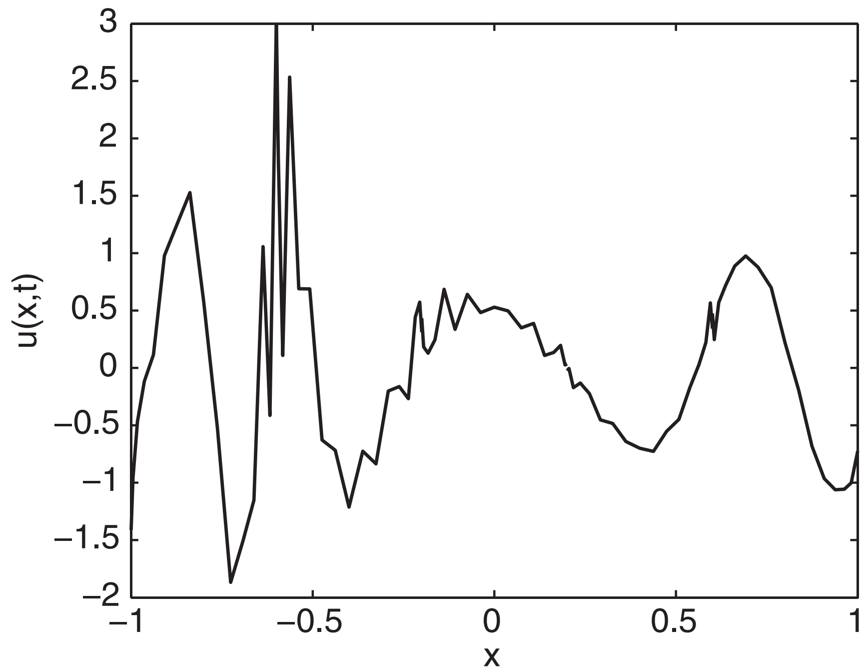
$$x \in \mathsf{D}^k : \; f_h^k(x,t) = \sum_{i=1}^{N_p} a(x_i^k)u_h^k(x_i,t)\ell_i^k(x);$$

Consider an example

Schemes I+II



Schemes III



What is the problem ?

$$f_h^k(x) = \mathcal{P}_N(a(x)u_h^k(x))$$

$$f_h^k(x,t) = \sum_{i=1}^{N_p} f_h^k(x_i^k)\ell_i^k(x),$$

is not $f_h^k(x,t) = \sum_{i=1}^{N_p} a(x_i^k)u_h^k(x_i,t)\ell_i^k(x);$

Aliasing

Consider an example

So we should just forget about scheme III ?

It is, however, very attractive:

✓ Scheme II requires special operators for each element

✓ Scheme III requires accurate integration all the time

And for more general non-linear problems, the situation is even less favorable.

Scheme III is simple and fast -- but (weakly) unstable!

May be worth trying to stabilize it

A second look

Consider

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(a(x)u\right) = 0.$$

Discretized as

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N(au_h) = 0.$$

interpolation

$$f_h^k(x,t) = \mathcal{I}_N(a(x)u_h^k(x,t)) = \sum_{i=1}^{N_p} a(x_i^k)u_h^k(x_i^k,t)\ell_i^k(x),$$

Express this as

$$\frac{\partial u_h}{\partial t} \quad + \frac{1}{2}\frac{\partial}{\partial x}\mathcal{I}_N(au_h) + \frac{1}{2}\mathcal{I}_N\left(a\frac{\partial u_h}{\partial x}\right)$$

skew symmetric part

$$+\frac{1}{2}\mathcal{I}_N\frac{\partial}{\partial x}au_h - \frac{1}{2}\mathcal{I}_N\left(a\frac{\partial u_h}{\partial x}\right)$$

low order term

$$+\frac{1}{2}\frac{\partial}{\partial x}\mathcal{I}_N(au_h) - \frac{1}{2}\mathcal{I}_N\frac{\partial}{\partial x}au_h = 0$$

aliasing term

A second look

One obtains the estimate

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{\Omega} \leq C_1\|u_h\|_{\Omega} + C_2(h,a)N^{1-p}|u|_{\Omega,p}.$$

$$\left\|\mathcal{I}_N\frac{\partial}{\partial x}au_h - \frac{\partial}{\partial x}\mathcal{I}_N(au_h)\right\|_{\Omega}^2$$

Aliasing driven instability
if u is not sufficiently smooth

What can we do? -- add dissipation

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N(au_h) = \varepsilon(-1)^{\tilde{s}+1}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]^{\tilde{s}} u_h.$$

$$\frac{1}{2}\frac{d}{dt}\|u_h\|_{\Omega}^2 \leq C_1\|u_h\|_{\Omega}^2 + C_2N^{2-2p}|u|_{\Omega,p}^2 - C_3\varepsilon|u_h|_{\Omega,\tilde{s}}^2.$$

This is enough to stabilize!

Filtering

So we can stabilize by adding dissipation as

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N(au_h) = \varepsilon(-1)^{\tilde{s}+1}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]^{\tilde{s}} u_h.$$

... but how do we implement this ?

Let us consider the split scheme

$$\frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N f(u_h) = 0, \qquad \frac{\partial u_h}{\partial t} = \varepsilon(-1)^{\tilde{s}+1}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]^{\tilde{s}} u_h.$$

and discretize the dissipative part in time

$$u_h^* = u_h(t+\Delta t) = u_h(t) + \varepsilon\Delta t(-1)^{\tilde{s}+1}\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]^{\tilde{s}} u_h(t).$$

Filtering

Now recall that

$$u_h(x,t) = \sum_{n=1}^{N_p} \hat{u}_n(t)\tilde{P}_{n-1}(x).$$

and the Legendre polynomials satisfy

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}\tilde{P}_n + n(n+1)\tilde{P}_n = 0,$$

so we obtain

$$u_h^*(x,t) \simeq u_h(x,t) + \varepsilon \Delta t(-1)^{\tilde{s}+1}\sum_{n=1}^{N_p} \hat{u}_n(t)(n(n-1))^{\tilde{s}}\tilde{P}_{n-1}(x)$$

$$\simeq \sum_{n=1}^{N_p} \sigma\left(\frac{n-1}{N}\right)\hat{u}_n(t)\tilde{P}_{n-1}(x), \quad \varepsilon \propto \frac{1}{\Delta t N^{2\tilde{s}}}.$$

The dissipation can be implemented as a filter

Filtering

We will define a filter as

$$\sigma(\eta) \begin{cases} = 1, & \eta = 0 \\ \leq 1, & 0 \leq \eta \leq 1 \\ = 0, & \eta > 1, \end{cases} \qquad \eta = \frac{n-1}{N}.$$

Polynomial filter of order 2s: $\sigma(\eta) = 1 - \alpha\eta^{2\tilde{s}},$

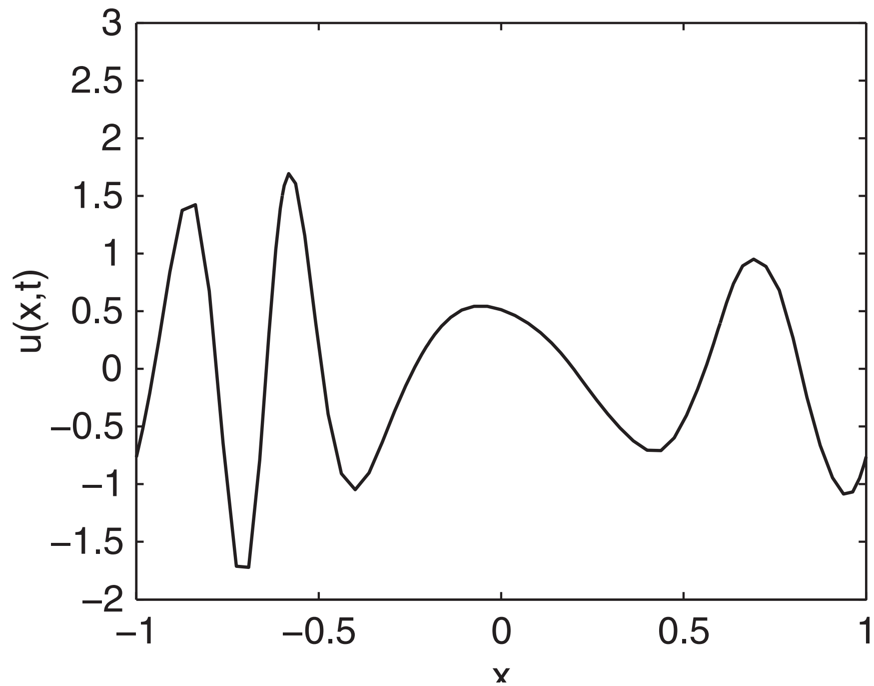
Exponential filter of order 2s: $\sigma(\eta) = \exp(-\alpha\eta^{2\tilde{s}}),$

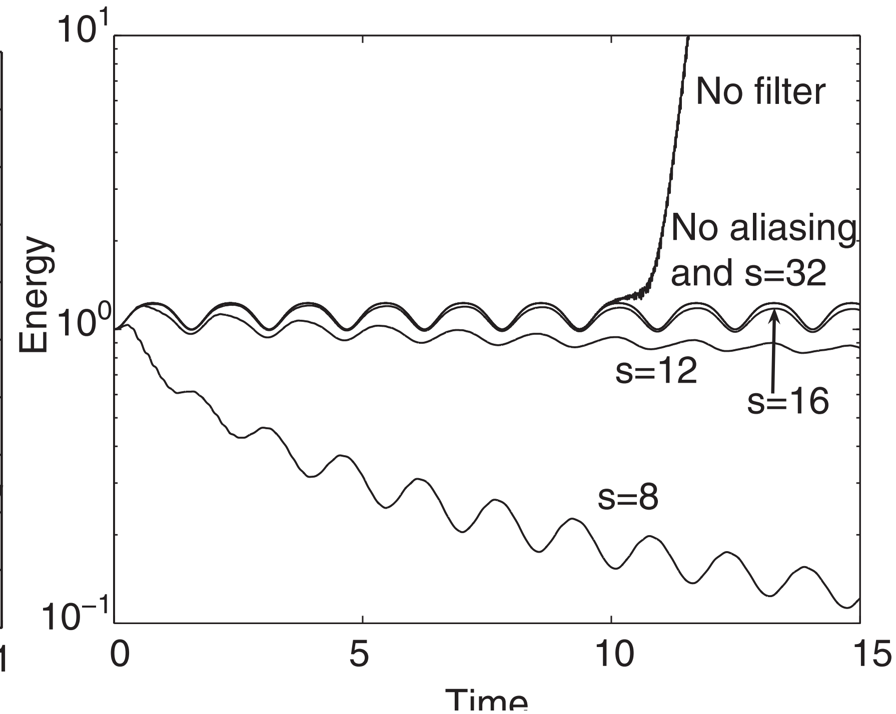
It is easily implemented as

$$\mathcal{F} = \mathcal{V}\Lambda\mathcal{V}^{-1}, \qquad \Lambda_{ii} = \sigma\left(\frac{i-1}{N}\right), \quad i = 1, \ldots, N_p.$$

Filtering

Does it work ?





A 2s-order filter is like adding a 2s dissipative term.

How much filtering:

As little as possible
... but as much as needed

Problems on non-conservative form

Often one encounters problems as

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0,$$

- ✓ Discretize it directly with a numerical flux based on f=au

- ✓ If a is smooth, solve

$$\frac{\partial u}{\partial t} + \frac{\partial au}{\partial x} - \frac{\partial a}{\partial x}u = 0,$$

- ✓ Introduce $v = \frac{\partial u}{\partial x}$ and solve

$$\frac{\partial v}{\partial t} + \frac{\partial av}{\partial x} = 0,$$

Basic results for smooth problems

Theorem 5.5. *Assume that the flux $f \in \mathsf{C}^3$ and the exact solution u is sufficiently smooth with bounded derivatives. Let u_h be a piecewise polynomial semidiscrete solution of the discontinuous Galerkin approximation to the one-dimensional scalar conservation law; then*

$$\|u(t) - u_h(t)\|_{\Omega,h} \leq C(t)h^{N+\nu},$$

provided a regular grid of $h = \max h^k$ is used. The constant C depends on u, N, and time t, but not on h. If a general monotone flux is used, $\nu = \frac{1}{2}$, resulting in suboptimal order, while $\nu = 1$ in the case an upwind flux is used.

The result extends to systems provided flux splitting is possible to obtain an upwind flux -- this is true for many important problems.

Lets summarize Part I

We have achieved a lot

✓ The theoretical support for DG for conservation laws is very solid.
✓ The requirements for ‘exact’ integration is expensive.
✓ It seems advantageous to consider a nodal approach in combination with dissipation.
✓ Dissipation can be implemented using a filter
✓ There is a complete error-theory for smooth problems.

... but we have ‘forgotten’ the unpleasant issue

What about discontinuous solutions?

Lecture 4

- ✓ Let's briefly recall what we know
- ✓ Part I: Smooth problems
  - ✓ Conservations laws and DG properties
  - ✓ Filtering, aliasing, and error estimates
- ✓ Part II: Nonsmooth problems
  - ✓ Shocks and Gibbs phenomena
  - ✓ Filtering and limiting
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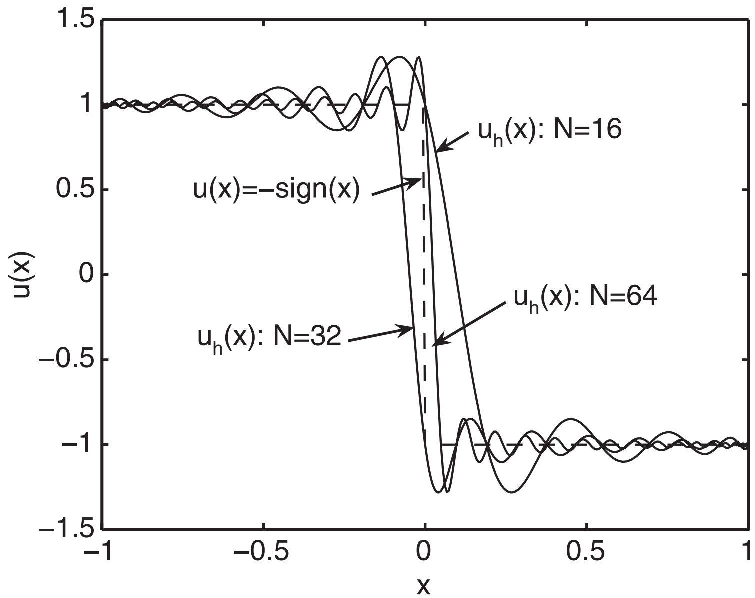
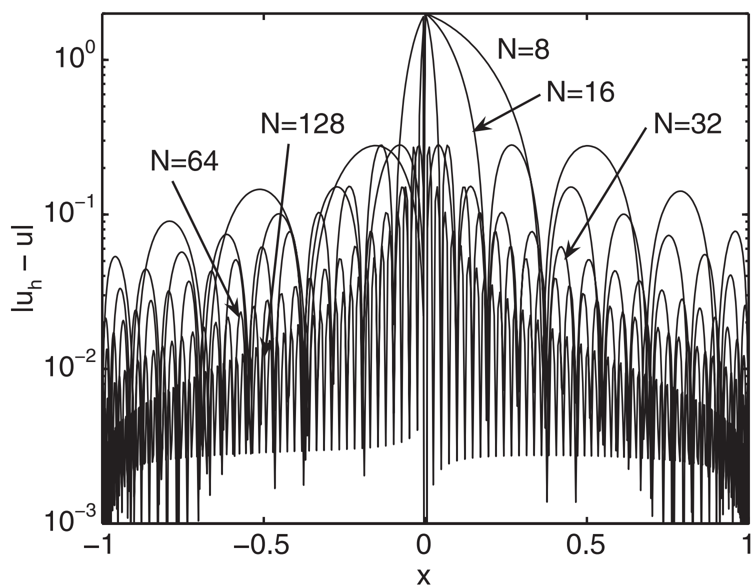
Gibbs Phenomenon

Let us first consider a simple approximation

$$u(x) = -\text{sign}(x), \quad x \in [-1, 1],$$





- ✓ Overshoot does not go away with N
- ✓ First order point wise accuracy
- ✓ Oscillations are global

Gibbs Phenomenon

Gibbs Phenomenon

But do the oscillations destroy the nice behavior?

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0, \quad = \frac{\partial u}{\partial t} + \mathcal{L}u = 0,$$

a(x) is smooth - but *u(x,0)* is not

Define the adjoint problem

$$\frac{\partial v}{\partial t} - \mathcal{L}^* v = 0,$$

solved with smooth *v(x,0)*

Clearly, we have

$$\frac{d}{dt}(u,v)_\Omega = 0 \quad \Rightarrow \quad (u(t),v(t))_\Omega = (u(0),v(0))_\Omega.$$

Gibbs Phenomenon

Using central fluxes, we also have

$$(u_h(t), v_h(t))_{\Omega,h} = (u_h(0), v_h(0))_{\Omega,h}.$$

Consider

$$(u_h(0), v_h(0))_{\Omega,h} = (u(0), v(0))_{\Omega} + (u_h(0) - u(0), v_h(0))_{\Omega,h}$$
$$+ (u(0), v_h(0) - v(0))_{\Omega,h}.$$

We also have

$$(u_h(0), v_h(0))_{\Omega,h} \leq (u(0), v(0))_{\Omega} + C(u)h^{N+1}N^{-q}|v(0)|_{\Omega,q}.$$

$$\|v(t) - v_h(t)\|_{\Omega,h} \leq C(t)\frac{h^{N+1}}{N^q}|v(t)|_{\Omega,q};$$

Combining it all, we obtain

$$(u_h(t), v(t))_{\Omega,h} = (u(t), v(t))_{\Omega} + \varepsilon,$$

Gibbs Phenomenon

The solution is spectrally accurate !

... but it is 'hidden'

This also shows that the high-order accuracy is maintained -- '*the oscillations are not noise*' !

How do we recover the accurate solution?

Recall

$$u_h(x) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(x), \quad \hat{u}_n = \int_{-1}^{1} u(x) \tilde{P}_{n-1}(x)\, dx.$$

One easily shows that

$$u(x) \in H^q \Rightarrow \hat{u}_n \propto n^{-q}$$

Filtering

So there is a close connection between smoothness and decay for the expansion coefficients.

Perhaps we can 'convince' the expansion do decay faster ?

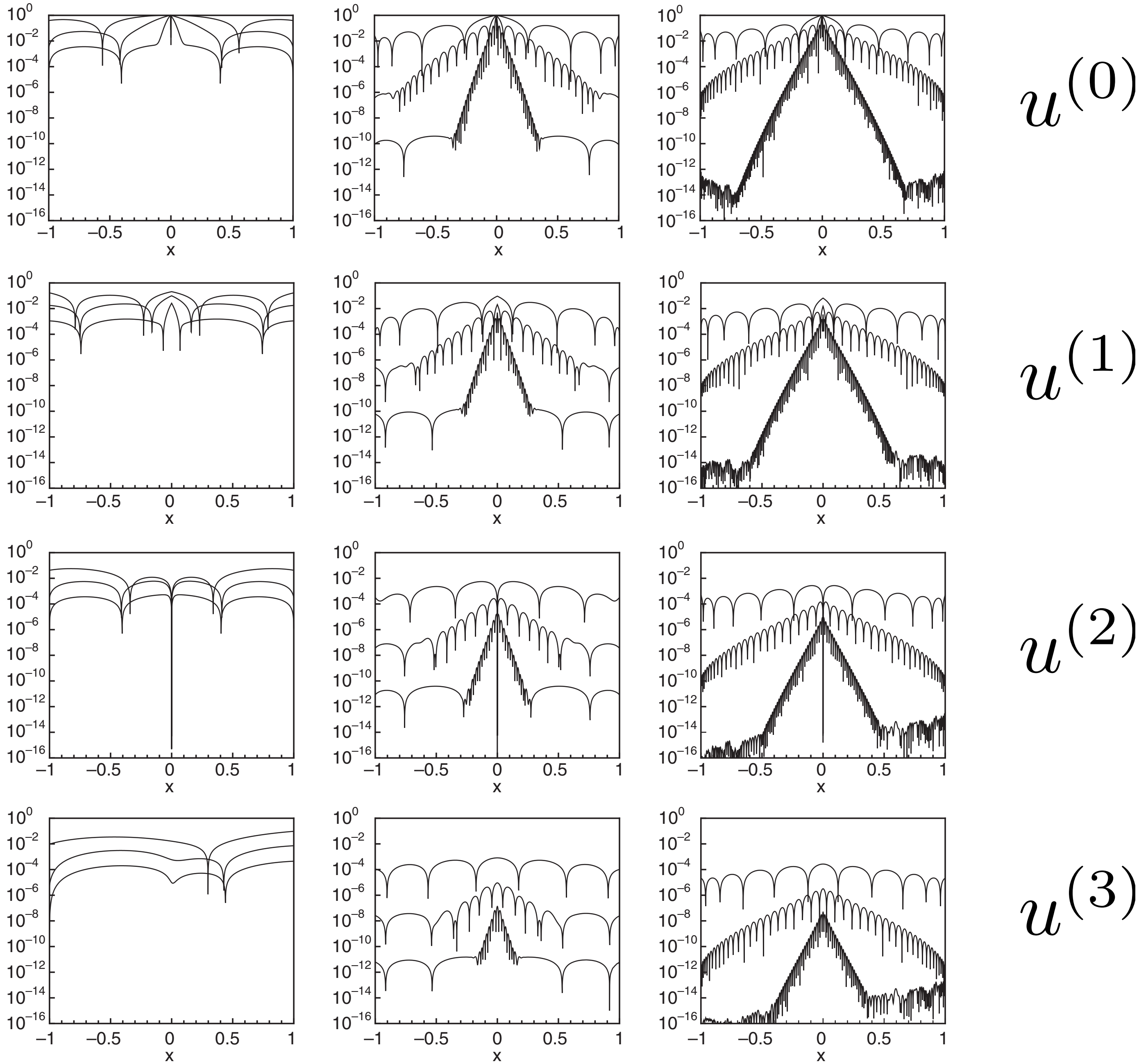
Consider

$$u_h^F(x) = \sum_{n=1}^{N_p} \sigma\left(\frac{n-1}{N}\right) \hat{u}_n \tilde{P}_{n-1}(x). \qquad \sigma(\eta) = \exp(-\alpha\eta^s)$$

Example

$$u^{(0)} = \begin{cases} -\cos(\pi x), & -1 \leq x \leq 0 \\ \cos(\pi x), & 0 < x \leq 1, \end{cases} \qquad u^{(i)} = \int_{-1}^{x} u^{(i-1)}(s)\, ds,$$

Filtering



Filtering

This achieves exactly what we hoped for

✓ Improves the accuracy away from the problem spot
✓ Does not destroy matter at the problem spot
... but does not help there.

This suggests a strategy:

✓ Use a filter to stabilize the scheme but do not remove the oscillations.
✓ Postprocess the data after the end of the computation.

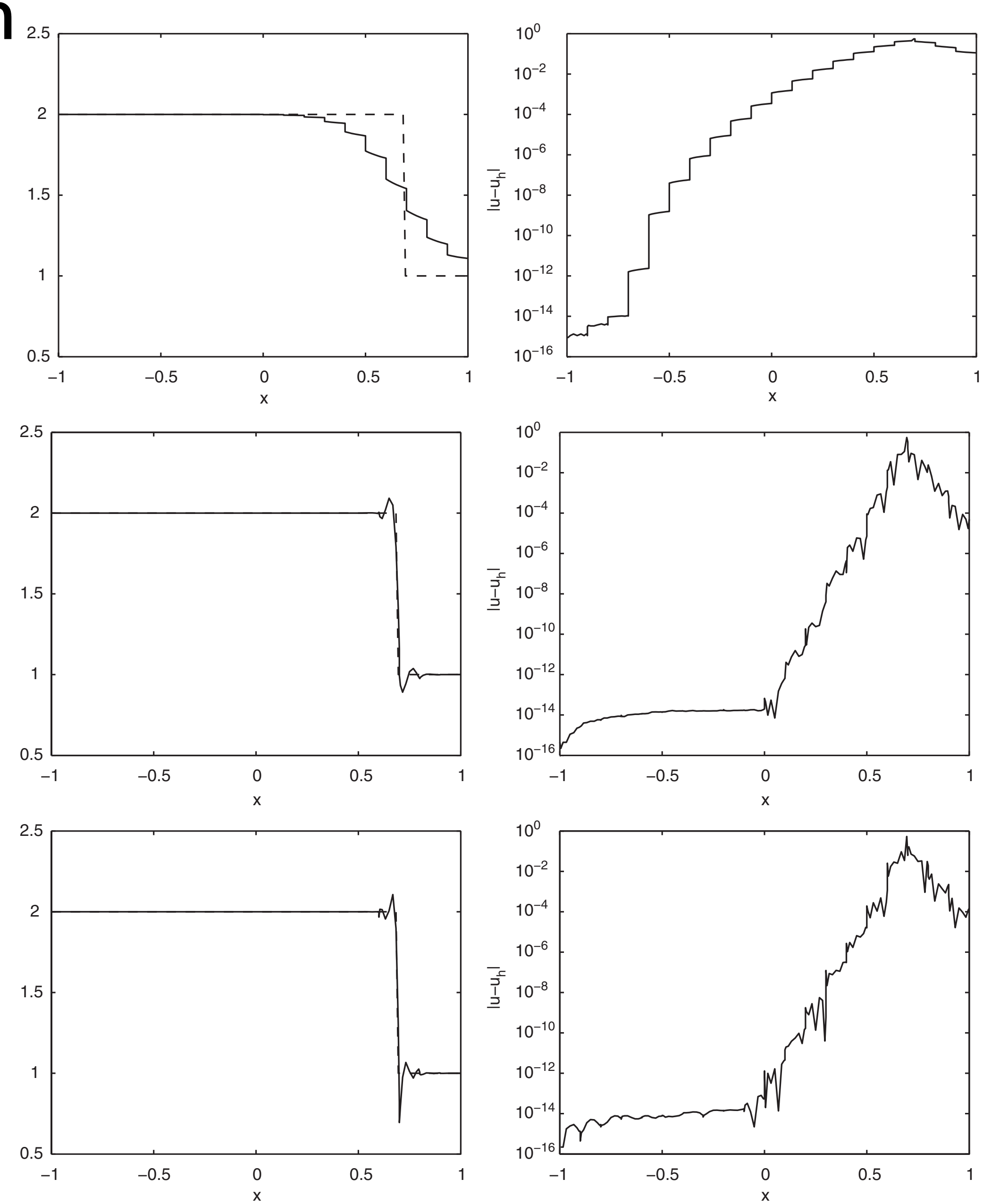
Filtering

Consider Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1],$$

$$u_0(x) = u(x,0) = \begin{cases} 2, & x \leq -0.5 \\ 1, & x > -0.5. \end{cases}$$

$$u(x,t) = u_0(x - 3t),$$



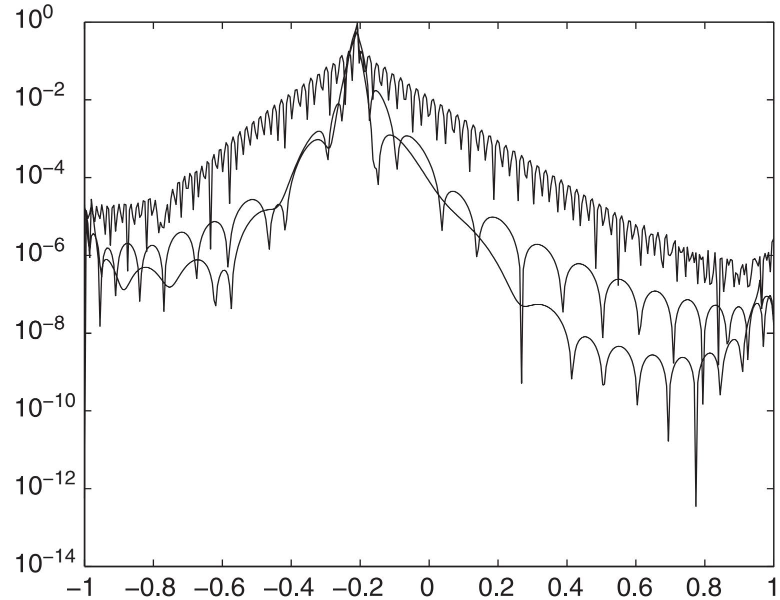
Overfiltering leads to severe smearing.

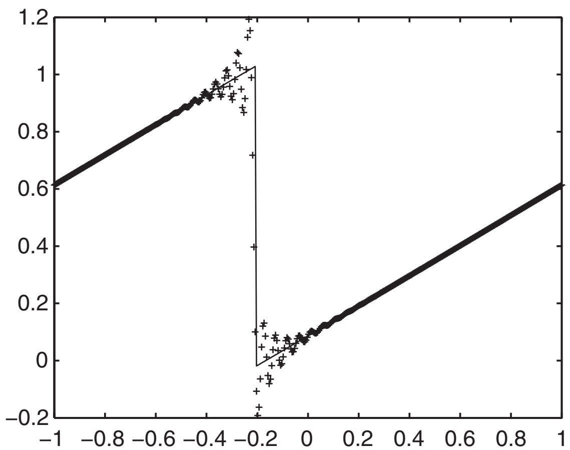
Limited filtering looks much better

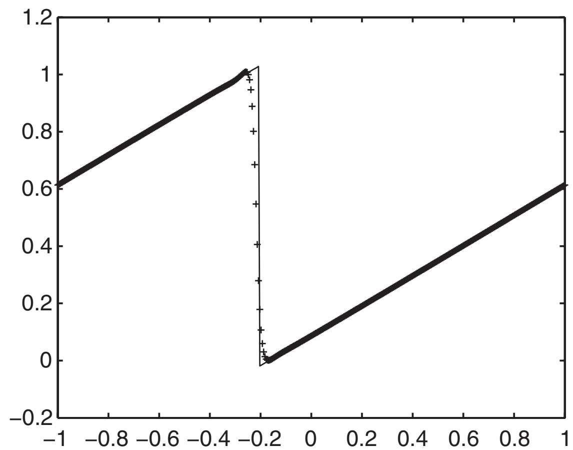
Filtering

An alternative - Pade filtering

$$u_h^k(x) = \frac{R_M(x)}{Q_L(x)},$$







To fully recover, the shock location is required (see text).

✓ Eliminates oscillations and improves accuracy

✓ .. but no improvement at the point

Limiting

So for some/many problems, we could simply leave the oscillations -- and then postprocess.

However, for some applications (.. and advisors) this is not acceptable

- ✓ Unphysical values (negative densities)
- ✓ Artificial events (think combustion)
- ✓ Visually displeasing (.. for the advisor).

So we are looking for a way to completely remove the oscillations:

Limiting

Limiting

We are interested in guaranteeing uniform boundedness

$$\|u\|_{L^1} \leq C, \quad \|u\|_{L^1} = \int_\Omega |u|\, dx.$$

Consider

$$\frac{\partial}{\partial t}u^\varepsilon + \frac{\partial}{\partial x}f(u^\varepsilon) = \varepsilon \frac{\partial^2}{\partial x^2}u^\varepsilon.$$ and define $$\eta(u) = |u|$$

We have

$$-\int_\Omega (\eta'(u_x))_x u_t \, dx = \int_\Omega \frac{u_x}{|u_x|} u_{xt}\, dx = \frac{d}{dt}\int_\Omega |u_x| dx = \frac{d}{dt}\|u_x\|_{L^1}.$$

and one easily proves

$$\frac{d}{dt}\|u_x^\varepsilon\|_{L^1} \leq 0.$$

Limiting

We would like to repeat this for the discrete scheme.

Consider first the N=0 FV scheme

$$h\frac{du_h^k}{dt} + f^*(u_h^k, u_h^{k+1}) - f^*(u_h^k, u_h^{k-1}) = 0,$$

Multiply with

$$v_h^k = -\frac{1}{h}\left[\eta'\left(\frac{u_h^{k+1} - u_h^k}{h}\right) - \eta'\left(\frac{u_h^k - u_h^{k-1}}{h}\right)\right]$$

and sum over all elements to get

$$\frac{d}{dt}|u_h|_{TV} + \sum_{k=1}^{K} v_h^k \left(f^*(u_h^k, u_h^{k+1}) - f^*(u_h^k, u_h^{k-1})\right) = 0,$$

$$|u_h|_{TV} = \sum_{k=1}^{K} |u_h^{k+1} - u_h^k|.$$

Limiting

Using that the flux is monotone, one easily proves

$$v_h^k \left(f^*(u_h^k, u_h^{k+1}) - f^*(u_h^k, u_h^{k-1}) \right) \geq 0$$

and therefore

$$\frac{d}{dt} |u_h|_{TV} \leq 0,$$

So for N=0 everything is fine -- but what about N>0

$$h \frac{d\bar{u}_h^k}{dt} + f^*(u_r^k, u_l^{k+1}) - f^*(u_l^k, u_r^{k-1}) = 0,$$

using a Forward Euler method in time, we get

$$\frac{h}{\Delta t} \left(\bar{u}^{k,n+1} - \bar{u}^{k,n} \right) + f^*(u_r^{k,n}, u_l^{k+1,n}) - f^*(u_l^{k,n}, u_r^{k-1,n}) = 0,$$

Limiting

Resulting in

$$|\bar{u}^{n+1}|_{TV} - |\bar{u}^n|_{TV} + \Phi = 0,$$

However, the monotone flux is not enough to guarantee uniform boundedness through $\Phi \geq 0$

That is the job of the limiter -- which must satisfy

- ✓ Ensures uniform boundedness/control oscillations
- ✓ Does not violate conservation
- ✓ Does not change the formal/high-order accuracy

This turns out to be hard !

Limiting

Two tasks at hand

- ✓ Detect troubled cells
- ✓ Limit the slope to eliminate oscillations

Define the *minmod* function

$$m(a_1, \ldots, a_m) = \begin{cases} s \min_{1 \le i \le m} |a_i|, & |s| = 1 \\ 0, & \text{otherwise}, \end{cases} \qquad s = \frac{1}{m} \sum_{i=1}^{m} \text{sign}(a_i).$$

If a are slopes, the minmod function

- ✓ Returns the minimum slope is all have the same sign
- ✓ Returns slope zero if the slopes are different

Limiting

Let us assume N=1 in which case the solution is

$$u_h^k(x) = \bar{u}_h^k + (x - x_0^k)(u_h^k)_x,$$

We have the classic MUSCL limiter

$$\mathit{\Pi}^1 u_h^k(x) = \bar{u}_h^k + (x - x_0^k) m \left((u_h^k)_x, \frac{\bar{u}_h^{k+1} - \bar{u}_h^k}{h}, \frac{\bar{u}_h^k - \bar{u}_h^{k-1}}{h} \right),$$

or a sligthly less dissipative limiter

$$\mathit{\Pi}^1 u_h^k(x) = \bar{u}_h^k + (x - x_0^k) m \left((u_h^k)_x, \frac{\bar{u}_h^{k+1} - \bar{u}_h^k}{h/2}, \frac{\bar{u}_h^k - \bar{u}_h^{k-1}}{h/2} \right),$$

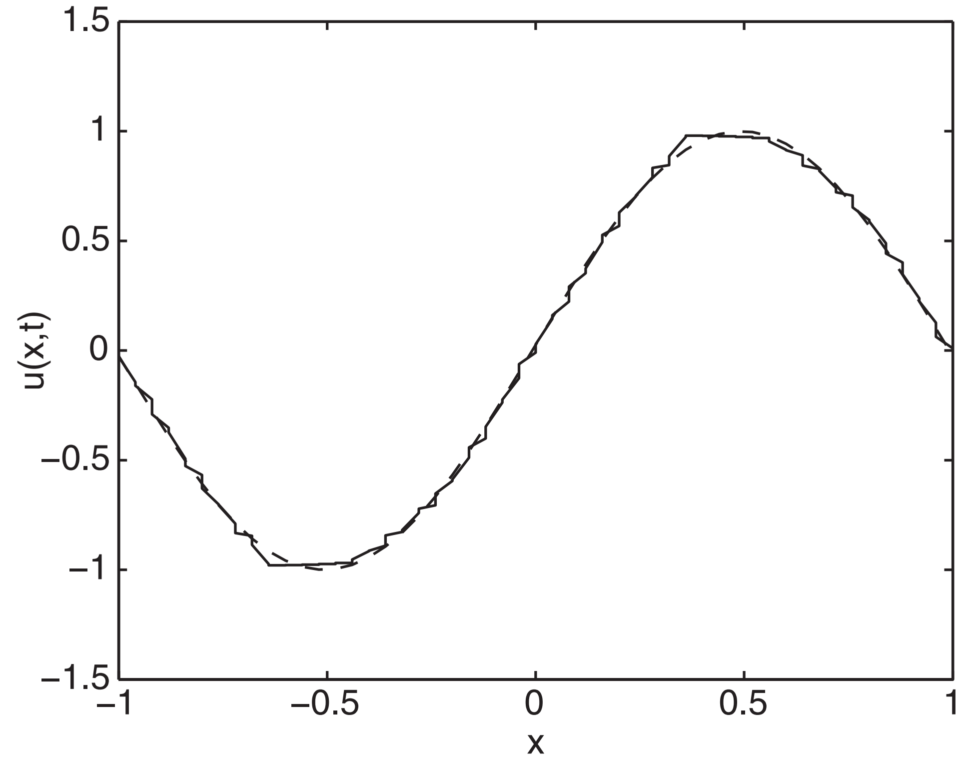
There are many other types but they are similar

Limiting

Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x \in [-1, 1],$$

Smooth initial condition



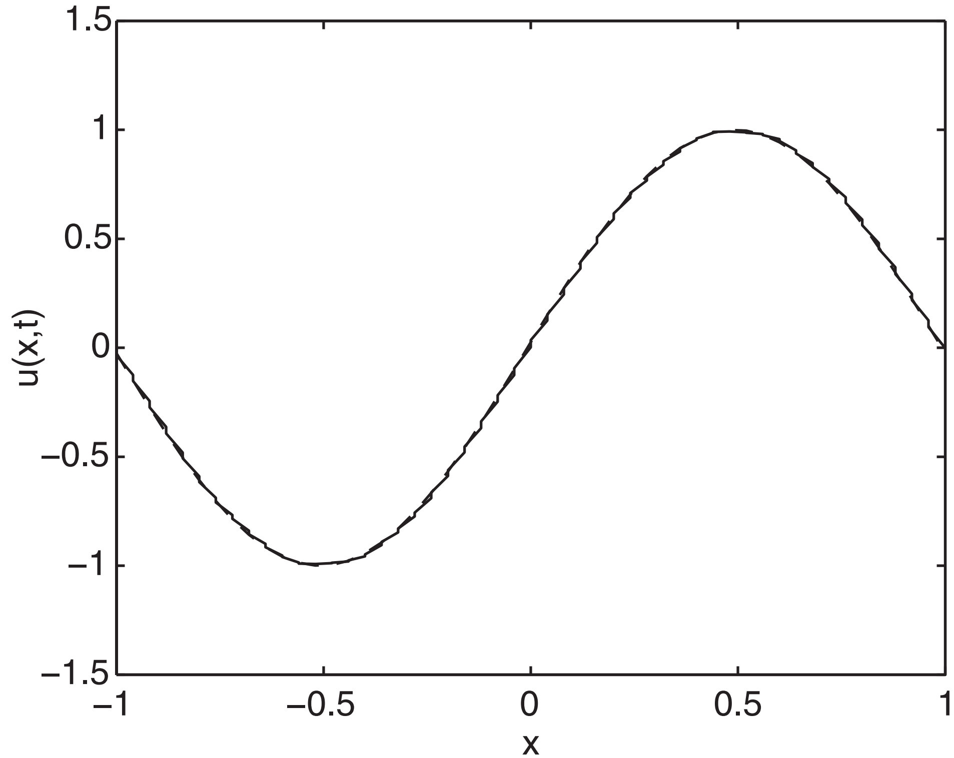
Reduction to 1st order at local smooth extrema

Limiting

Introduce the TVB minmod

$$\bar{m}(a_1,\ldots,a_m) = m\left(a_1, a_2 + Mh^2\text{sign}(a_2),\ldots,a_m + Mh^2\text{sign}(a_m)\right),$$

M estimates maximum curvature



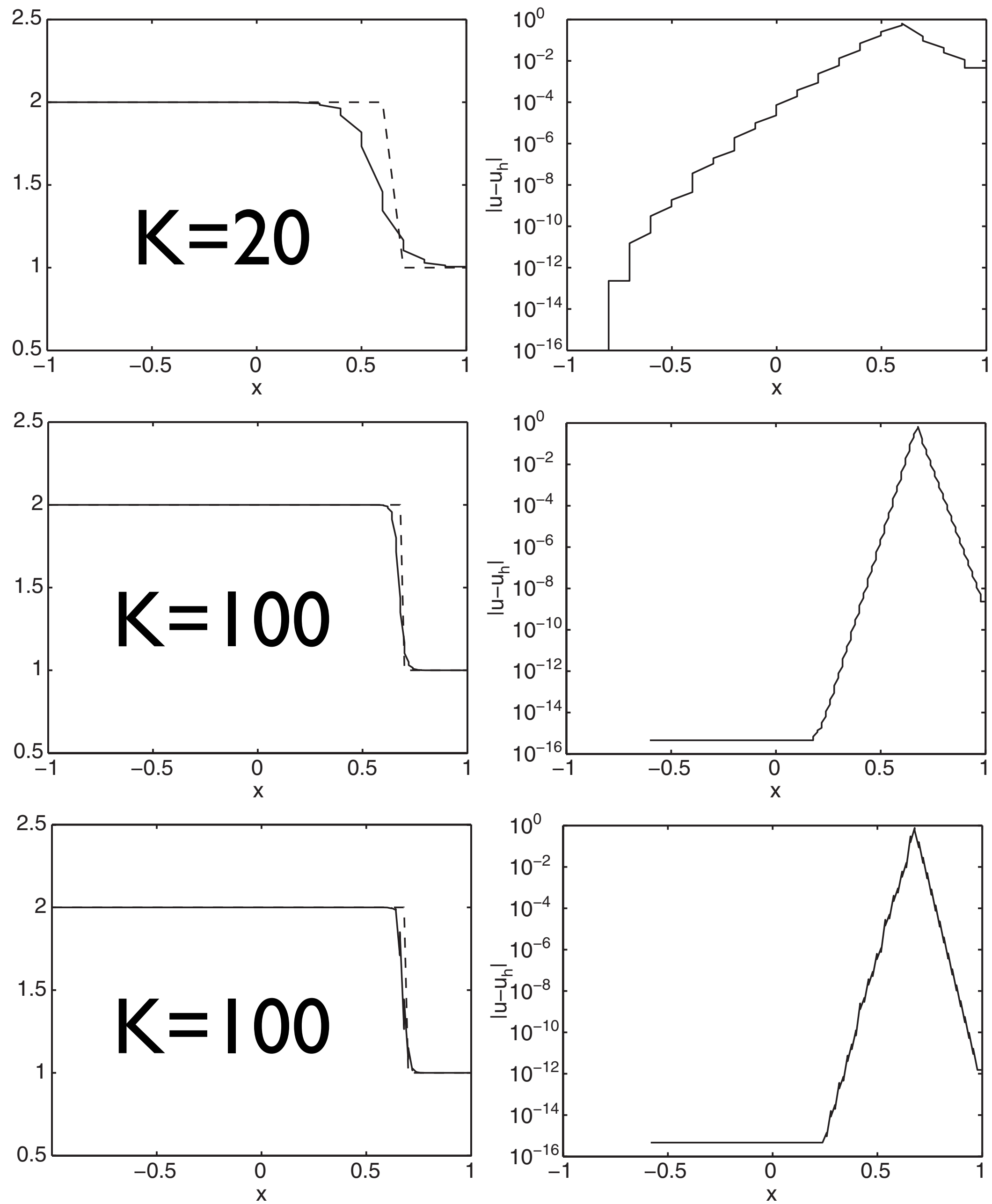
Limiting

Consider Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1],$$

$$u_0(x) = u(x, 0) = \begin{cases} 2, & x \leq -0.5 \\ 1, & x > -0.5. \end{cases}$$

$$u(x, t) = u_0(x - 3t),$$



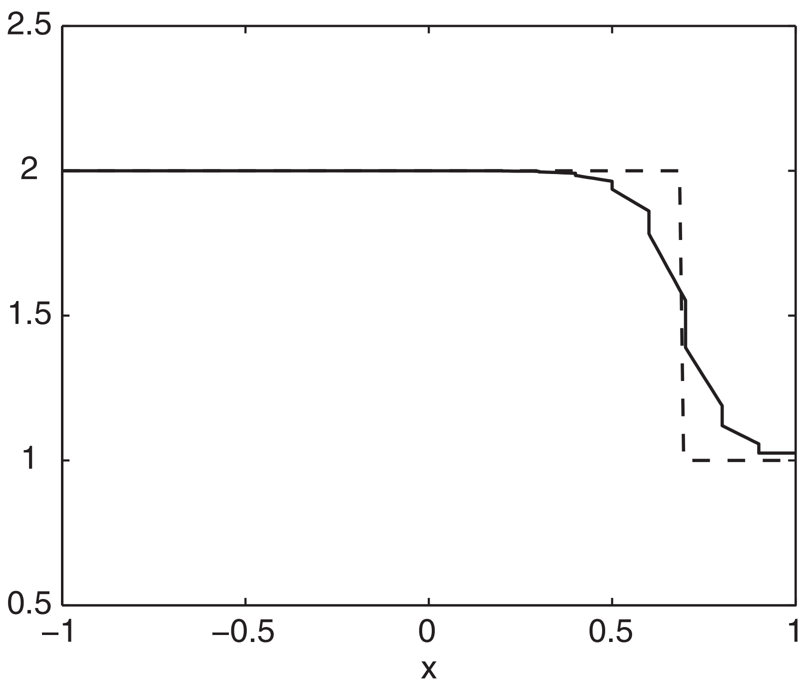
Too dissipative limiting leads to severe smearing.

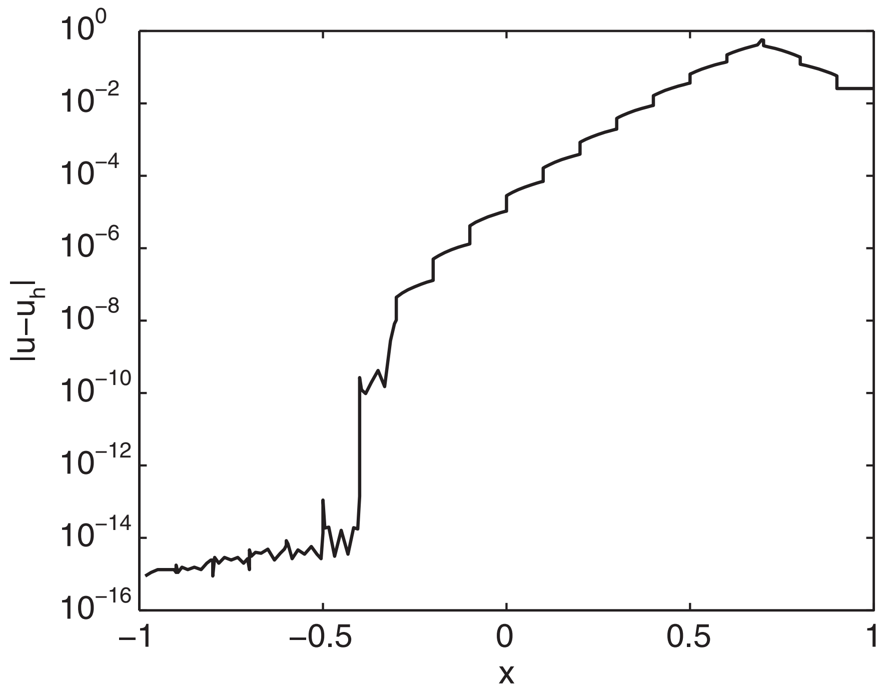
.. but no oscillations!

Limiting

But what about N>1?

✓ Compare limited and nonlimited interface values
✓ If equal, no limiting is needed.
✓ If different, reduce to N=1 and apply slope limiting





Limiting

General remarks on limiting

- ✓ The development of a limiting technique that avoid local reduction to 1st order accuracy is likely the most important outstanding problem in DG

- ✓ There are a number of techniques around but they all have some limitations -- restricted to simple/equidistant grids, not TVD/TVB etc

- ✓ The extensions to 2D/3D and general grids are very challenging

TVD Runge-Kutta methods

Consider again the semi-discrete scheme

$$\frac{d}{dt}u_h = \mathcal{L}_h(u_h, t),$$

For which we just discussed TVD/TVB schemes as

$$u_h^{n+1} = u_h^n + \Delta t \mathcal{L}_h(u_h^n, t^n), \quad |u_h^{n+1}|_{TV} \leq |u_h^n|_{TV}.$$

.. but this is just 1st order in time -- we want
high-order accuracy

Do we have to redo it all ?

TVD Runge-Kutta methods

Assume we can find a ERK method on the form

$$\begin{cases} v^{(0)} = u_h^n \\ i = 1, \ldots, s: \; v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} v^{(j)} + \beta_{ij} \Delta t \mathcal{L}_h(v^{(j)}, t^n + \gamma_j \Delta t) \\ u_h^{n+1} = v^{(s)} \end{cases} .$$

Coefficients found to satisfy order conditions

Write this as

$$v^{(i)} = \sum_{j=0}^{i-1} \alpha_{ij} \left(v^{(j)} + \frac{\beta_{ij}}{\alpha_{ij}} \Delta t \mathcal{L}_h(v^{(j)}, t^n + \gamma_j \Delta t) \right) .$$

Clearly if $\alpha_{ij}, \beta_{ij} > 0$ →

The scheme is a convex combination of Euler steps and the stability of the high-order methods follows

TVD Runge-Kutta methods

... but do such schemes exits ?

2nd order

$$v^{(1)} = u_h^n + \Delta t \mathcal{L}_h(u_h^n, t^n),$$
$$u_h^{n+1} = v^{(2)} = \frac{1}{2}\left(u_h^n + v^{(1)} + \Delta t \mathcal{L}_h(v^{(1)}, t^n + \Delta t)\right),$$

3rd order

$$v^{(1)} = u_h^n + \Delta t \mathcal{L}_h(u_h^n, t^n),$$
$$v^{(2)} = \frac{1}{4}\left(3u_h^n + v^{(1)} + \Delta t \mathcal{L}_h(v^{(1)}, t^n + \Delta t)\right),$$
$$u_h^{n+1} = v^{(3)} = \frac{1}{3}\left(u_h^n + 2v^{(2)} + 2\Delta t \mathcal{L}_h\left(v^{(2)}, t^n + \frac{1}{2}\Delta t\right)\right).$$

No 4th order, 4 stage scheme is possible - but there are other options (not implicit)

With filter/limiting

$$v^{(i)} = \Pi^p\left(\sum_{l=0}^{i-1} \alpha_{il} v^{(l)} + \beta_{il} \Delta t \mathcal{L}_h(v^{(l)}, t^n + \gamma_l \Delta t)\right).$$

TVD Runge-Kutta methods

Example

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \quad x \in [-1, 1],$$

$$u_0(x) = u(x,0) = \begin{cases} 2, & x \le -0.5 \\ 1, & x > -0.5. \end{cases} \qquad u(x,t) = u_0(x - 3t),$$

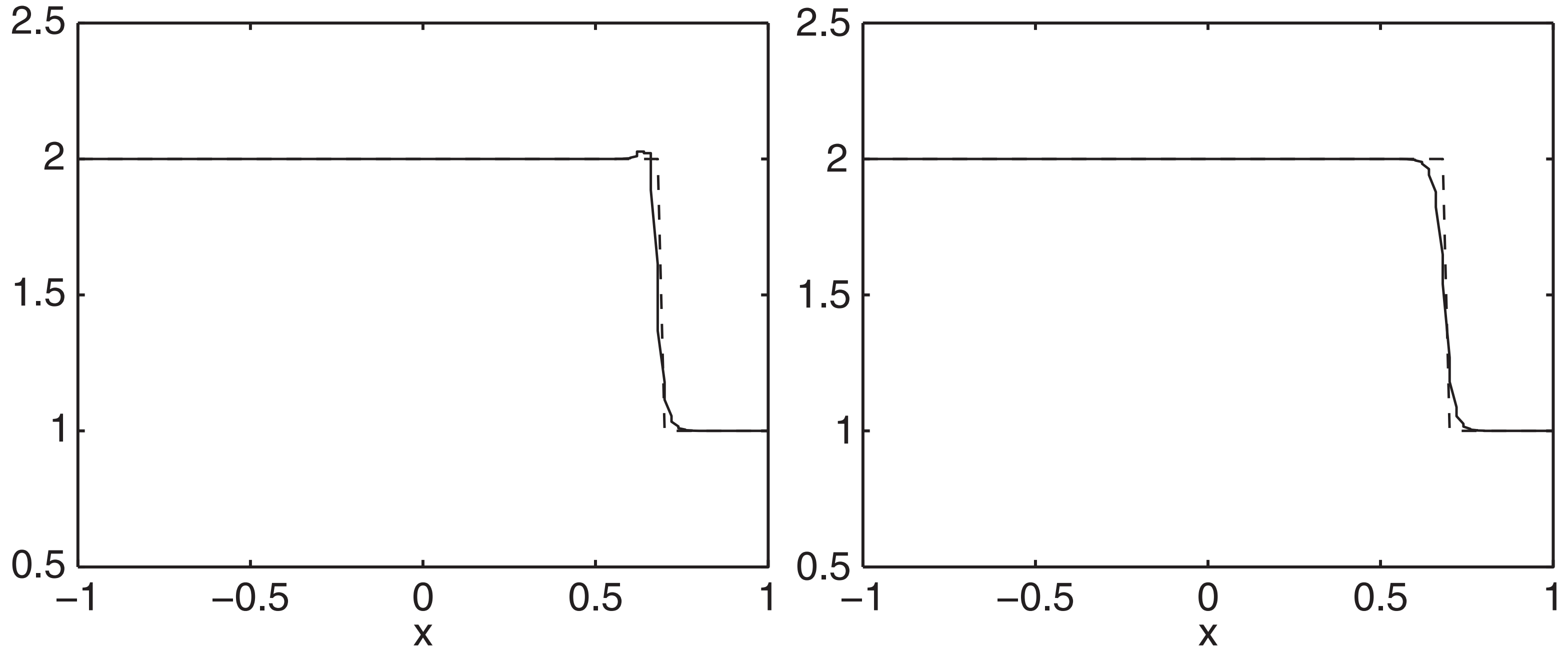
Use 'standard' 2nd order ERK

$$v^{(1)} = u_h^n - 20\Delta\mathcal{L}_h(u_h^n),$$
$$u_h^{n+1} = u_h^n + \frac{\Delta t}{40}\left(41\mathcal{L}_h(u_h^n) - \mathcal{L}_h(v^{(1)})\right).$$

Compare to 2nd order TVD-RK

MUSCL limiting in space, i.e., no oscillations

TVD Runge-Kutta methods



The oscillation is caused by time-stepping!

The 2nd order ERK is a bit unsual and 'reasonable' ERK method typically do not show this.

However, only with TVD-RK can one guarantee it

A few theoretical results

Theorem 5.12. *Assume that the limiter, Π, ensures the $TVDM$ property; that is,*

$$v_h = \Pi(u_h) \quad \Rightarrow \quad |v_h|_{TV} \leq |u_h|_{TV},$$

and that the SSP-RK method is consistent.

Then the DG-FEM with the SSP-RK solution is $TVDM$ as

$$\forall n : \quad |u_h^n|_{TV} \leq |u_h^0|_{TV}.$$

Theorem 5.14. *Assume that the slope limiter, Π, ensures that u_h is $TVDM$ or $TVBM$ and that the SSP-RK method is consistent.*

Then there is a subsequence, $\{\bar{u}_h'\}$, of the sequence $\{\bar{u}_h\}$ generated by the scheme that converges in $L^\infty(0,T;L^1)$ to a weak solution of the scalar conservation law.

Moreover, if a $TVBM$ limiter is used, the weak solution is the entropy solution and the whole sequence converges.

Finally, if the generalized slope limiter guarantees that

$$\|\bar{u}_h - \Pi\bar{u}_h\|_{L^1} \leq Ch|\bar{u}_h|_{TV},$$

then the above results hold not only for the sequence of cell averages, $\{\bar{u}_h\}$, but also for the sequence of functions, $\{u_h\}$.

Solving the Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,$$ Mass

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0,$$ Momentum

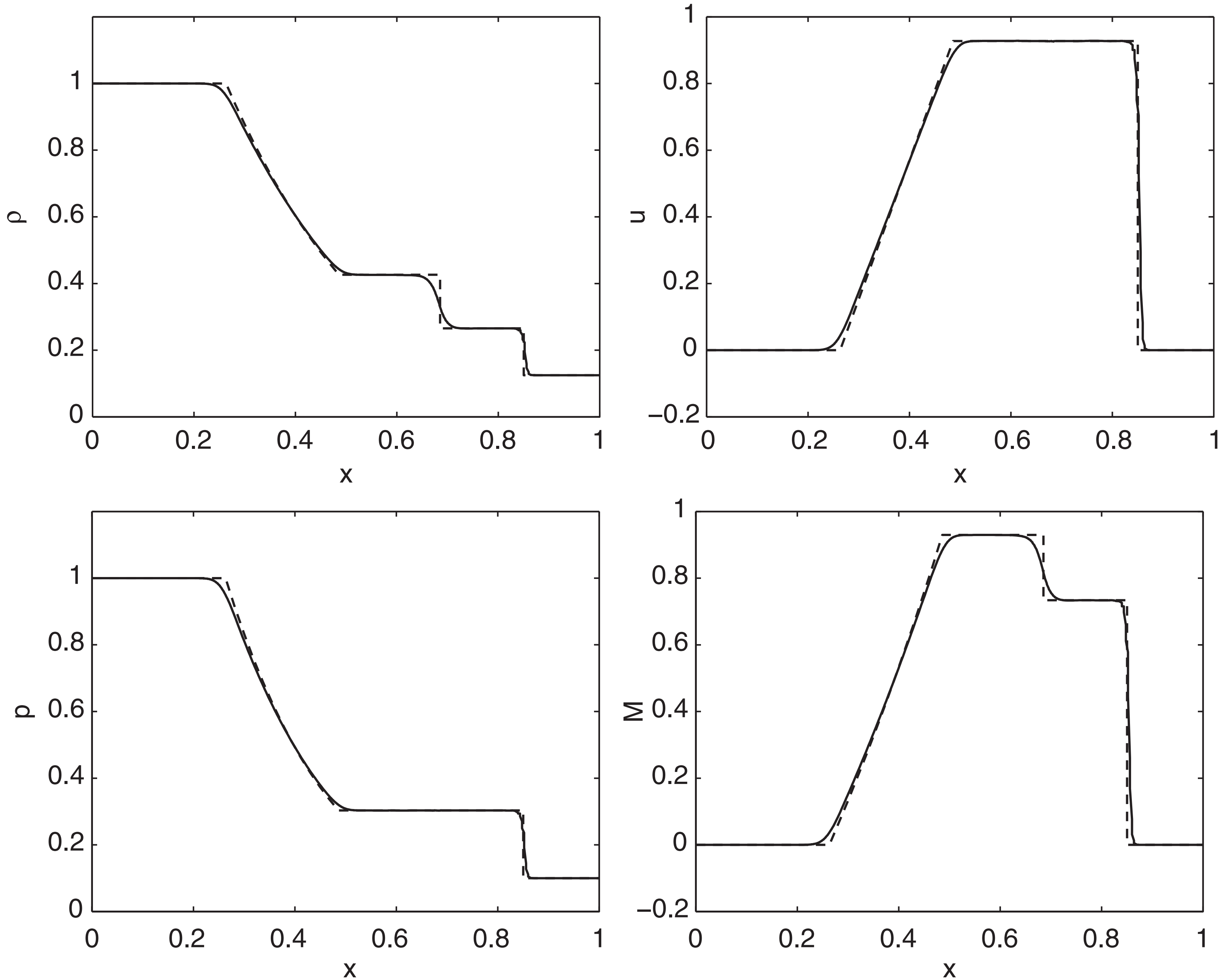
$$\frac{\partial E}{\partial t} + \frac{\partial (E + p)u}{\partial x} = 0,$$ Energy

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right), \quad c = \sqrt{\frac{\gamma p}{\rho}},$$ Ideal gas

Sod's Problem

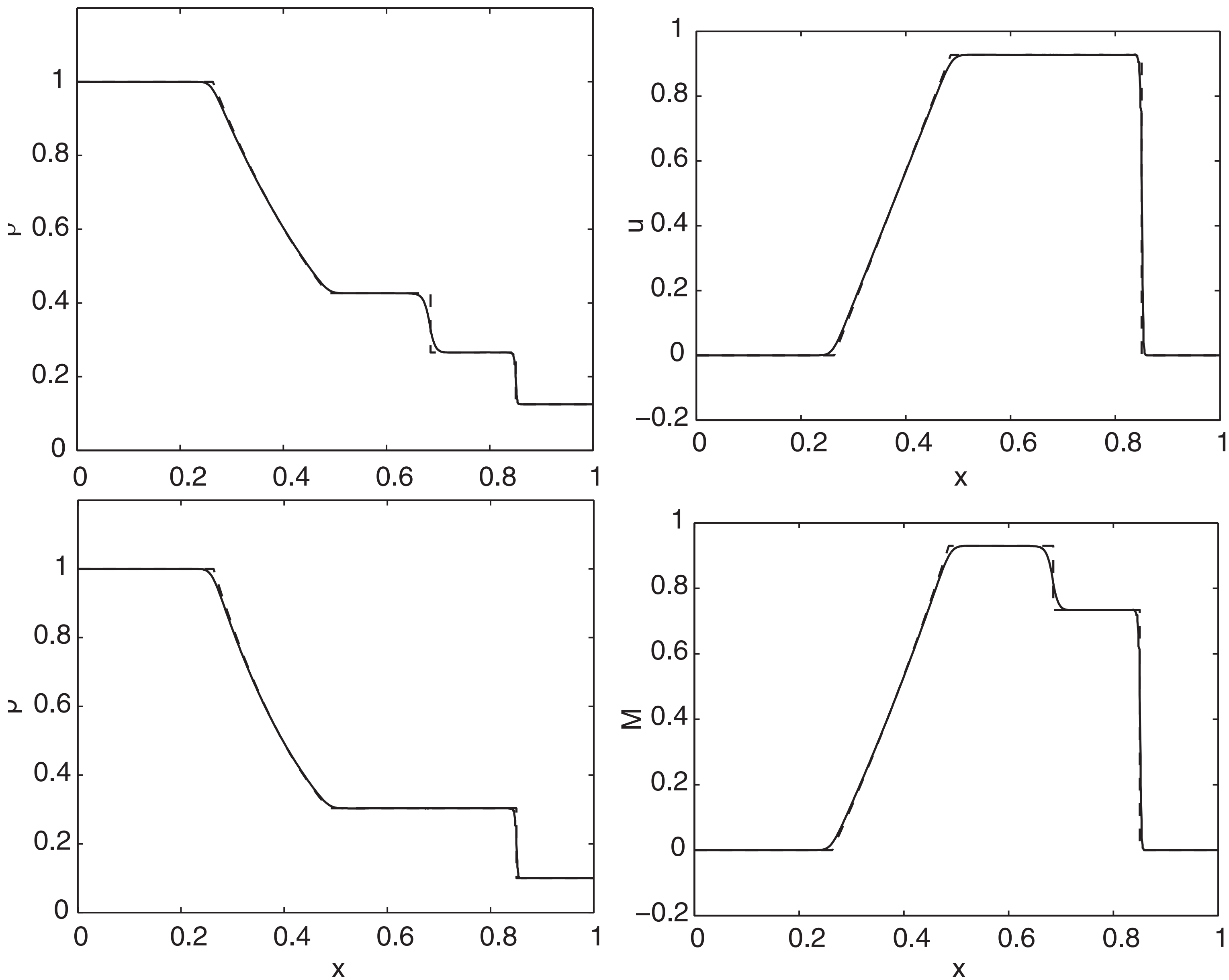
$$\rho(x,0) = \begin{cases} 1.0, & x < 0.5 \\ 0.125, & x \geq 0.5, \end{cases} \qquad \rho u(x,0) = 0 \quad E(x,0) = \frac{1}{\gamma - 1}\begin{cases} 1, & x < 0.5 \\ 0.1, & x \geq 0.5. \end{cases}$$

Solving the Euler equations



K=250
N=1
MUSCL

Solving the Euler equations



K=500
N=1
MUSCL

Fluxes - a second look

For the linear problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}_x \frac{\partial \boldsymbol{u}}{\partial x} + \mathcal{A}_y \frac{\partial \boldsymbol{u}}{\partial y} = 0,$$

we could derive the exact upwind flux - Riemann Pro.

Let us now consider a general nonlinear problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0,$$

For this we have used Lax-Friedrich fluxes -- but when used with limiting, this is too dissipative.

We need to consider alternatives

Fluxes - a second look

Let us locally assume that

$$\boldsymbol{f}^* = \hat{\mathcal{A}}\boldsymbol{u}^*,$$

where $\hat{\mathcal{A}}$ and $\mathbf{u}^*$ depends on $\mathbf{u}^\pm$

Let us assume that $\hat{\mathcal{A}}$ can diagonalized as

$$\hat{\mathcal{A}}\boldsymbol{r}_i = \lambda_i \boldsymbol{r}_i,$$

Use these waves to represent the solution

$$\boldsymbol{u}^* = \boldsymbol{u}^- + \sum_{\lambda_i \leq 0} \alpha_i \boldsymbol{r}_i = \boldsymbol{u}^+ - \sum_{\lambda_i \geq 0} \alpha_i \boldsymbol{r}_i.$$

Taking the average gives

$$\hat{\mathcal{A}}\boldsymbol{u}^* = \hat{\mathcal{A}}\{\{u\}\} + \frac{1}{2}|\hat{\mathcal{A}}|[\![u]\!], \qquad |\hat{\mathcal{A}}| = \mathcal{S}|\Lambda|\mathcal{S}^{-1},$$

Fluxes - a second look

.. but what is $\hat{\mathcal{A}}$?

We must require that

.. consistency: $\hat{\mathcal{A}}(\boldsymbol{u}^-, \boldsymbol{u}^+) \rightarrow \dfrac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial \boldsymbol{u}}$

.. diagonizable: $\hat{\mathcal{A}} = \mathcal{S}\Lambda\mathcal{S}^{-1}.$

Write

$$\boldsymbol{f}(\boldsymbol{u}^+) - \boldsymbol{f}(\boldsymbol{u}^-) = \int_0^1 \frac{d\boldsymbol{f}(\boldsymbol{u}(\xi))}{d\xi}\, d\xi = \int_0^1 \frac{d\boldsymbol{f}(\boldsymbol{u}(\xi))}{d\boldsymbol{u}} \frac{d\boldsymbol{u}}{d\xi}\, d\xi.$$

Assume:

$$\boldsymbol{u}(\xi) = \boldsymbol{u}^- + (\boldsymbol{u}^+ - \boldsymbol{u}^-)\xi,$$

Roe linearization

Fluxes - a second look

This results in the Roe condition

$$\boldsymbol{f}(\boldsymbol{u}^+) - \boldsymbol{f}(\boldsymbol{u}^-) = \hat{\mathcal{A}}\left(\boldsymbol{u}^+ - \boldsymbol{u}^-\right), \qquad \hat{\mathcal{A}} = \int_0^1 \frac{d\boldsymbol{f}(\boldsymbol{u}(\xi))}{d\boldsymbol{u}}\, d\xi.$$

One clear option

$$\boldsymbol{f}^* = \{\{\boldsymbol{f}\}\} + \frac{1}{2}|\hat{\mathcal{A}}|[\![\boldsymbol{u}]\!].$$

Like LF in 1D

.. but not computable in general

Approximations

$$\hat{\mathcal{A}} = \boldsymbol{f}_{\boldsymbol{u}}(\{\{\boldsymbol{u}\}\}),$$

$$\hat{\mathcal{A}} = \{\{\boldsymbol{f}_{\boldsymbol{u}}\}\}.$$

Summary

Dealing with discontinuous problems is a challenge

- ✓ The Gibbs oscillations impact accuracy
- ✓ .. but it does not destroy it, it seems
- ✓ So they should not just be removed
- ✓ One can the try to postprocess by filtering or other techniques.
- ✓ For some problems, true limiting is required
- ✓ Doing this right is complicated -- and open
- ✓ TVD-RK allows one to prove nonlinear results
- ✓ ... and it all works :-)

Time to move beyond 1D - Next week !