Feedback on Problem Set 2

Ph.D. Course:
An Introduction to DG-FEM
for solving partial differential equations

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We want to solve the wave equation in 1D using DG-FEM

$$\partial_{tt} u - \partial_{xx} u = 0, \quad x \in \Omega([-\pi, \pi])$$

For the discretization we rewrite into

$$\begin{array}{l} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u = 0 \end{array} , \quad x \in \Omega([-\pi, \pi])$$

We have exact solutions of the form

$$\begin{aligned} u(x,t) &= f(x+t) + g(x-t) \\ v(x,t) &= -f(x+t) + g(x-t) \end{aligned}$$

with $f(\cdot)$ and $g(\cdot)$ arbitrary functions.

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By the energy method we find

$$\frac{d}{dt}(||u||_\Omega^2 + ||v||_\Omega^2) = 2\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix}$$
$$= \int_{-\pi}^{\pi} \partial_x(uv)dx$$
$$= uv|_{x=\pi} - uv|_{x=-\pi}$$

If solution is periodic

$$u(-\pi, t) = u(\pi, t)$$
$$v(-\pi, t) = v(\pi, t)$$

we have energy conservation...

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Make setup periodic, such that

$$u(-\pi, t) = u(\pi, t)$$
$$v(-\pi, t) = v(\pi, t)$$

Simple change in code

```
vmapP(1)   = vmapM(end);
vmapP(end) = vmapM(1);
```

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DG-FEM discretization with central fluxes, i.e.

$$f^*(u^-, u^+) = \frac{f^- + f^+}{2}$$

In the strong formulation, the surface integral term of the first PDE is of the form

$$\int_{\partial D^k} \hat{n}^- \cdot (v^- - v^*) dx$$

where the integrand is found to be

$$\hat{n}^- \cdot (v^- - v^*) = \hat{n}^- \cdot (v^- - v^*) = \hat{n}^- \cdot \left(\frac{v^- - v^+}{2} \right)$$

Implement as

```
df(:) = 0.5*nx(:).*( v(vmapM) - v(vmapP) );
```

Now, all interfaces treated in the same way.

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```
function [rhsu,rhsv] = WaveRHS1DCentralPeriodic(u,v,time)

% function [rhsu,rhsv] = WaveRHS1DCentralPeriodic(u,v,time)
% Purpose  : Evaluate RHS flux in 1D wave equation system
%            using central fluxes

Globals1D;

% form flux differences at faces
df    = zeros(Nfp*Nfaces,K);
df(:) = 0.5*nx(:).*(u(vmapM)-u(vmapP));

dg    = zeros(Nfp*Nfaces,K);
dg(:) = 0.5*nx(:).*(v(vmapM)-v(vmapP));

% compute right hand sides of the semi-discrete PDE
rhsu = -rx.*(Dr*v) + LIFT*(Fscale.*(dg));
rhsv = -rx.*(Dr*u) + LIFT*(Fscale.*(df));
```

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From the quasilinear form

$$\partial_t \mathbf{q} + \partial_x(\mathcal{A}\mathbf{q}) = 0, \quad \mathbf{q} = (u, v)^T$$

It is possible to show that the system is strictly hyperbolic. Thus, the flux jacobian can be diagonalized as

$$\mathcal{A} = \mathcal{R}\mathcal{D}\mathcal{R}^{-1}$$

with

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Furtheremore, we can split the left and right states into $\mathcal{D} = \mathcal{D}^- + \mathcal{D}^+$ where

$$\mathcal{D}^- = \frac{1}{2}(\mathcal{D} - |D|) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D}^+ = \frac{1}{2}(\mathcal{D} + |D|) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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We then find the characteristic variables

$$\mathbf{w} = \mathcal{R}^{-1}\mathbf{q} = \frac{1}{\sqrt{2}} \begin{pmatrix} v - u \\ u + v \end{pmatrix}$$

and the characteristic equations

$$\partial_t \mathbf{w} + \mathcal{D}\partial_x \mathbf{w} = 0, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Waves propagating to the left and right have flux functions

$$f_l^c(\mathbf{w}) = \mathcal{D}^-\mathbf{w}$$
$$f_r^c(\mathbf{w}) = \mathcal{D}^+\mathbf{w}$$

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We can transform back to variables of the system using

$$\mathbf{q} = \mathcal{R}\mathbf{w} = \frac{1}{\sqrt{2}} \begin{pmatrix} w_2 - w_1 \\ w_1 + w_2 \end{pmatrix}$$

and similarly the flux functions becomes

$$f_l(\mathbf{q}) = \mathcal{R}\mathcal{D}^{-}\mathcal{R}^{-1}\mathbf{q}$$
$$f_r(\mathbf{q}) = \mathcal{R}\mathcal{D}^{+}\mathcal{R}^{-1}\mathbf{q}$$

which can account for fluxes to the left and right in the physical domain.

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With appropriate upwinding the net fluxes across interfaces are

- Left-side interface, $u^+|u^-$:

$$\begin{aligned} f(\mathbf{q}^-,\mathbf{q}^+) &= f_r(\mathbf{q}^+) + f_l(\mathbf{q}^-) \\ &= \frac{1}{2}\mathcal{A}(\mathbf{q}^- + \mathbf{q}^+) - \frac{1}{2}\mathcal{R}|\mathcal{D}|\mathcal{R}^{-1}(\mathbf{q}^- - \mathbf{q}^+) \end{aligned}$$

- Right-side interface, $u^-|u^+$:

$$\begin{aligned} f(\mathbf{q}^-,\mathbf{q}^+) &= f_r(\mathbf{q}^-) + f_l(\mathbf{q}^+) \\ &= \frac{1}{2}\mathcal{A}(\mathbf{q}^- + \mathbf{q}^+) - \frac{1}{2}\mathcal{R}|\mathcal{D}|\mathcal{R}^{-1}(\mathbf{q}^+ - \mathbf{q}^-) \end{aligned}$$

Thus, on a general 1D interface the flux is a Lax-Friedrichs-type flux of the form

$$f(\mathbf{q}^-,\mathbf{q}^+) = \frac{1}{2}\mathcal{A}(\mathbf{q}^- + \mathbf{q}^+) - \frac{\hat{n}_x}{2}\mathcal{R}|\mathcal{D}|\mathcal{R}^{-1}(\mathbf{q}^+ - \mathbf{q}^-)$$

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```
function [rhsu,rhsv] = WaveRHS1DUpwindFinite(u,v,time,f,g)
% function [rhsu] = WaveRHS1DUpwindFinite(u,v,time)
% Purpose  : Evaluate RHS flux in 1D wave equation system
%            by upwinding using Lax-Friedrichs-type flux
Globals1D;

% jumps in u and v
du = u(vmapM)-u(vmapP); dv = v(vmapM)-v(vmapP);

% form flux differences at faces
df = zeros(Nfp*Nfaces,K); df(:) = 0.5*nx(:).*dv-0.5*du;
dg = zeros(Nfp*Nfaces,K); dg(:) = 0.5*nx(:).*du-0.5*dv;

% Impose left outer boundary
x0 = min(x(:));
uM = u(vmapI); vM = v(vmapI);
uP =  f(x0+time) + g(x0-time);  vP = -f(x0+time) + g(x0-time);
us = 0.5*( uP + vP - vM + uM ); vs = 0.5*( uP + vP + vM - uM );
df(mapI) = 0.5*nx(mapI)*(vM-vs)-0.5*(uM-us);
dg(mapI) = 0.5*nx(mapI)*(uM-us)-0.5*(vM-vs);

% Impose right outer boundary
x0 = max(x(:));
uM = u(vmapO); vM = v(vmapO);
uP =  f(x0+time) + g(x0-time);  vP = -f(x0+time) + g(x0-time);
us = 0.5*( uM + vM - vP + uP ); vs = 0.5*( vP - uP + uM + vM );
df(mapO) = 0.5*nx(mapO)*(vM-vs)-0.5*(uM-us);
dg(mapO) = 0.5*nx(mapO)*(uM-us)-0.5*(vM-vs);

% compute right hand sides of the semi-discrete PDE
rhsu = -rx.*(Dr*v) + LIFT*(Fscale.*(df));
rhsv = -rx.*(Dr*u) + LIFT*(Fscale.*(dg));
```

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Direction and speed of propagation for the characteristic variables are determined by the corresponding eigenvalues. Upwinding considerations suggest that we impose

- Left outer boundary, $x = -\pi$:

$$\lambda = -1, : \quad w_1^* = w_1^- = \frac{1}{\sqrt{2}}(v^- - u^-)$$
$$\lambda = 1, : \quad w_2^* = w_2^+ = \frac{1}{\sqrt{2}}(u^+ + v^+)$$

- Right outer boundary, $x = \pi$:

$$\lambda = -1, : \quad w_1^* = w_1^+ = \frac{1}{\sqrt{2}}(v^+ - u^+)$$
$$\lambda = 1, : \quad w_2^* = w_2^- = \frac{1}{\sqrt{2}}(u^- + v^-)$$

Now, the numerical fluxes can be determined from these contributions

$$\mathbf{q}^* = \mathcal{R}\mathbf{w}^*$$

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We find

- $x = -\pi$:

$$u^* = \frac{1}{\sqrt{2}}(w_2^* - w_1^*) = \frac{1}{2}[u^+ + v^+ - v^- + u^-]$$
$$v^* = \frac{1}{\sqrt{2}}(w_1^* + w_2^*) = \frac{1}{2}[u^+ + v^+ + v^- - u^-]$$

- $x = \pi$:

$$u^* = \frac{1}{\sqrt{2}}(w_2^* - w_1^*) = \frac{1}{2}[u^- + v^- - v^+ + u^+]$$
$$v^* = \frac{1}{\sqrt{2}}(w_1^* + w_2^*) = \frac{1}{2}[v^+ - u^+ + u^- + v^-]$$

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```
close all, clear all, clc
% Driver script for solving the 1D wave equations
% on either a finite or periodic domain
Globals1D;

% Order of polymomials used for approximation
N = 6; K = 4;
FinalTime = 100;
Periodic = 1; % 0 = no, 1 = yes

% Generate simple mesh
[Nv, VX, K, EToV] = MeshGen1D(0.0,2*pi,K);

% Initialize solver and construct grid and metric
StartUp1D;

if Periodic
   vmapP(1)   = vmapM(end);  vmapP(end) = vmapM(1);
end

% Define initial wave functions
f = @(s) sin(s)*0;   % wave form propagating to the left
g = @(s) sin(s); % wave form propagating to the right

% Set initial conditions
u =  f(x) + g(x); % intial disturbance
v = -f(x) + g(x); %

% Solve Problem
[u,v,time] = Wave1D(u,v,FinalTime,f,g,Periodic);
```

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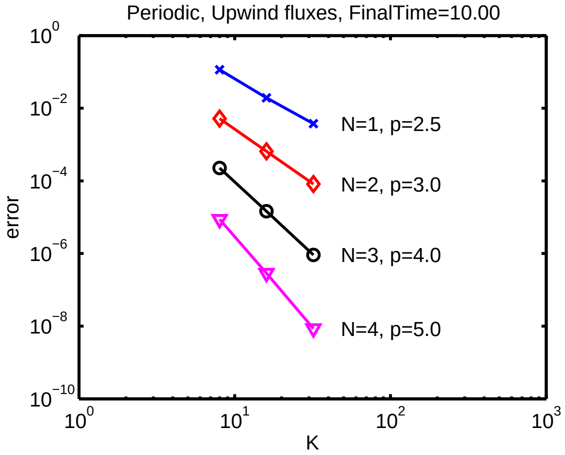

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