Assignment - Part I

Ph.D. Course 2009:
An Introduction to DG-FEM for solving partial differential equations

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If you have not already done so, please download all the Matlab codes from the book from

http://www.nudg.org/

and store and unpack them in a directory you can use with Matlab.

This is the first part of the assignment which is mandatory to complete to pass the course.

The goal will be to solve the nonlinear shallow water equations[1] in one spatial dimension stated in conservation form as

$$\partial_t \boldsymbol{q} + \partial_x \boldsymbol{F} = \boldsymbol{S}(\boldsymbol{q}), \tag{1}$$

with

$$\boldsymbol{q} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{2}$$

Here $\boldsymbol{q}$ is a vector of conserved variables, $F(\mathbf{q})$ is a flux vector and $S(\mathbf{q})$ a source term vector accounting for, e.g., slope and bottom friction terms. The conserved variables are expressed in terms of the total height of the water column $h = h_0 + \eta$ with h_0 the still-water depth and η the perturbation from the still water level, the depth-averaged velocity u and g the gravitational acceleration constant.

A source term accounting for a bottom with slope can be stated in the form

$$s_2 = -gh\partial_x b \tag{3}$$

where $b(x)$ is the profile of the bed measured positive in the upward direction from some fixed reference level in the vertical.

Your first task is to solve the linearized shallow water equations (SWE) stated as

$$\begin{aligned} \partial_t \eta &= -h_0 \partial_x u \\ \partial_t u &= -g \partial_x \eta \end{aligned}, \quad x \in [0, 2\pi] \tag{4}$$

[1]Sometimes referred to as the Saint-Venant equations.

which are valid for small perturbations η (meters) of the free surface from the still water level h_0 (meters), i.e. $\eta << h_0$. Consider a single linear travelling wave over a plane bed with exact solution

$$\begin{aligned} \eta(x,t) &= A\cos(\omega t - kx) \\ u(x,t) &= \frac{\omega}{kh_0}\eta(x,t) \end{aligned} \tag{5}$$

where A is the amplitude (meters), $k = 2\pi/l$ is the wave number with l the wave length (meters), $\omega = 2\pi/T$ the angular frequency with T the wave period (seconds), and h_0 is the still-water depth (meters). The following parameters may be used in the following

$$l = 2\pi, \quad A = 1, \quad T = 2\pi, \quad h_0 = \tfrac{\omega^2}{k^2 g}, \quad g = 9.81\ m/s^2$$

for validation purposes. Note that the amplitude is scaled for the purposes in this context.

- Show that the energy is conserved for the exact solution on a periodic domain.
- Derive and implement a periodic DG-FEM scheme for the linear SWE and justify your choice of numerical fluxes and time step size.
- Validate the accuracy of the method by using the exact solution for a linear travelling wave. Does it match your expectations? Discuss.
- Describe and implement what changes are needed to solve the same problem on a finite domain with your code and validate the accuracy.

Having established a solver for the linear SWE, proceed with an implementation of the nonlinear shallow water equations (NSWE), which can be used as basis for the simulation of shock waves of various wave phenomena.

- Give expressions for the characteristic wave speeds of the NSWE. Is the system hyperbolic?
- Make appropriate changes to turn your linear solver into a nonlinear SWE solver and use Lax-Friedrichs-type fluxes to connect the elements.

With the nonlinear SWE solver, consider the so-called *dam-break problem* which is a model of a dam which suddenly seperates two levels of water at $t = 0$ in a burst. For the setup the initial condition is

$$h(x,0) = \begin{cases} h_l & ,\text{if } x < 20 \\ h_r & ,\text{if } x > 20 \end{cases} \tag{6}$$

For the initial condition you can use water heights $(h_l, h_r) = (3.5, 1.25)$ meters. An exact solution for total water height h and momentum hu at time $t = 2.5$ seconds is provided in dambreakdata.mat which can be loaded into matlab and visualized using the commands

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>> load dambreakdata
>> showme
```

This data can be used for validation of your solver.

Finally, select at least one of the following two subjects

- Take into account a varying bed slope and make use of filters/limiters for stabilization.
- Local resolution in space and time of a moving shock.

In either case demonstrate a robust and accurate solution strategy.

Enjoy!