DG-FEM for PDE's
Lecture 3

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A brief overview of what's to come

- Lecture 1: Introduction and DG-FEM in 1D
- Lecture 2: Implementation and numerical aspects
- Lecture 3: Insight through theory
- Lecture 4: Nonlinear problems
- Lecture 5: Extension to two spatial dimensions
- Lecture 6: Introduction to mesh generation
- Lecture 7: Higher order/Global problems
- Lecture 8: 3D and advanced topics

Lecture 3

- Let's briefly recall what we know
- Why high order methods?
- Part I:
  - Constructing fluxes for linear systems
  - Approximation theory on the interval
- Part II:
  - Convergence and error estimates
  - Dispersive properties
  - Discrete stability and how to overcome

Let us recall

We already know a lot about the basic DG-FEM

- **Stability** is provided by carefully choosing the numerical flux.
- **Accuracy** appears to be given by the local solution representation.
- We can utilize major advances on monotone schemes to design fluxes.
- The scheme generalizes with very few changes to very general problems -- multidimensional systems of conservation laws.
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We already know a lot about the basic DG-FEM

- **Stability** is provided by carefully choosing the numerical flux.
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- We can utilize major advances on monotone schemes to design fluxes.
- The scheme generalizes with very few changes to very general problems -- multidimensional systems of conservation laws.

At least in principle -- but what can we actually prove?

**Why high-order accuracy?**

How do I solve a wave-problem to a given accuracy, $\varepsilon_p$, for a specific period of time, $\nu$, most efficiently?

$\text{Memory} \propto \left(\frac{\nu}{\varepsilon_p}\right)^{\frac{d}{2}}$, $\text{Work} \propto (2m)^d\nu\left(\frac{\nu}{\varepsilon_p}\right)^{\frac{d}{2}}$

<table>
<thead>
<tr>
<th>Memory</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_p = 0.1$</td>
<td>$\varepsilon_p = 0.01$</td>
</tr>
<tr>
<td>$\nu = 100$</td>
<td>$\nu = 1000$</td>
</tr>
</tbody>
</table>

2nd order FD

Infinite order FD

**Why high-order accuracy?**

Let us just make sure we understand why high-order accuracy/methods is a good idea

**General concerns/criticism:**

- High-order accuracy is not needed for real appl.
- The methods are not robust/flexible
- They only work for smooth problems
- They are hard to do in complex geometries
- They are too expensive

*After having worked on these methods for 15 years, I have heard them all*

**Why high-order accuracy?**

High-order is important if

- High accuracy is required - and it increasingly is!
- Long time integration is needed
- High-dimensional problems (3D) are considered
- Memory restrictions become a bottleneck
Added benefit of high-order support

High-order takes 'some' of the pain out of grid generation

A bit more on fluxes

Let us briefly look a little more carefully at linear systems

\[ Q(x) \frac{\partial u}{\partial t} + \nabla \cdot F = Q(x) \frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0, \]

\[ F = [F_1, F_2] = [A_1(x)u, A_2(x)u]. \]

Prominent examples are

- Acoustics
- Electromagnetics
- Elasticity

In such cases we can derive exact upwind fluxes

Linear systems and fluxes

Assume first that all coefficients vary smoothly

\[ Q(x) \frac{\partial u}{\partial t} + A_1(x) \frac{\partial u}{\partial x} + A_2(x) \frac{\partial u}{\partial y} + B(x)u = 0, \]

The flux along a normal \( \hat{n} \) is then

\[ \Pi = (\hat{n}_x A_1(x) + \hat{n}_y A_2(x)). \]

Now diagonalize this as

\[ Q^{-1} \Pi = SAS^{-1}, \]

\[ A = A^+ + A^-, \]

and we obtain

\[ (\hat{n} \cdot F)^* = QS (A^+ S^{-1} u^- + A^- S^{-1} u^+). \]

Linear systems and fluxes

For non-smooth coefficients, it is a little more complex

Consider the problem

\[ \frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \quad x \in [a, b]. \]

Then we clearly have

\[ \frac{d}{dt} \int_a^b u \, dx = \int_a^b u(\lambda b - a) + (b - \lambda a) u^+ = \lambda (u^- - u^+). \]
Linear systems and fluxes

Hence, by simple mass conservation, we achieve
\[-\lambda(u^- - u^+) + (f^- - f^+) = 0.\]
for \( a \rightarrow x^-, b \rightarrow x^+ \)
These are the Rankine-Hugoniot conditions
For the general system, these are
\[\forall i : -\lambda_i Q[u^- - u^+] + [(\Pi u)^- - (\Pi u)^+] = 0,\]
They must hold across each wave and can be used to connect across the interface

Linear systems and fluxes -- an example

Consider
\[\frac{\partial q}{\partial t} + A \frac{\partial q}{\partial x} = 0,\]
Following the general approach, we have
\[a^- (q^* - q^-) + (\Pi q)^* - (\Pi q)^- = 0,\]
\[a^+ (q^* - q^+) + (\Pi q)^* - (\Pi q)^+ = 0,\]
with \((\Pi q)^\pm = \hat{n} \cdot (Aq)^\pm = \hat{n} \cdot [a^\pm 0 -a^\pm] [a^\pm 0 -a^\pm].\]
Solving this yields
\[(\Pi q)^* = \frac{2a^+a^-}{a^+ + a^-} \hat{n} \cdot \left( \frac{\{q\} + \frac{1}{2} \{u\}}{\{v\}} \right),\]
Intermediate velocity
\[a^* = \frac{2a^-a^+}{a^+ + a^-},\]
Linear systems and fluxes

So for the 3-wave problem we have
\[\lambda Q^- (u^* - u^-) + [(\Pi u)^* - (\Pi u)^-] = 0,\]
\[[(\Pi u)^* - (\Pi u)^*] = 0,\]
\[-\lambda Q^+ (u^+ - u^*) + [(\Pi u)^* - (\Pi u)^+] = 0,\]
and the numerical flux is given as
\[(\hat{n} \cdot F)^* = (\Pi u)^* = (\Pi u)^*;\]
This approach is general and yields the exact upwind fluxes -- but requires that the system can be solved!

Linear systems and fluxes -- an example

Consider Maxwell’s equations
\[\begin{bmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} E \\ H \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} E \\ H \end{bmatrix} = 0.\]
The exact same approach leads to
\[H^* = \frac{1}{2} (|2H| + |2E|), \quad E^* = \frac{1}{2} \left( |2Y| + \frac{1}{2} |2F| \right).\]
Now assume smooth materials:
\[H^* = \{H\} + \frac{Y}{2} |2E|, \quad E^* = \{|E|\} + \frac{Z}{2} |2H|,\]
We have recovered the LF flux!
Consider Maxwell's equations
\[ \varepsilon(x) \frac{\partial E}{\partial t} = -\frac{\partial H}{\partial x}, \quad \mu(x) \frac{\partial H}{\partial t} = -\frac{\partial E}{\partial x}, \]

On the DG form
\[ \frac{dE^k}{dt} + \frac{1}{\varepsilon(x)} \mathbf{n} \cdot \mathbf{D}(H^k) = \frac{1}{\varepsilon(x)} M^{-1} i \mathbf{E}(x)(H^k - H^*), \]
with the flux
\[ H^- - H^* = \frac{1}{2} \left( Z^+ [H] - [E] \right), \]
\[ E^- - E^* = \frac{1}{2} \left( Y^+ [E] - [H] \right). \]

\[ \text{An example} \]

Test example is cavity problem

Material

PEC

Vacuum

PEC

[Diagram of vacuum and material with labels for PEC and vacuum, showing boundary conditions and fluxes]

Wednesday, August 8, 12
Let's move on

At this point we have a good understanding of stability for linear problems -- through the flux.

Let's now look at accuracy in more detail.

Recall
\[ \Omega \simeq \Omega_h = \bigcup_{k=1}^{K} \mathbb{D}^k, \quad u(x, t) \simeq u_h(x, t) = \bigoplus_{k=1}^{K} u^k_h(x, t), \]
we assume the local solution to be
\[ x \in \mathbb{D}^k = [x_k^l, x_k^r]: \quad u^k_h(x, t) = \sum_{n=1}^{N_p} \hat{u}^k_n(t) \psi_n(x) = \sum_{i=1}^{N_p} u^k_i(x, t) t^k_i(x). \]

A second look at approximation

We will need a little more notation

Regular energy norms
\[ \|u\|_{\Omega}^2 = \int_{\Omega} u^2 \, dx, \quad \|u\|_{\Omega_h}^2 = \sum_{k=1}^{K} \|u\|_{\mathbb{D}^k}^2, \quad \|u\|_{\mathbb{D}^k}^2 = \int_{\mathbb{D}^k} u^2 \, dx. \]

Sobolev norms
\[ \|u\|_{\mathbb{D}^k, q}^2 = \sum_{|\alpha|<q} \|u^{(\alpha)}\|_{\mathbb{D}^k}^2, \quad \|u\|_{\mathbb{D}^k, q, h}^2 = \sum_{k=1}^{K} \|u\|_{\mathbb{D}^k, q}^2, \quad \|u\|_{\mathbb{D}^k, q}^2 = \sum_{|\alpha|=q} \|u^{(\alpha)}\|_{\mathbb{D}^k}^2. \]

Semi-norms
\[ \|u\|_{\mathbb{D}^k, q, h}^2 = \sum_{k=1}^{K} \|u\|_{\mathbb{D}^k, q}^2, \quad \|u\|_{\mathbb{D}^k, q}^2 = \sum_{|\alpha|=q} \|u^{(\alpha)}\|_{\mathbb{D}^k}^2. \]

Local approximation

To simplify matters, introduce local affine mapping
\[ x \in \mathbb{D}^k: \quad x(r) = x^k_t + \frac{1+r}{2} h^k, \quad h^k = x^k_r - x^k_t, \quad r \in [-1, 1] \]

We have already introduced the Legendre polynomials
\[ u(r) \simeq u_h(r) = \sum_{n=1}^{N_p} \hat{u}_n \hat{P}_{n-1}(r) = \sum_{n=1}^{N_p} u_n(t) \ell_n(r), \]
\[ u = \mathcal{V} u, \quad \mathcal{V}^T \ell(r) = \tilde{P}(r), \quad \mathcal{V}_{ij} = \tilde{P}_j(r_i). \]

and \( r_i \) are the Legendre Gauss Lobatto points:

It is robust -- but is it accurate?

Approximation theory

Recall
\[ \Omega \simeq \Omega_h = \bigcup_{k=1}^{K} \mathbb{D}^k, \quad u(x, t) \simeq u_h(x, t) = \bigoplus_{k=1}^{K} u^k_h(x, t), \]
we assume the local solution to be
\[ x \in \mathbb{D}^k = [x^k_t, x^k_r]: \quad u^k_h(x, t) = \sum_{n=1}^{N_p} \hat{u}^k_n(t) \psi_n(x) = \sum_{i=1}^{N_p} u^k_i(x, t) t^k_i(x). \]

The question is in what sense is \( u(x, t) \simeq u_h(x, t) \)

We have observed improved accuracy in two ways

- Increase \( K \)/decrease \( h \)
- Increase \( N \)

Wednesday, August 8, 12
Prior to discussing interpolation, used throughout this text, we consider

\[ v(r) = u(hr) = u(x); \]

We consider expansions as

\[ v_h(r) = \sum_{n=0}^{N} \tilde{v}_n \tilde{P}_n(r), \quad \tilde{P}_n(r) = \frac{P_n(r)}{\sqrt{\gamma_n}}, \quad \gamma_n = \frac{2}{2n + 1}. \quad \tilde{v}_n = \int_{1}^{r} v(r) \tilde{P}_n(r) \, dr. \]

**Theorem 4.1.** Assume that \( v \in H^p(I) \) and that \( v_h \) represents a polynomial projection of order \( N \). Then

\[ \| v - v_h \|_{L^q} \leq N^{\rho-p} \| v \|_{L^p}, \]

where

\[ \rho = \begin{cases} \frac{2q}{2q-1}, & 0 \leq q \leq 1 \\ 2q - \frac{1}{2}, & q \geq 1 \end{cases} \]

and \( 0 \leq q \leq p \).

A sharper result can be obtained by using

\[ \| v^{(q)} - v_h^{(q)} \|_{L^1} \leq \left( \frac{(N+1-\sigma)!}{(N+1+\sigma-4q)!} \right)^{1/2} \| v \|_{L^p}, \]

where \( \sigma = \min(N+1,p) \) and \( q \leq p \).

Note that in the limit of \( N \gg p \) we recover

\[ \| v^{(q)} - v_h^{(q)} \|_{L^1} \leq N^{2q-p} \| v \|_{L^p}. \]

A minor issue arises -- these results are based on projections and we are using interpolations?

Consider this term

\[ \hat{P}_n^T(r) = \int_{n}^{n+1} \tilde{v}_n \tilde{P}_n(r) = \sum_{n=N+1}^{\infty} \tilde{v}_n \tilde{P}_n(r), \quad \hat{P}_n^T(r) = \int_{n}^{n+1} \tilde{v}_n \tilde{P}_n(r) \int_{n}^{n+1} \tilde{P}_n(r). \]

Caused by interpolation of high-frequency unresolved modes

**Aliasing**

Caused by the grid
Approximation theory

This has a some impact on the accuracy

**Theorem 4.5.** Assume that \( v \in H^p(\Omega) \), \( p > \frac{1}{2} \), and that \( v_h \) represents a polynomial interpolation of order \( N \). Then

\[
\|v - v_h\|_{L^q} \leq N^{2q-p+1/2}|v|_{L^p},
\]

where \( 0 \leq q \leq p \).

To also account for the cell size we have

**Theorem 4.7.** Assume that \( u \in H^p(\Omega^h) \) and that \( u_h \) represents a piecewise polynomial approximation of order \( N \). Then

\[
\|u - u_h\|_{\Omega^h} \leq C h^{\sigma-q}|u|_{\Omega^h},
\]

for \( 0 \leq q \leq \sigma \), and \( \sigma = \min(N + 1, p) \).

Let us revisit Example 3.2 where we considered the error when

**Fluxes:**
- For linear systems, we can derive exact upwind fluxes using Rankine-Hugenoit conditions.

**Accuracy:**
- Legendre polynomials are the right basis
- Local accuracy depends on elementwise smoothness
- Aliasing appears due to the grid but is under control
- For smooth problems, we have a spectral method
- Convergence can be recovered in two ways
  - Increase \( N \)
  - Decrease \( h \)

Convergence of the solution at all times?
Lecture 3

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  - Dispersive properties
  - Discrete stability and how to overcome

Convergence and all that

Let us introduce the error
\[ \varepsilon(x,t) = u(x,t) - u_h(x,t), \]
What we really seek is convergence
\[ \forall t \in [0,T] : \lim_{\text{dof} \to \infty} \| \varepsilon(t) \|_{\Omega,h} \to 0. \]
This is often a little complicated to get to due to the requirement for all \( t \).

Let us get to it in a different way.

Let’s recall convergence etc

We consider the system
\[ \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0, \]
which we assume is wellposed in the sense
\[ \| u(t) \|_\Omega \leq C \exp(\alpha t) \| u(0) \|_\Omega. \]
The semi-discrete scheme is given as
\[ \frac{d u_h}{d t} + L_h u_h = 0. \]
Inserting the exact solution \( u \) into the scheme yields
\[ \frac{d u}{d t} + L_h u = T(u(x,t)), \]
\[ \text{truncation error} \]

Convergence and all that

Let us consider the error equation
\[ \frac{d}{d t} \varepsilon + L_h \varepsilon = T(u(x,t)), \]
The solution is given as
\[ \varepsilon(t) - \exp(-L_h t) \varepsilon(0) = \int_0^t \exp(-L_h (s-t)) T(u(s)) \, ds, \]
Now consider
\[ \| \varepsilon(t) \|_{\Omega,h} \leq \| \exp(-L_h t) \varepsilon(0) \|_{\Omega,h} + \left\| \int_0^t \exp(-L_h (s-t)) T(u(s)) \, ds \right\|_{\Omega,h} \]
\[ \leq \int_0^t \| \exp(-L_h (s-t)) T(u(s)) \|_{\Omega,h} ds, \]
Convergence and all that

So if we require consistency

\[
\begin{align*}
\lim_{\text{dof} \to \infty} \| \varepsilon(0) \|_{\Omega,h} &= 0, \\
\lim_{\text{dof} \to \infty} \| T(u(t)) \|_{\Omega,h} &= 0
\end{align*}
\]

and stability

\[
\lim_{\text{dof} \to \infty} \| \exp(-L_h t) \|_{\Omega,h} \leq C_h \exp(\alpha_h t), \ t \geq 0,
\]

we obtain convergence

\[
\forall t \in [0,T]: \lim_{\text{dof} \to \infty} \| \varepsilon(t) \|_{\Omega,h} \to 0.
\]

This is of course part of the celebrated Lax-Richtmyer equivalence theorem.

Back to the example

Consider again the simple example

\[
\frac{\partial u}{\partial t} - 2\pi \frac{\partial u}{\partial x} = 0, \ x \in [0,2\pi], \ u(x,0) = \sin(lx), \ l = \frac{2\pi}{\lambda},
\]

<table>
<thead>
<tr>
<th>N \backslash K</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>Convergence rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0E-01</td>
<td>1.0E-02</td>
<td>9.1E-02</td>
<td>2.3E-02</td>
<td>5.7E-03</td>
<td>1.4E-03</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>2.0E-01</td>
<td>4.3E-02</td>
<td>6.3E-03</td>
<td>2.1E-03</td>
<td>5.7E-03</td>
<td>1.4E-03</td>
<td>3.0</td>
</tr>
<tr>
<td>4</td>
<td>3.3E-03</td>
<td>3.1E-04</td>
<td>9.9E-06</td>
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<td>1.0E-08</td>
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<tr>
<td>8</td>
<td>2.1E-07</td>
<td>2.5E-09</td>
<td>4.8E-12</td>
<td>2.2E-13</td>
<td>5.0E-15</td>
<td>6.0E-15</td>
<td>9.0</td>
</tr>
</tbody>
</table>

The error clearly behaves as

\[
\| u - u_h \|_{\Omega,h} \leq C h^{N+1}.
\]

Convergence and all that

Recall

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0,
\]

for which we proved stability as

\[
\frac{1}{2} \frac{d}{dt} \| u_h \|_{\Omega,h}^2 \leq c \| u_h \|_{\Omega,h}^2,
\]

This generalizes easily to systems when upwinding is used on the characteristic variables.

Combining this with the accuracy analysis yields

\[
\| u - u_h \|_{\Omega,h} \leq \frac{h^N}{N^{p-5/2}} | u |_{\Omega,p,h},
\]

Back to the example

What about time dependence

<table>
<thead>
<tr>
<th>Final time (T)</th>
<th>(\pi)</th>
<th>(10\pi)</th>
<th>(100\pi)</th>
<th>(1000\pi)</th>
<th>(2000\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N.K)=((2,4))</td>
<td>4.3E-02</td>
<td>7.8E-02</td>
<td>5.6E-01</td>
<td>&gt;1</td>
<td>&gt;1</td>
</tr>
<tr>
<td>(N.K)=((4,2))</td>
<td>3.3E-03</td>
<td>4.4E-03</td>
<td>2.8E-02</td>
<td>2.6E-01</td>
<td>4.8E-01</td>
</tr>
<tr>
<td>(N.K)=((4,4))</td>
<td>3.1E-04</td>
<td>3.3E-04</td>
<td>3.4E-04</td>
<td>7.7E-04</td>
<td>1.4E-03</td>
</tr>
</tbody>
</table>

The error behaves as

\[
\| u - u_h \|_{\Omega,h} \leq C(T) h^{N+1} \sim (c_1 + c_2 T) h^{N+1},
\]
Convergence and all that

Recall
\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \]
for which we proved stability as
\[ \frac{1}{2} \frac{d}{dt} \| u_h \|^2_{\Omega,h} \leq c \| u_h \|^2_{\Omega,h}, \]
This generalizes easily to systems when upwinding is used on the characteristic variables.

Combining this with the accuracy analysis yields
\[ \| u - u_h \|_{\Omega,h} \leq \frac{h^{N_{\text{opt}}}}{N_{d-\text{opt}}^{5/2}} \| u \|_{\Omega,p,h}, \]
but we observed
\[ \| u(T) - u_h(T) \|_{\Omega,h} \leq h^{N+1}(C_1 + TC_2). \]

Error estimates

We will now mimic this for the semi-discrete problem
\[ B_h(u, \phi_h) = (u_0, \phi_h)_{\Omega,h} + a(u, \phi_h)_{\Omega,h} - (v_0 \cdot O(\partial u_h - (au_h)^*))_{\Omega,h}, \]
Let us use a central flux
\[ (au)^* = \{\{au\}\}, \]
to obtain
\[ B_h(u, \phi_h) = (u_0, \phi_h)_{\Omega,h} + a(u, \phi_h)_{\Omega,h} - \frac{1}{2} (\{au\})_{\Omega,h} = 0. \]
Observe
\[ B_h(u, \phi_h) = 0, \quad B_h(\varepsilon, \phi_h) = 0, \quad \varepsilon = u - u_h. \]
Using
\[ B_h(\varepsilon, \varepsilon) = \frac{1}{2} \frac{d}{dt} \| \varepsilon \|^2_{\Omega,h}, \]
Error estimates

To get closer to the observed behavior, we need to be a little more careful.

Define
\[ B(u, \phi) = (u_0, \phi)_{\Omega} + a(u, \phi)_{\Omega} = 0 \]
we have
\[ B(u, u) = 0 = \frac{1}{2} \frac{d}{dt} \| u \|^2_{\Omega}; \]
For two different solutions we have
\[ \varepsilon(t) = u_1(t) - u_2(t) \]
\[ \frac{1}{2} \frac{d}{dt} \| \varepsilon \|^2_{\Omega} = 0, \quad \| \varepsilon(T) \|_{\Omega} = \| u_1(0) - u_2(0) \|_{\Omega}, \]

Better -- but not quite there
Error estimates

The observe full order

\[ \| u(T) - u_h(T)\|_{\Omega,h} \leq h^{N+1}(C_1 + TC_2). \]

is in fact a special case!

It only works when

✓ When full upwinding on all characteristic variables are used
✓ Proof is only valid for the linear case
✓ Proof relies on 1D superconvergence results

In spite of this, optimal convergence is observed in many problems - why?

Dispersive properties

Consider again

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad u(x,0) = \exp(\imath l x), \]

\[ u(x,t) = \exp(\imath (lx - \omega t)). \]

The scheme is given as

\[ \frac{h}{2} M \frac{du_h^k}{dt} + a S u^k = e_N \left[ (au_h^k) - (au_h^k)^* \right] \delta t - e_0 \left[ (au_h^k) - (au_h^k)^* \right] \delta t, \]

\[ (au)^* = \{\{au\}\} + [a] \frac{1 - \alpha}{2} [a]. \]

Look for solutions of the form

\[ u_h^k(x^k, t) = U_h^k \exp[\imath (lx^k - \omega t)], \]

Why often optimal anyway?

Assume stability

\[ \lim_{\text{dof} \to \infty} \| \exp(-\mathcal{L}_h t) \|_{\Omega,h} \leq C_h \exp(\alpha h t), \quad t \geq 0. \]

Recall

\[ \| \varepsilon(t) \|_{\Omega,h} \leq \| \exp(-\mathcal{L}_h t) \varepsilon(0) \|_{\Omega,h} \]

\[ \| \exp(\mathcal{L}_h (s-t)) T(u(s)) de \|_{\Omega,h} \]

Error in I.C.

\[ \| u - u_h \|_{\Omega,q,h} \leq C_{\text{h}} h^{\sigma-q} \frac{h^{2-q-1/2}}{N^{p-2q-1/2}} |u|_{\Omega,q,h}, \]

\[ \sigma = \min(N + 1, p). \]

Dispersive properties

We recover

\[ \left[ 2S - \alpha e_N (e_N^T - \exp(i L (N+1)) e_0^T) \right. \]

\[ \left. + (2 - \alpha) e_0 (e_0^T - \exp(-i L (N+1)) e_N^T) \right] U_h^k = i \Omega M U_h^k. \]

Where

\[ L = \frac{h}{N+1} = \frac{2\pi}{\lambda (N+1)} = 2 \pi p^{-1}, \quad \Omega = \frac{\omega h}{a}, \]

\[ p = \frac{\lambda}{h/(N+1)} = \text{DoF per wavelength} \]

So for a fixed L we solve the eigenvalue problem

.. and the eigenvalue will tell us how the wave propagates
Dispersive properties

Upwind fluxes

Central fluxes

Discrete stability

So far we have not done anything to discretize time.

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{d u_h}{d t} + L_h u_h = 0.
\]

We shall consider the use of ERK methods

\[
k^{(1)} = L_h (u^n_h, t^n),
\]

\[
k^{(2)} = L_h \left( u^n_h + \frac{1}{2} \Delta t k^{(1)}, t^n + \frac{1}{2} \Delta t \right),
\]

\[
k^{(3)} = L_h \left( u^n_h + \frac{1}{2} \Delta t k^{(2)}, t^n + \frac{1}{2} \Delta t \right),
\]

\[
k^{(4)} = L_h \left( u^n_h + \Delta t k^{(3)}, t^n + \Delta t \right),
\]

\[
u_{n+1}^h = u^n_h + \frac{1}{6} \Delta t \left( k^{(1)} + 2k^{(2)} + 2k^{(3)} + k^{(4)} \right).
\]

Dispersive properties

There are some analytic results available (upwind)

\[
\left| R(\bar{h}) - R(h) \right| \approx \frac{1}{2} \left[ \frac{N!}{(2N+1)!} \right]^2 (h)^{2N+3},
\]

\[
\left| I(\bar{h}) \right| \approx \frac{1}{2} \left[ \frac{N!}{(2N+1)!} \right]^2 (1 - \alpha)^{-N} (h)^{2N+2},
\]

The dispersive accuracy is excellent!

Define the relative phase error

\[
\rho_N = \frac{\exp(\bar{h}l) - \exp(\bar{l}h)}{\exp(\bar{l}h)}.
\]

\[
\rho_N \approx \begin{cases} 2N+1 < h - C(l)^{1/3}, & \text{no convergence} \\ lh - o(lh)^{1/3} < 2N+1 < lh + o(lh)^{1/3}, & \mathcal{O}(N^{-1/3}) \text{ convergence} \\ 2N+1 \gg lh, & \mathcal{O}(hl/(2N+1))^{2N+2} \text{ convergence} \end{cases}
\]

Convergence for

\[2 \approx \frac{lh}{N+1} = 2\pi p^{-1}; \quad p \geq \pi\]

Dispersive properties

and also a Low Storage form

\[
U^{(0)} = u^n,
\]

\[
i \in [1, \ldots, 5]: \begin{cases} k^{(i)} = a_i k^{(i-1)} + \Delta t L_h \left( p^{(i-1)}, t^n + \epsilon_i \Delta t \right), \\ p^{(i)} = p^{(i-1)} + b_i k^{(i)}; \end{cases}
\]

Consider

\[u_t = \lambda u, \quad \text{Real}(\lambda) \leq 0,\]

The stability region defines the timestep that gives stability.
**Discrete stability**

Consider

\[ \mathcal{L}_h = \frac{2a}{h} \mathcal{M}^{-1} \left[ S - \mathcal{E} \right], \]

We have

\[ \frac{h^2}{4a^2} \| \mathcal{L}_h \|^2 = \frac{h^2}{4a^2} \sup_{\| u_h \|=1} \| \mathcal{L}_h u_h \|^2 \]

\[ \leq \| D_r \|^2 + \| \mathcal{M}^{-1} \mathcal{E} \|^2 + 2 \sup_{\| u_h \|=1} (D_r u_h, \mathcal{M}^{-1} \mathcal{E} u_h) \]

\[ \leq C_1 N^4 + C_2 N^2 + C_3 N^3 \leq CN^4, \]

So we should expect

\[ \| \mathcal{L}_h \|_{D^k} \leq C \frac{a}{h^k} N^2 \]

Which would indicate

\[ \Delta t \leq C \frac{h}{a N^2} \]

---

**Local time-stepping**

**Problem:** Small cells, even just one, cause a very small global time-step in an explicit scheme.

\[ \Delta t \leq C' \Delta x \leq C_1 \frac{h}{N^2} \]

A significant problem for large scale complex applications

**Old idea:** take only time-steps required by local restrictions.

**Old problems:** accuracy and stability
Local time-stepping

Recall the ERK scheme

\[ u_{n+1} = u_n + \frac{\Delta t}{12} \left( 23F(u_n) - 16F(u_{n-1}) + 5F(u_{n-2}) \right) \]

We consider a multi-step scheme

\[ \Delta t = n \cdot \Delta t \]

This generalizes to many levels and arbitrary time-step fractions

Challenge: Achieving this at high-order accuracy

For all interior cells

\[ u_{n+1} = u_n + \frac{\Delta t}{12} \left( 23F(u_n) - 16F(u_{n-1}) + 5F(u_{n-2}) \right) \]

At interface cells

\[ u_{n+1/2} = u_n + \frac{\Delta t}{12} \left( 7F(u_n) - 7F(u_{n-1}) + 2F(u_{n-2}) \right) \]

This generalizes to many levels and arbitrary time-step fractions

Substantial recent work by Cohen, Grote, Lanteri, Piperno, Gassner, Munz etc

Most of the recent work is based on LF-like schemes, restricted to 2nd order in time.

Layout for multi-rate local time-stepping

\[ t_{n+1} \]
\[ t_{n+1/2} \]
\[ t_n \]
\[ t_{n+1/4} \]
\[ t_{n+3/4} \]
\[ t_{n+1} \]

Execution Times with Local Time-Stepping

- \( N = 23342 \)
- \( N = 23391 \) (8%)
- \( N = 151 (10\%) \)
- \( N = 151 (<1\%) \)
- \( N = 21632 (91\%) \)
- \( N = 151 (<1\%) \)

Computations by HyperComp Inc
Local time-stepping

Segmentation is done in preprocessing

Ideally suited for local DG scheme

Known problems:
- No known stability proof
- Time-step is not optimal (about 80%)

The potential speed up is considerable -- and the more complex the better!

<table>
<thead>
<tr>
<th>Example</th>
<th>Simulation time with</th>
<th>Adams-Bashford</th>
<th>Adams-Bashford</th>
<th>LSERK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(global time step)</td>
<td>(local time step)</td>
<td>(global time step)</td>
</tr>
<tr>
<td>Resonator</td>
<td>100%</td>
<td>59%</td>
<td>45%</td>
<td></td>
</tr>
<tr>
<td>3dB-Coupler</td>
<td>100%</td>
<td>29%</td>
<td>45%</td>
<td></td>
</tr>
<tr>
<td>Airplane</td>
<td>100%</td>
<td>15%</td>
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<td></td>
</tr>
</tbody>
</table>

Computations by Nico Godel, Hamburg

A brief summary

We now have a good understanding all key aspects of the DG-FEM scheme for linear first order problems

- We understand both accuracy and stability and what we can expect.
- The dispersive properties are excellent.
- The discrete stability is a little less encouraging.
  A scaling like
  \[ \Delta t \leq C \frac{h}{aN^2} \]
  is the Achilles Heel -- but there are ways!

... but what about nonlinear problems?