

Optimal Mortgage Loan Diversification

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Abstract

Homebuyers in several countries may finance the purchase of their properties using different variants of either adjustable-rate mortgages (ARMs) or fixed-rate mortgages (FRMs). The variety and complexity of these loan products poses a risk management task for mortgage bank advisors to recommend the right mortgage loan strategy for the individual mortgagor; almost all mortgage banks advise their customers to take a single loan product. This argument is often justified by the fact that trade frictions make it unattractive to hold a portfolio of loans as a private home owner. Even with transaction costs, however, we show in this paper that most mortgagors with some degree of risk aversion benefit from holding a mortgage portfolio. To do so we develop a multi stage Mean-Conditional Value at Risk (MCVaR) model to consider the risk of the mortgage payment frequency function explicitly using a coherent risk measure. In addition to the diversification benefits we also show that the multistage model produces superior results as compared to single stage models and that the solutions are robust with regards to changes in uncertainty parameters in particular for risk averse mortgagors. Finally, we show how the model can be used to calculate fair premia for adjustable rate mortgages with interest rate guarantees (caps) which are becoming increasingly popular as a hybrid product between the existing ARM and FRM mortgages.

Keywords: Mortgage loans products, CVaR modeling, stochastic programming.

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1 Introduction

Most homebuyers across the world finance the purchase of their houses by taking a mortgage loan. Mortgage banks in several countries offer several mortgage products with different payment schemes and risk profiles. This complicates the job of the mortgage advisor who has to account for the credit worthiness of the mortgagor and its effect on the amount and type of mortgage loan that should be granted. This paper deals with modeling the mortgage choice problem to account for some of the most significant uncertainties of this problem.

Even though the model in this paper can be applied to any mortgage market (perhaps with some modifications) the cases considered are based on the Danish mortgage market. The Danish mortgage banks are highly specialized institutions whose main focus is, on the one hand, to collect investments from investors in mortgage-backed securities, and on the other hand, to pool the investments together and issue mortgage loans to home buyers. The Danish mortgage-backed security legislation requires equal payments on the investor and the mortgagor side – “the balance principle”. This law in effect limits the financial risks assumed by the mortgage banks to credit-default risk on the mortgagor side and mortgage-bond-liquidity risk on the investor side.

A feature unique to the Danish mortgage market is that mortgagors, via the mortgage banks, are virtually trading mortgage bonds and may exercise all the embedded options in the underlying bonds. For a fixed-rate mortgage (FRM) this includes a call option of Bermudan type with a strike price of 100 and a buy-back delivery option which gives the mortgagor the right to redeem the mortgage at the actual market price of the underlying bond. These two features provide security on the mortgagor side. In the case of falling interest rates the mortgagor can exercise the call option and refinance the existing FRM, with a new FRM with lower coupon payments. Furthermore for rising interest rates the mortgagor can reduce the outstanding debt by prepaying the mortgage at a market price lower than the original issuance price. This has an impact on FRMs with large durations. Small movements in interest rates result in big movements in prices of FRMs and this has an important impact on the debt-free value of the property. The buy-back may be refinanced either by selling the house or by taking a new loan with an underlying FRM with higher coupon payments, or an adjustable-rate mortgage (ARM). The exercise of the buy-back delivery option is useful in case the mortgagor needs to move or in case he or she believes the interest rates will fall again in the near future. These features of the FRM, and the security they offer, however come at a price; the effective interest rate payments are often considerably

higher than those of adjustable-rate mortgages.

Since the mid 1990's the Danish mortgage market has been growing rapidly and a number of new mortgage products have been introduced in addition to FRMs. The two most popular products have been ARM with varying adjustment intervals, and the *capped rate mortgages* (CRM) where the interest rate cannot grow higher than a predetermined level (cap). All these loans may be issued with or without principal payments (interest payments only) for a period of up to 10 years. The interest-only period is renewable after the initial 10 years.

ARMs are financed by issuing underlying bullet bonds with maturities of 1 to 10 years. For example an ARM1 is a mortgage with annual interest rate adjustments, whereas the rate of an ARM2 is readjusted every other year, etc. Since there are no embedded options available with an ARM, the average payments of an ARM are lower than an FRM but payment volatility is considerably greater for very long horizons. CRMs are financed by a variable rate security with an embedded cap. The interest rate follows a 6-month CIBOR (Copenhagen Inter Bank Offered Rate) plus a premium. Should the CIBOR plus premium increase to a level higher than the interest rate guarantee level, the rate will be fixed at the guarantee level. Should the CIBOR plus premium fall below the guarantee level again, the CRM's rate will follow accordingly.

The extra features of the CRM come at a price, i.e. higher interest rate payments than alternative floating rate products such as an ARM1. For more details on the workings of the Danish mortgage market see Svenstrup and Willemann (2005). For a review of recent innovations in the mortgage-backed security market see Piskorski and Tchisty (2006).

It must be evident by now that it is a non-trivial task to advise a mortgagor on choices of mortgage loan. Indeed wrong advice together with unfavorable market behavior might result in financial ruin for a large pool of mortgagors and this in turn will have unprecedented macro-economic effects such as a devaluation of the housing market. The research interest in this problem is well justified.

Nielsen and Poulsen (2004) design a trinomial scenario tree using an underlying two-factor model of interest rates for pricing existing and synthetic mortgage bonds. Furthermore they introduce a stochastic programming model to find the optimal initial loan strategy and to advise the mortgagor on optimal readjustments along the way. Their optimization model, however, does not include a risk measure and the effects of fixed-mortgage origination costs were ignored. Rasmussen and Clausen (2004) further develop this model to

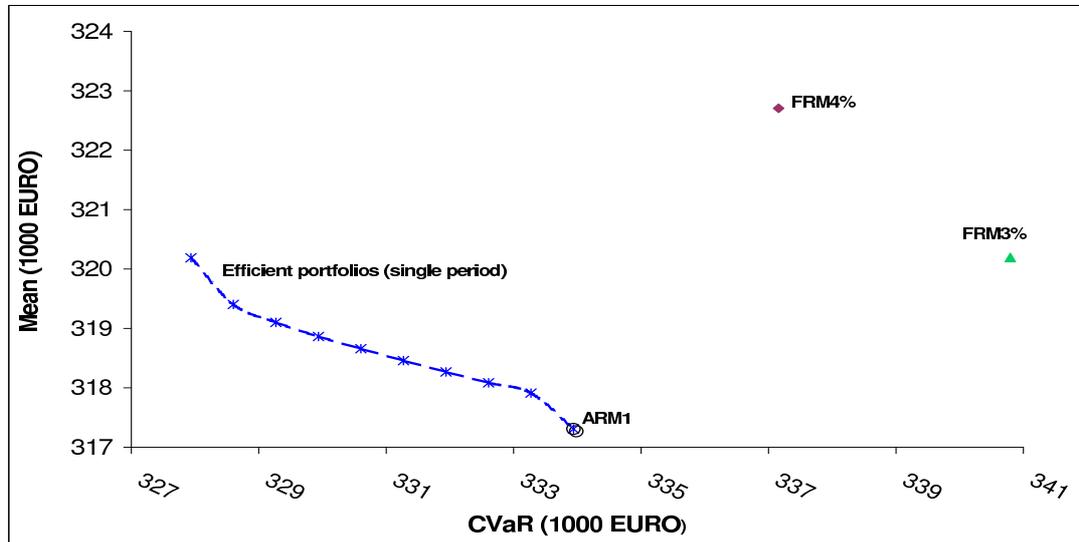


Figure 1: For a mortgagor with a seven year horizon a mix of variable and fixed-rate mortgages provide low payments and low risk, here measured by the 10% CVaR value.

include fixed-mortgage origination costs and budget constraints. Their conclusion is that a mortgagor with budget constraints benefits from choosing an initial portfolio of an ARM and a FRM, given that there are only these two types of products to choose from. The budget constraints provide indirect means for risk control, but no explicit risk measure is considered in this paper either.

An explicit risk measure for this class of problems was introduced by Rasmussen and Zenios (2007) who develop a single-period stochastic programming model to trade off the present value of average mortgage payments against the Conditional Value at Risk (CVaR¹) value. They use a Mean/CVaR efficient frontier to show that diversified mortgage loan strategies outperform single mortgage loan strategies; Figure 1 highlights their findings which speak strongly in favor of diversification.

In this paper we develop a multi-stage version of our earlier model and show that improved results can be obtained by introducing dynamic trading into the model. It will be seen that the budget-constrained model of Rasmussen

¹For a review of CVaR as a coherent risk measure see Artzner et al. (1999), Rockafellar and Uryasev (2000) and Zenios (2007).

and Clausen is subsumed by the bilinear Mean/CVaR minimizing model. Furthermore, we consider CRMs as part of our universe of loans and suggest a simple approach to determine whether the cap option comes at a fair price for a given mortgagor with a certain risk appetite.

2 Single mortgage strategies

Today the advisors in the Danish mortgage market recommend homebuyers to take single mortgage loans only. Until 2005 this included only ARMs or FRMs. Since 2005 CRMs have been among the favorites of the Danish advisors as well. When comparing the cashflows of these loans only the first year payments are quantified. This leaves out important information on the uncertain cashflows from year 1 on. In this section we suggest a scenario analysis approach which gives a quantitative comparison of different loans across a number of representative scenarios.

We generate an event tree with a seven-year horizon, using the one factor Vasicek model; see Appendix A for details of the interest rate model and Nielsen and Poulsen (2004) for its discrete implementation. Price calculations are performed using the RIO application which is a specialized commercial system for pricing Danish mortgage-backed securities, (see www.scanrate.dk).

2.1 FRMs and ARMs

Even though the interest rate payments on an FRM are fixed, the overall payment distribution is not. This is due to the fact that the price of an FRM changes as the general level of interest rates changes and the (Danish) mortgagor has a buy-back delivery option, meaning that the mortgagor can prepay the mortgage at any time at the market price. So, unless the mortgagor keeps the FRM until maturity, the overall payments remain uncertain. Figure 2 shows the density functions for the total payments of two 30-year FRMs, given that the mortgages are repaid after seven years.

It is noteworthy that at present no mortgage banks outside Denmark offer this buy-back delivery option to an FRM mortgagor. Should the FRM mortgagor repay the mortgage loan early, it occurs at par. Most mortgage banks across the world, however, offer a call option, so the mortgagor may repay the mortgage early at most at a predetermined price (usually par). The call option introduces an asymmetry in the payment density functions of FRMs

with prices close to par as seen in Figure 2. The density function of the 4% FRM has a longer left tail than the right tail due to the call option at par.

Comparing the payments of two FRMs with different rates and prices it is easily seen that the FRM 4% has a smaller volatility but a higher average payment that compensates for the upper bound on the payments due to the embedded call option. Most mortgagors are willing to make higher payments on average in order to avoid the very high payments that might occur if the initial price of the FRM is significantly below par and interest rates fall.

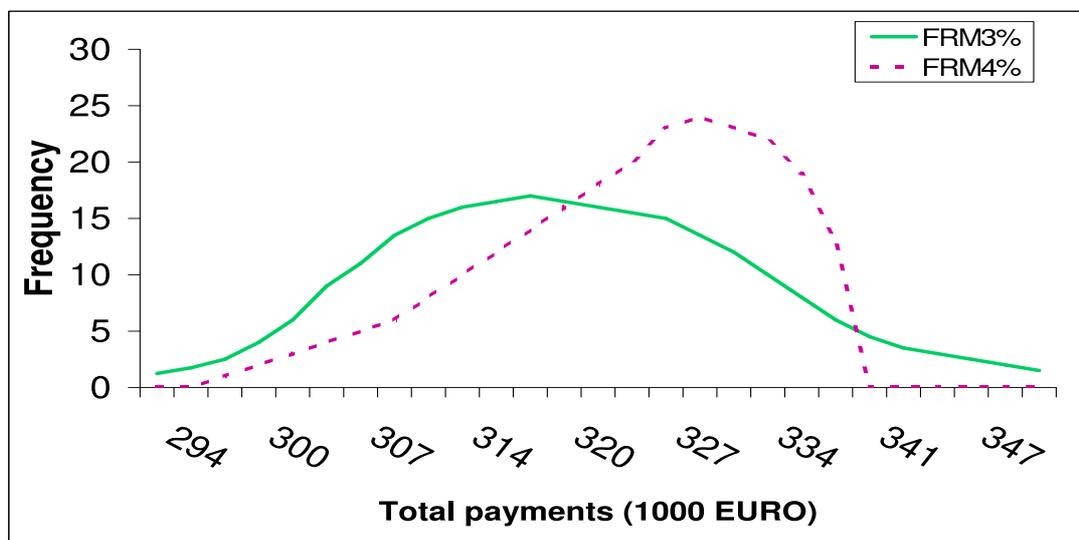


Figure 2: The density of total payments with early repayment at year 7, of two 30-year FRMs with different coupon rates.

Adjustable-rate mortgages (ARMs) have both a varying rate and a varying price, resulting in uncertain payments. Figure 3 (top) shows that ARM1 has not only a lower average payment than the FRM 4% but it also offers a lower risk (shorter right tail). Comparing the payments of ARM1 and FRM 4% for each scenario, however, you will notice a very strong negative correlation, see Figure 3 (bottom), and this has implication if the mortgage must be prepaid.

The high payment scenarios for an FRM occur when interest rates are decreasing, so that the mortgagor is both paying high interest rates and a high repayment. These same low interest rate scenarios are obviously low payment scenarios for an ARM. The scenarios in which the interest rates increase, however, are low payment scenarios for FRMs, due to low repayments, whereas they are high payment scenarios for ARMs due to upward adjustment of interest rates. There is, in other words, a negative correlation

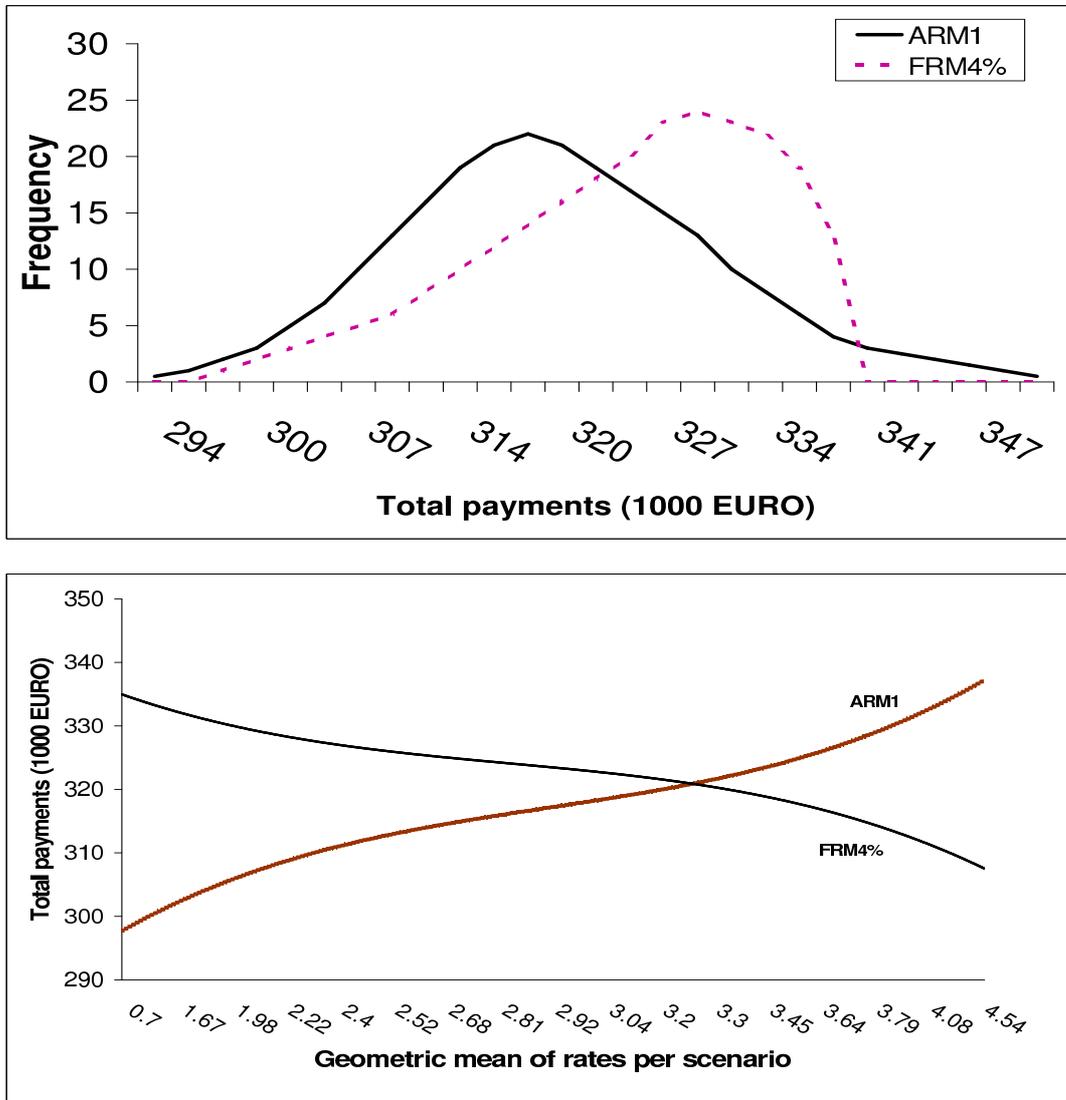


Figure 3: Comparison of payment densities (top) and scenario payments (bottom) for an ARM1 and an FRM4%. In the bottom figure the scenarios are ranked according to the geometric average of the short rates on the scenario paths.

between the payment of FRMs and ARMs making it potentially attractive to hold a diversified portfolio of mortgages. The interesting question now raised is how big the amount of the loan should be before the diversification can pay off, considering transaction and mortgage-origination costs, and what is a good mix for a mortgagor with a certain risk attitude or limited budget. We will answer these questions in the rest of this paper.

2.2 CRMs

In 2005 one of the leading Danish mortgage banks released a new mortgage product under the commercial name "garantilån", where the loan starts as a floating rate loan with an agreed cap level (the guarantee level). If interest rates increase so that the floating rate reaches the guarantee level, then the loan is transferred into a fixed-rate loan with the guarantee rate as the coupon for the rest of the loan's lifetime. The mortgagor pays a premium on top of the underlying floating level for the optionality embedded in this mortgage construction. Other Danish banks responded by introducing similar products and in particular a competing bank introduced a product where an additional feature was built into the loan so that if the interest rates fall below cap again then the mortgagor's coupon payments will decrease accordingly. This type of loan has the commercial name "RenteMax" and its popularity inspired other mortgage banks to provide similar constructions. The main feature of these products is that they offer the best of the two worlds to customers; they are a hybrid of an ARM and FRM. But the extra optionality comes at a cost and this cost might be too high for a mortgagor with a short horizon. In Figure 4 (top) we compare payment densities for an ARM1, an FRM4% and CRM5%.²

The payment density of the CRM5% is shifted to the right as compared to that of an ARM1 with the exception of the right tail. This indicates that the mortgagor pays an extra premium without getting anything in exchange for most scenarios but that for extreme scenarios the cap is exercised and very large payments can be avoided. In Figure 4 (bottom) it can further be seen that CRMs are in part negatively correlated with ARMs and in part with FRMs. Hence there is a potential that a combination of the three mortgage types may offer more diversification than a combination of two.

Our preliminary studies, however, indicate that the cap option of the CRM5% as released in February 2005 is not very beneficial for a mortgagor with a horizon of up to 7 years, i.e. if the mortgagor is willing to assume that interest rates will stay within the range estimated by the Vasicek model. Even for a mortgagor with a longer horizon (20 to 30 years), it is a good idea to consider the alternative of making a tailored replication of a CRM product using plain ARMs and FRMs, where the bond series are more liquid and therefore more fairly priced than the CRM.

²CRM5% is a Capped Rate Mortgage with a 5% cap. In this paper we only consider CRMs of type "RenteMax".

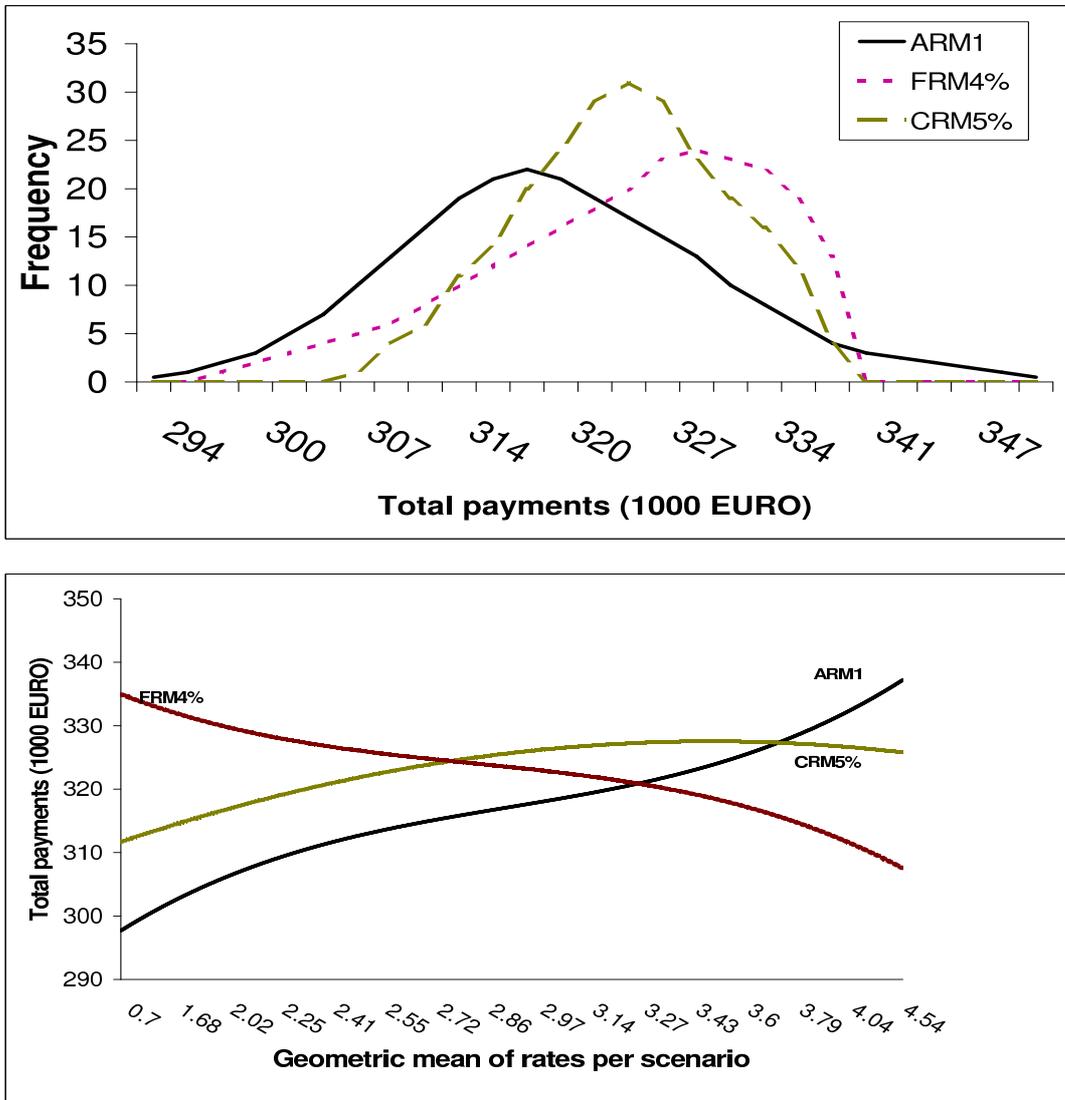


Figure 4: Comparison of payment densities (top) and scenario payments (bottom) for an ARM1, FRM4% and a CRM5%.

3 The mortgage choice model

In this section we will first introduce the event tree notation and then develop the mortgage choice model. See Zenios for event trees, and Nielsen and Poulsen for event tree scenarios for mortgage products.

3.1 Presentation of mortgage rate and price scenarios

To present the mortgage rates and price scenarios we use the notion of an event tree. An event tree is a directed graph $\mathcal{G} = (\Sigma, \mathcal{E})$ where nodes Σ denote time and state, and links \mathcal{E} indicate possible transitions between states as time evolves. At each time t we have one or more states, $s \in \Sigma_t$, representing the underlying stochastic variables. There is a unique path of states, $s \in \mathcal{E}_t^l$, from the root to any one of the leaf states, where $l \in \Sigma_\tau$ denotes a scenario.

An example of such an event tree is seen in Figure 5. Any node of the tree is populated with a number of loans, each with a set of specific data (LoanID:LoanName–Rate/Price) connected to them.

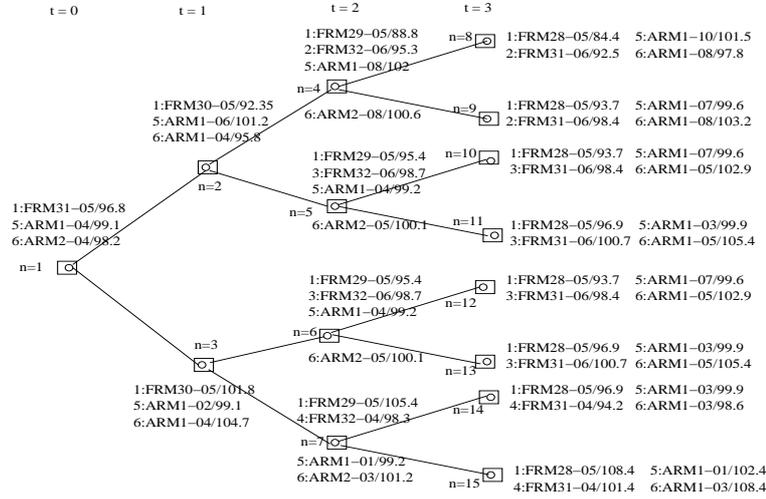


Figure 5: A binomial event tree, representing the uncertainty on bond prices and coupon rates.

Note that throughout this paper we operate with two time horizons, namely the mortgage maturity and the mortgage early repayment horizons. Set \mathcal{T} includes all the time periods up to the maturity of the loans $\mathcal{T} = \{0, \dots, \tau, \dots, T\}$.

T is, in other words, the length of the loans considered in the model and it is needed to determine the cashflows of the loans in question. Most loans are, however, repaid early at sometime, $\tau \leq T$ which is why we only need to estimate a scenario tree of length τ .

In the event tree, every state $s \in \Sigma_t$, for $1 \leq t \leq \tau$, has a unique parent denoted by $s^- \in \Sigma_{t-1}$, and every state $s \in \Sigma_t$ for $0 \leq t \leq \tau - 1$ has a non-empty set $s^+ \in \Sigma_{t+1}$ of child states. The probability distribution P is modeled by attaching weights $p_t^s > 0$ to each leaf node Σ_τ so that $\sum_{s \in \Sigma_\tau} p_t^s = 1$. For each non-terminal node one has, recursively,

$$p_t^s = \sum_{s^+ \in \Sigma_{t+1}} p_{t+1}^{s^+}, \quad \text{for all } s \in \Sigma_t, t = \tau - 1, \dots, 0,$$

and so each node receives a probability mass equal to the combined mass of the paths passing through it.

At every node of this tree we need estimates of interest rates and prices associated with any mortgage loan in the considered market. In order to obtain these estimates we need a stochastic process to represent the uncertainties on the dynamics of interest rates and we need pricing methods to determine mortgage loan prices consistently with the estimated term structures of interest rates.

We use a one-factor Vasicek model as the underlying stochastic process for the interest rates (see Appendix A). The model is discretized in a trinomial fashion as described in Nielsen and Poulsen (2004). Mortgage-backed securities are then priced using pricing system RIO; See www.scanrate.dk.

3.2 A dynamic stochastic mortgage choice model

The model in this section finds an efficient portfolio of loans that trades off the expected net present value (NPV) of total payments against the Conditional Value at Risk (CVaR) of the payments.

Given an event tree with τ stages and its corresponding mortgage loan rate and price information on a set of loans $i \in I$ we define the following parameters:

p_t^s , probability at state s , time t ,

d_t^s , discount factor at state s , time t ,

P_0 , current price of loan raised initially by the mortgagor,

r_{ti}^s , coupon rate for loan i at state s , time t ,

P_{ti}^s , price of loan i at state s , time t ,

K_{ti}^s , call price of loan i at state s , time t . We have $K_{ti}^s = \min\{1, P_{ti}^s\}$ for callable loans and $K_{ti}^s = P_{ti}^s$ for non-callable loans,

γ , tax reduction rate from interest rate and administration fee payments,

c_a , administration costs (in percent),

c , variable transaction costs (in percent),

c_f , fixed costs associated with mortgage origination and re-balancing,

λ , degree of risk aversion; 1 for very high, and 0 for no risk aversion,
 α , confidence level for the Value at Risk (VaR),
 M , a big constant.

Next we define the variables used in our model:

z_{ti}^s , outstanding debt of loan i at state s , time t ,

y_{ti}^s , units sold (originated) of loan i at state s , time t ,

x_{ti}^s , units bought back of loan i at state s , time t ,

$Z_{ti}^s = \begin{cases} 1, & \text{if loan } i \text{ is originated at state } s, \text{ time } t, \\ 0 & \text{otherwise,} \end{cases}$

A_{ti}^s , principal payment of loan i at state s , time t ,

CF_t^s , total net payment at state s , time t ,

ζ , Value-at-Risk (VaR) at the $100\alpha\%$ confidence level,

$\text{CVaR}(y; \alpha)$, Conditional Value-at-Risk of a portfolio with loans $y = (y_i)_{i \in U}$ at the $100\alpha\%$ confidence level,

y_+^l , amount of payment under scenario l exceeding the VaR level ζ .

We are now ready to formulate the multi-stage stochastic model for the mortgage choice problem. The objective is to trade off the total expected present value of payments (repayment included) against the Conditional Value-at-Risk (CVaR) of the payments as weighted by λ :

$$\min (1 - \lambda) \left[\sum_{t=1}^{\tau} \sum_{s \in \Sigma_t} p_t^s d_t^s CF_t^s + \sum_{s \in \Sigma_\tau} p_\tau^s d_\tau^s PP_\tau^s \right] + \lambda \text{CVaR}(y; \alpha) \quad (1)$$

We also need to make sure that we sell enough bonds to raise an initial amount, P_0 , to buy the house and pay the mortgage-origination costs as follows:

$$\sum_{i \in U} P_{0i}^0 y_{0i}^0 \geq P_0 + \sum_{i \in U} (c y_{0i}^0 + c_f Z_{0i}^0), \quad (2)$$

In eqn. (3) we initialize the outstanding debt:

$$z_{0i}^0 = y_{0i}^0, \quad \text{for all } i \in U. \quad (3)$$

Eqn. (4) is the balance equation, where the outstanding debt at any child node for any bond equals the outstanding debt at the parent node minus principal payment and possible repayment (buying back), plus possible origination of new bonds to establish a new loan.

$$z_{ti}^s = z_{t-1,i}^{s-} - A_{ti}^s - x_{ti}^s + y_{ti}^s, \quad \text{for all } i \in U, s \in \Sigma_t, t = 1, \dots, \tau. \quad (4)$$

Eqn. (5) is a cashflow equation which guarantees that the money used to repay existing mortgages (in case of re-adjustments), plus the transaction fees for sale and purchase of bonds, and fixed costs for establishing new mortgage loans come from the sale of new bonds:

$$\sum_{i \in U} (P_{ti}^s y_{ti}^s) = \sum_{i \in U} (K_{ti}^s x_{ti}^s + c(y_{ti}^s + x_{ti}^s) + c_f Z_{ti}^s), \quad \text{for all } s \in \Sigma_t, t = 1, \dots, \tau. \quad (5)$$

The principal payment is defined in eqn. (6) as an annuity payment.

$$A_{ti}^s = z_{t-1,i}^{s-} \left[\frac{r_{t-1,i}^{s-} (1 + r_{t-1,i}^{s-})^{-T+t-1}}{1 - (1 + r_{t-1,i}^{s-})^{-T+t-1}} \right], \quad \text{for all } i \in U, s \in \Sigma_t, t = 1, \dots, \tau. \quad (6)$$

The total net payment at each node, CF_t^s , is defined as the sum of principal payments, interest payments net of tax and administration fees in eqn. (7), whereas the total net prepayment amount for each leaf node is defined in eqn. (8).

$$CF_t^s = \sum_{i \in U} (A_{ti}^s + (1 - \gamma)(r_{t-1,i}^{s-} + c_a)z_{t-1,i}^s), \quad \text{for all } s \in \Sigma_t, t = 1, \dots, \tau, \quad (7)$$

$$PP_\tau^s = \sum_{i \in U} (z_{\tau i}^s K_{\tau i}^s), \quad \text{for all } s \in \Sigma_\tau. \quad (8)$$

The next constraint uses the binary variables Z_{ti}^s to ensure that the fixed cost indicator is set to 1 in case of re-financing along the way.³

³The constant M might be set to a value slightly greater than the initial amount raised; If a too large value is used, numerical problems may arise.

$$MZ_{ti}^s - y_{ti}^s \geq 0, \quad \text{for all } i \in U, s \in \Sigma_t, t = 0, \dots, \tau. \quad (9)$$

Constraints (10) and (11) together define the VaR and CVaR at the $100\alpha\%$ confidence level using the standard linear programming formulation (See Rockafellar and Uryasev and Zenios).

$$y_+^l \geq \left[\left(\sum_{t=0}^{\tau} \sum_{s \in \mathcal{E}_t^l} d_t^s CF_t^s \right) + d_\tau^l PP_\tau^l \right] - \zeta, \quad \text{for all } l \in \Sigma_\tau, \quad (10)$$

$$\text{CVaR}(y; \alpha) = \zeta + \frac{\sum_{l \in \Sigma_\tau} p_\tau^l y_+^l}{1 - \alpha}. \quad (11)$$

Finally non-negativity and binary constraints are introduced:

$$z_{ti}^s, y_{ti}^s, x_{ti}^s, y_+^l, \zeta \geq 0, Z_{ti}^s \in \{0, 1\} \quad \text{for all } i \in U, l \in \Sigma_\tau, s \in \Sigma_t, t = 0, \dots, \tau. \quad (12)$$

3.3 Generalization of the Rasmussen and Clausen models

Rasmussen and Clausen (2004) introduce a family of models which together cover risk preferences among mortgagors. Their risk-neutral model (minimizing average payments across scenarios) and the minmax model (minimizing the maximum payment) represent the two poles of risk preferences considered in their paper. Their budget-constrained models are then used to find mortgage strategies resulting in cashflows between the two poles. One of the main contributions of our paper is to consider an explicit risk measure (CVaR) and thereby generalize the models of Rasmussen and Clausen within a common model framework.

The two parameters (λ and α) can be used to generate any solution found in Rasmussen and Clausen (2004). By setting $\lambda = 0$ in the objective function eqn. (1), the model turns into a risk-neutral model. For $\lambda = 1$ and $\alpha = 1$ the model turns into a minmax model. The budget-constrained model takes as input a predefined maximum budget level that the mortgagor does not wish to violate either for any given single scenario or for several scenarios. The model has a hard and a soft constraint which ensure that the mortgagor's wishes are respected. The soft budget constraint may be violated but if this

occurs a penalty is incurred, whereas the hard budget constraint may by no means be violated. In our model the confidence level α implicitly decides the Value at Risk (VaR) level ζ which may be interpreted as a budget constraint. The level of λ corresponds to the penalty level.

The main advantages of using our Mean/CVaR model as compared to the budget-constrained model are the following:

1. Reasonable budget constraint levels are often hard to find. An inappropriate choice of budget levels may result in either infeasible problems or leave out interesting regions of the solution domain.
2. CVaR is a widely acceptable risk measure and its use is becoming increasingly popular, while it has the property of being coherent.
3. The Mean/CVaR models are easier to solve since they do not use hard constraints which usually add to the non-convexity of the problem.

4 Model testing and validity

Stochastic Programming models are discretizations of the uncertain world, hence in testing such models we should be concerned about convergence of solutions for different discretizations. Likewise we should test for robustness with respect to errors in the parameters representing uncertainty.

4.1 Convergence of solutions

Mortgagors pay⁴ for having the right to re-balance their mortgage portfolios, so it is crucial that the model facilitates this option. Ideally we would like to have as many stages in the model as in real life, for instance in a quarterly or yearly basis. However given our trinomial discretization of the interest-rate model and the path-dependent nature of the problem we need to limit the number of stages to less than 6, i.e. 729 scenarios or 1093 nodes. Bigger problems are not computationally tractable on a standard personal computer within reasonable time. A relevant question to investigate is the necessary number of stages and decision nodes in order to obtain best possible solutions given the underlying model of uncertainty.

⁴Options embedded in the mortgage backed securities have a price. The price is not paid upfront but it is either recalculated as an extra premium on top of interest rate payments or as a higher initial outstanding debt resulting on higher future payments

Figure 6 compares the mean/CVaR efficient frontiers found by one to five period models for our example with a seven-year horizon. As seen in Figure 6 the efficient frontiers tend to converge for four to five re-balancing stages with period lengths of one to two years. Likewise, as seen in Figure 7 the first stage solutions converge after only two to three stages. This behavior is expected since only some of the improvement in the efficient frontier is due to the structure of the first stage solutions and the rest of the improvement comes from extra re-balancing activities resulting from adding extra decision stages.

4.2 Stability

The solutions found by the stochastic program are dependent on the parameters of the stochastic process used to generate the scenario tree. So we need to study to what extent changes in the parameters depicting uncertainty have an influence on the solutions found. The stochasticity in the model comes from the underlying interest-rate model, so we are interested in observing changes in the solution structure as a result of changes in the interest-rate model parameters. The three parameters of the Vasicek model include: (i) The long-run mean value of the interest rate θ ; (ii) the volatility of the

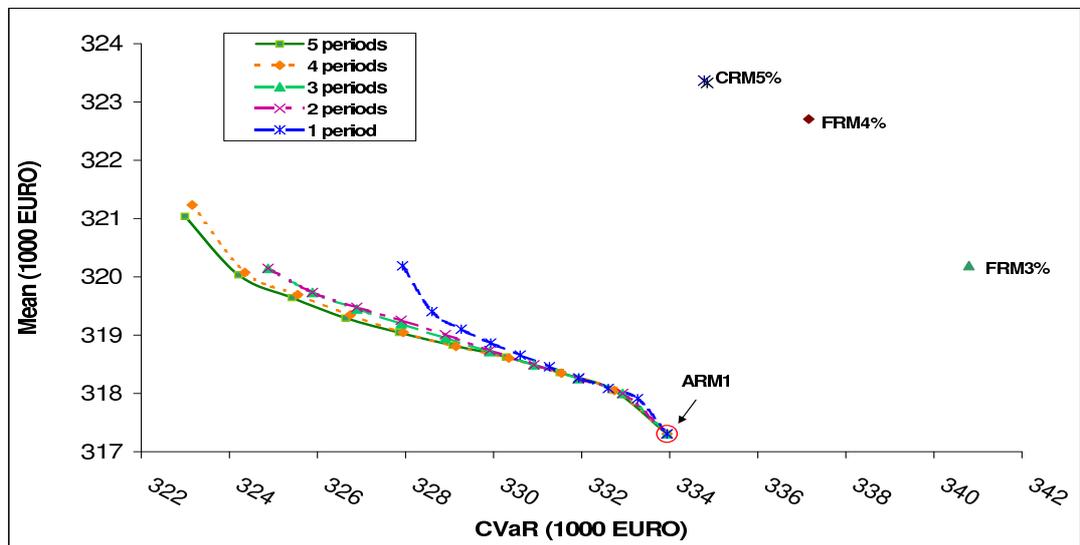


Figure 6: As more decision stages are added to the problem the solution quality is improved. The improvement is, however, marginal after adding three extra decision stages.

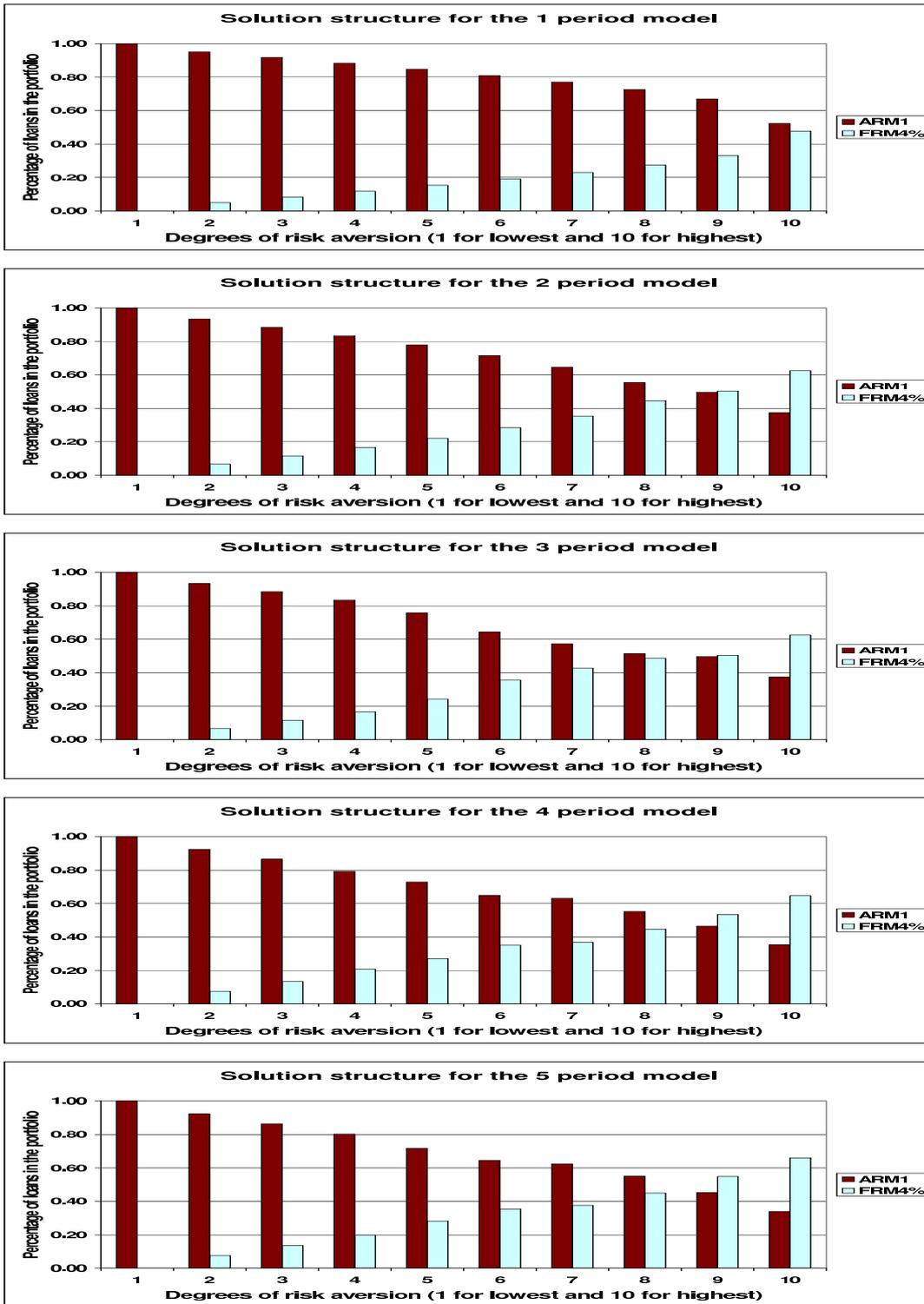


Figure 7: First stage solutions for different degrees of risk aversion and increasing number of decision stages.

interest rate σ ; (iii) the dispersion of interest rates at each step κ .

We perturb the two most significant parameters, namely the long-run mean θ and the volatility σ ; the calibrated parameters based on historical time series were 0.042 and 0.010 respectively. We generate 100 different scenario trees based on uniformly random θ values in the interval of $[0.032; 0.052]$ and σ values in the interval $[0.008; 0.012]$. The mean/CVaR model is then run for all 100 scenario trees and the results are analyzed.

By studying the first stage solutions we find that the risk-averse mortgage strategies, i.e. those found for high λ values in the objective function are robust with regards to parameter uncertainty. Figure 8 gives the intuition behind this finding. The model chooses to combine mortgage loans for high risk aversion no matter what the parameters are, whereas single mortgage loans are chosen for the mortgagor with low degree of risk aversion. As a result the diversified portfolio is more robust with respect to changes in parameters. This is an additional argument why we should choose a portfolio of mortgage loans rather than a single mortgage product, as is normal practice today.

5 Analysis of the model applications

We now use the model to analyze the underlying application of mortgage choice. In particular we investigate the use of the model for the advice offered to individual homeowners, for developing new products, and for estimating fair premia for CRMs. Throughout this section a five-period model with four rebalancing stages at years 1,2,3 and 5, and prepayment horizon at year seven is used.

5.1 Personal advice

Consider the bottom graph in Figure 7. Ten different first stage (initial) solutions are represented. Except in the case of the risk-neutral mortgagor (minimizing the average payments only) the optimal solution involves mixing an ARM1 with FRM4%. For the more risk-averse mortgagors a greater part of the mix comes from the fixed-rate mortgage. Each of these solutions corresponds to a Mean/CVaR point on the five-period frontier of Figure 6. The main lesson here is that diversification pays off for individual mortgagors regardless of existence of fixed mortgage origination costs. Considering Figures 6 and 7 together it also becomes clear that rebalancing pays off as well,

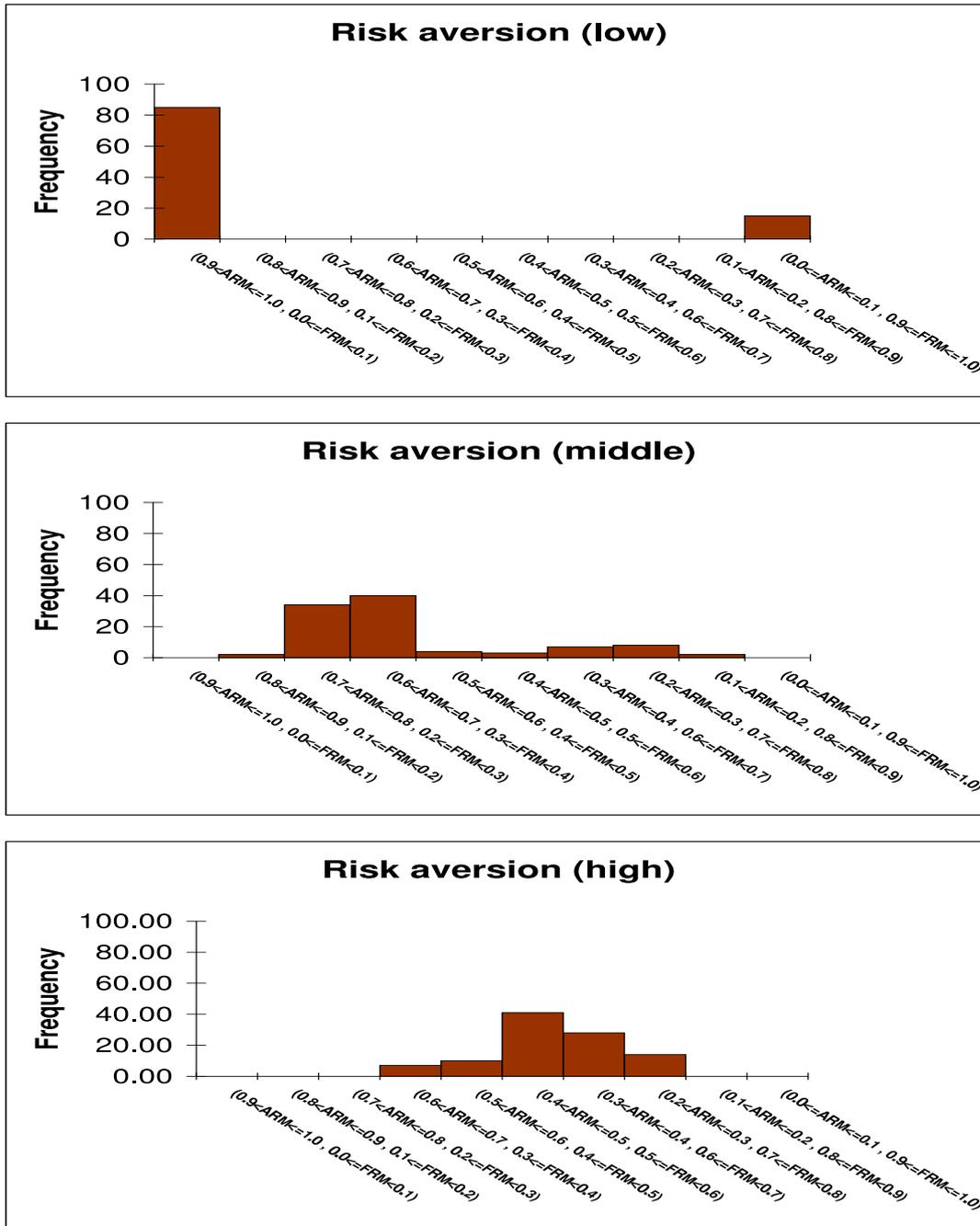


Figure 8: The first stage solutions are very sensitive to changes in uncertainty parameters for a model with little or no embedded risk aversion whereas they become more robust as the degree of risk aversion increases.

regardless of both fixed and variable transaction costs. It is noteworthy, however, that diversification and rebalancing are not usually relevant for small mortgage loans (below 100,000 EURO) or for very short horizons (under 3 years). These issues were further exemplified in Rasmussen and Zenios even with the use of a simpler single-period model.

5.2 Product development

So far we have used the model for finding optimal mortgage strategies based on FRMs with different coupon payments and ARM1. We can, however, easily add new products as input in order to quantify the value added by the new product. This is particularly useful before launching a new product. A synthetic equivalent of the new product may be tested within the model framework in order to find out whether the product adds value to certain segments of the market. The marketing of the new product may then be concentrated on these segments only.

Figure 9 illustrates this use of the model. Starting with our standard products (ARM1 and FRMs) we add new mortgage products one at a time and observe their effect on the Mean/CVaR efficient frontier. For any mortgagor with a 7-year repayment horizon, adding an ARM2 (adjustable-rate mortgage with bi-annual rate adjustments) does not add much value, whereas adding ARM3 and in particular ARM4 makes some contribution. Adding ARM5 adds more value for the very risk-averse mortgagors, but it adds less value for mortgagors with some appetite for risk.

A noteworthy observation was made when we continued the experiments adding the more exotic products CRM5% and CRM6%. These new products had no influence on the original efficient frontier indicating too high premia on their embedded cap options. At their current prices these new products do not add value to homebuyers. In the next section we use the model to estimate fair premia so that the new products become attractive for mortgagors.

5.3 Deciding fair premia for CRMs

CRMs are designed so that they follow the six-month CIBOR rate with a fixed premium on top of that for the embedded cap option. The premium for the CRM5% in November 2006 was for instance 0.8%. We have already observed that CRMs add no value to the existing mortgage products – at least for a mortgagor with a seven-year horizon. In Figure 10 we illustrate

how to use our model in order to find out fair premia for a given CRM for mortgagors with different risk appetites.

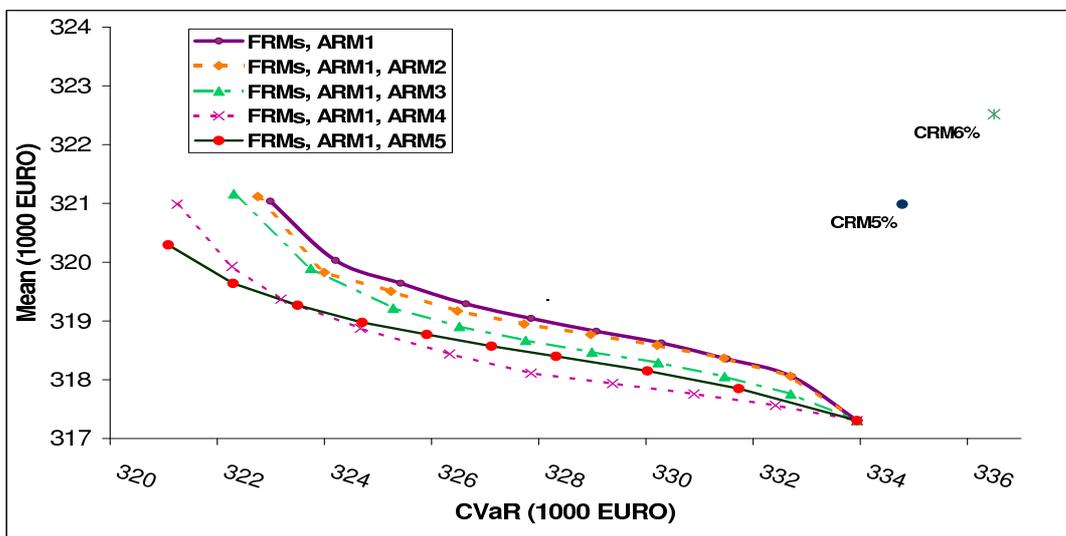


Figure 9: Adding one mortgage at a time to the existing universe of mortgages.

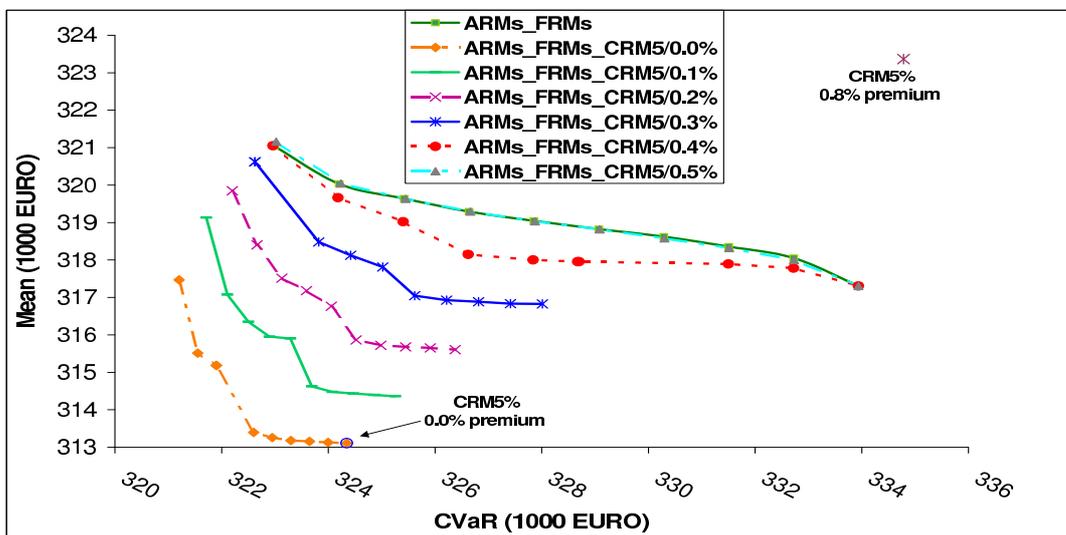


Figure 10: We introduce a CRM5% without any premium to begin with and then increase the premium in small increments until the CRM is no longer part of the efficient frontier.

We add a CRM5% to the existing universe of ARM1 and FRMs, but we remove the premium of 0.8% (so that we follow the 6-month CIBOR rate

with a 5% cap) in order to make sure the CRM5% is chosen in the efficient frontier. As expected such a loan would significantly improve the efficient frontier. Notice, however, that even without any premium on CRM5% the model still suggests diversification (initially with FRM4%) in order to reduce the CVaR. Only the point furthest right on the frontier corresponds to a strategy of holding only CRM5%.

Then we add some small premium in incremental steps of 0.1% and rerun the model, until the CRM5% is no longer chosen as part of the optimal solution. This happens at around 0.4% for the risk neutral mortgagor and surprisingly for the very risk averse mortgagor as well.⁵ For all other mortgagors the CRM5% is not attractive anymore at a premium of about 0.5%. Hence, a fair premium for the CRM5% would be 0.4–0.5%.

6 Conclusion

In this paper we showed that diversification and re-balancing of the mortgage loans pay off for a typical mortgagor. Building on the single-period model of Rasmussen and Zenios (2007) we observed that adding stages implies more diversification in the initial portfolio. This is in contrast to the existing practice where mortgagors hold one type of mortgage loan only. We exemplified how the results of the multi-stage Mean/CVaR model may be used for advanced analysis prior to mortgage choice recommendations as well as for product development.

Finally we used the model to show that CRMs, as offered today, are not attractive. Hence a reduction in premia for CRMs with shorter horizons and using CRMs as a component of the mortgage portfolio is recommended.

Considering income and house price dynamics would add some insights for even more individualized advice. Likewise extensive studies for different repayment horizons, different initial loan values and different mortgage loan combinations are needed to establish some rules of thumb which could be used on a daily basis for personal advice on the mortgage choice. These issues remain as future work.

⁵The intuition behind this is that the features of CRMs appeal to mortgagors with a medium risk appetite, so exactly these mortgagors would accept a higher premium for this product.

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A The interest rate model

We use a variant of the Vasicek interest rate model as used in Jensen and Poulsen (2002) the underlying stochastic process to generate estimates of future short rates:

$$dr(t) = \kappa(\theta^P - r(t))dt + \sigma dW^P(t),$$

where $r(t)$ is the short rate, W is a brownian motion and κ , θ and σ are the Vasicek model's parameters controlling the height of the interest rate jumps, the long run mean level of interest rates and the volatility of the interest rates. The model is given under real-world probability measure P . This can be shifted to the risk free measure Q using the transformation:

$$\theta^Q = \theta^P + \pi, \quad \pi \in \mathcal{R}$$

where π is the risk premium, so the Vasicek model under the risk free probability measure Q becomes:

$$dr(t) = \kappa(\theta^Q - r(t))dt + \sigma dW^Q(t)$$

The expectation of short rates are then found as follows:

$$E^Q(r(t)) = r_0 \cdot \exp(-\kappa t) + \theta^Q(1 - \exp(-\kappa t))$$