Stochastic Control
External Models

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Abstract

This note is devoted to control of stochastic systems described in discrete time. We are concerned with external descriptions or transfer function model, where we have a dynamic model for the input output relation only (i.e.. no direct internal information). The methods are based on LTI systems and quadratic costs.

We will start with the basic minimal variance problem and then move on to more complex and applicable strategies such as GMV, GPC and LQG control. These methods can be regarded as extension to the basic minimal variance strategy and have all a close relation to prediction. Consequently a section on that topic can be found in appendix.

Notice, this version has not got a proper list of references.

1 Introduction

It is assumed, that the system to be controlled is a linear, time invariant (*LTI system*) *SISO* system (single input single output system). SISO systems are also denoted as *scalar systems* and has one *control signal* (*input signal*), u_t, and one *output signal*, y_t.

$$y_t = q^{-k} \frac{B(q^{-1})}{A(q^{-1})} u_t + v_t \tag{1}$$

The signal, v_t, models the total effect of the disturbances.

In general the *time delay*, $k \geq 0$ due to the causality. If for example u_t is a measured input signal then $k = 0$ might be the case. In a control application, where the sampling of the output is carried out before the determination and the effectuating of the control action, the time delay is larger than zero (i.e. $k \geq 1$). If the underlying continuous time system do not contain any time delays then the discrete time have $k = 1$.

If the total effect of the disturbances is weak *stationary process* and has a *rational spectrum* the we can model the effect as:

$$v_t = \frac{C(q^{-1})}{D(q^{-1})} e_t$$

where $e_t \in \mathbb{F}\left(0, \sigma^2\right)$ and is a white noise sequence that is uncorrelated with past output signals ($y_{t-i},\ i = 1,\ 2,\ ...$).

In these notes we will assume the system is given by a *ARMAX structure* (autoregressive moving average model with external input) or the *CARMA* (ccontrolled autoregressive moving average model) which can be written as

$$A(q^{-1})y_t = q^{-k}B(q^{-1})u_t + C(q^{-1})e_t \tag{2}$$

or as

$$y_t = q^{-k}\frac{B(q^{-1})}{A(q^{-1})}u_t + \frac{C(q^{-1})}{A(q^{-1})}e_t$$

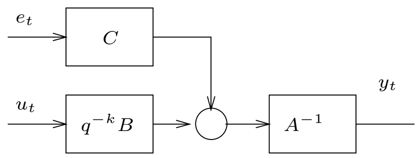

Figure 1. Stochastic system in the ARMAX form

The driving noise sequence, $e_t \in \mathbb{F}\left(0, \sigma^2\right)$, is a white noise sequence and is uncorrelated with past output signals ($y_{t-i},\ i = 1,\ 2,\ ...$). The 3 polynomials

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \ ... \ + a_n q^{-n} \\ B(q^{-1}) &= b_0 + b_1 q^{-1} + \ ... \ + b_n q^{-n} \\ C(q^{-1}) &= 1 + c_1 q^{-1} + \ ... \ + c_n q^{-n} \end{aligned}$$

is without loss of generality assumed to have the same *order* (n). The two polynomials, A and C, are assumed to be *monic* i.e. $A(0) = 1$ and $C(0) = 1$. Furthermore $\mathbf{C}(z) = z^n C(z^{-1})$ has no roots outside the unit circle. This latter assumption is justified by the spectral representation Theorem.

Remark: 1 The ARMAX (above) and the BJ structure:

$$y_t = q^{-k}\frac{B(q^{-1})}{F(q^{-1})}u_t + \frac{C(q^{-1})}{D(q^{-1})}e_t$$

can be regarded as extreme version of the more general L-structure:

$$A(q^{-1})y_t = q^{-k}\frac{B(q^{-1})}{F(q^{-1})}u_t + \frac{C(q^{-1})}{D(q^{-1})}e_t$$

If we are willing to accept common factors, we can always transform a description from one structure to another. □

2 Minimal Variance Control

In Appendix A we have investigated methods for optimal prediction.This facilitate the ability to evaluate the effect of a given control sequence. In this section we will solve the inverse problem which consists in finding that control sequence that in an optimal way brings the system to a desired state.

We will start with the basic minimal variance controller, which aim to (in stationarity) to minimize the following cost function:

$$\bar{J} = E\{y_{t+k}^2\} \tag{3}$$

We assume the system and the disturbance is given by the ARMAX in (2).

Furthermore, we assume that the polynomials **B** and **C** have all their roots inside the unit disk. In that situation we have the following theorem.

Theorem: 2.1: Assume the system is given by (2). The solution to the basic minimal variance control problem is by the controller:

$$B(q^{-1})G(q^{-1})u_t = -S(q^{-1})y_t \qquad (4)$$

where G and S are polynomials with order:

$$ord(G) = k-1 \qquad ord(S) = n-1$$

and is the solution to the Diophantine equation:

$$C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \qquad (5)$$

In stationarity the closed loop is characterized by:

$$y_t = G(q^{-1})e_t \qquad u_t = -\frac{S}{B}e_t$$

Notice the control error (y_t) is a MA(k)-process. □

Proof: Consider the situation in an instant of time t. Since the time delay through the system is k, the control action, u_t, can only effect the situation at the instant $t+k$ and further on. According to Theorem A.2 we have the following:

$$y_{t+k} = \frac{1}{C}\left[BGu_t + Sy_t\right] + Ge_{t+k}$$

and consequently:

$$\bar{J}_{t+k} = E\{y_{t+k}^2\} = E\left\{\left[\frac{1}{C}(BGu_t + Sy_t)\right]^2\right\} + E\left\{[Ge_{t+k}]^2\right\}$$

since $Ge_{t+k} = e_{t+k} + \cdots + g_{k-1}e_{t+1}$ is independent of $\bar{Y}_t$. Especially, is the last term independent on u_t. The optimum of $\bar{J}$ occur if the first term is canceled (equal to zero). This is valid for the given controller if the polynomial **C** has all its zeroes inside the unit disk. If the first term is zero, then the output is in closed loop (and under stationary conditions)

$$y_t = Ge_t$$

The closed loop expression for the control comes directly from this and the control law (4). □

Remark: 2 Notice, this control is equivalent to ensure (by a proper choice of u_t) that the (k-step ahead) prediction of y_t to zero. □

Remark: 3 Notice, the poles of the closed loop is roots for:

$$1 + \frac{S}{BG}z^{-k}\frac{B}{A}$$

or to:

$$\mathcal{C} = ABG + z^{-k}BS = B(AG + z^kS) = BC$$

That means that the basic minimal variance controller is only able to stabilize system with a stable inverse (discrete time minimal phase systems), i.e. system with zeroes (to the **B** polynomial) well inside the unit disk. Furthermore the **C** polynomial must have the same properties. □

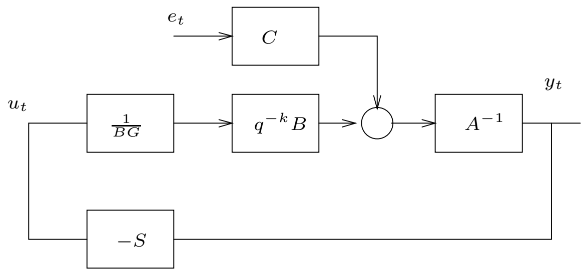

Figure 2. Basic minimal variance control and a ARMAX system.

Example: 2.1 *Assume, that the result of an analysis of a dynamic system and its disturbances are resulted in a model as in (2) with:*

$$A = 1 - 1.7q^{-1} + 0.7q^{-2}$$
$$B = 1 + 0.5q^{-1} \qquad k = 1$$
$$C = 1 + 1.5q^{-1} + 0.9q^{-2} \qquad e_t \in \mathbb{F}(0, \sigma^2)$$

Firstly, we will investigate the situation for $k = 1$. In the design we have the Diophantine equation (5) which in this case is:

$$(1 + 1.5q{-}1 + 0.9q^{-2}) = (1 - 1.7q^{-1} + 0.7q^{-2})1 + q^{-1}(s_0 + s_1q^{-1})$$

The solution can be found in different ways. The most strait forward is to identify the coefficient to q^{-i}, which results in:

$$0 \quad : \quad 1 = 1 \tag{6}$$
$$1 \quad : \quad 1.5 = -1.7 + s_0 \tag{7}$$
$$2 \quad : \quad 0.9 = 0.7 + s_1 \tag{8}$$

or in $s_0 = 3.2$ and $s_1 = 0.2$. The minimal variance controller is therefore given by:

$$u_t = -\frac{S}{BG}y_t = -\frac{3.2 + 0.2q^{-1}}{1 + 0.5q^{-1}}y_t$$

or by:

$$u_t = -0.5u_{t-1} - 3.2y_t - 0.2y_{t-1}$$

With this strategy the error will in stationarity be $y_t = e_t$.

We will now focus on how much the performance of the controller will be deteriorated if the time delay is increased e.g.. to $k = 2$. In this situation the Diophantine equation becomes:

$$(1 + 1.5q{-}1 + 0.9q^{-2})y_t = (1 - 1.7q^{-1} + 0.7q^{-2})(1 + g_1q^{-1}) + q^{-1}(s_0 + s_1q^{-1})$$

and the solution to the equation system:

	VS.	*HS.*
0	*1*	1
1	*1.5*	$-1.7 + g_1$
2	*0.9*	$0.7 - 1.7g_1 + s_0$
3	*0*	$0.7g_1 + s_1$

is:

$$g_1 = 3.2 \qquad s_0 = 5.64 \qquad s_1 = -2.24$$

The minimal variance controller is in this situation:

$$u_t = -\frac{S}{BG} = -\frac{5.64 - 2.224q^{-1}}{1 + 3.7q^{-1} + 1.6q^{-2}}y_t$$

or:

$$u_t = -5.64y_t + 2.24y_{t-1} - 3.7u_{t-1} - 1.6u_{t-2}$$

The stationary error is:

$$\tilde{y}_t = e_t + 3.2e_{t-1}$$

which has a variance equal:

$$Var\{\tilde{y}_t\} = (1 + 3.2^2)\sigma^2 = 11.24\sigma^2$$

In this example the variance of the error will increase if the time delay is increased. □

Example: 2.2 *Consider a system given in the ARMAX form:*

$$A = 1 - 1.5q^{-1} + 0.95q^{-2} \qquad B = 1 + 0.5q^{-1} \qquad k = 1$$

$$C = 1 - 0.95q^{-1} \qquad \sigma^2 = (0.1)^2$$

For this system the basic minimal variance controller is given by:

$$R = BG = 1 + 0.5q^{-1} \qquad S = 0.55 - 0.95q^{-1}$$

or as:

$$u_t = -0.5u_{t-1} - 0.55y_t + 0.95y_{t-1}$$

The output signal and the control are shown in the stationary situation in Figure 3. The transient phase (after cut in) can be studied in Figure 4. Notice, the reduction in variance just after cut in. □

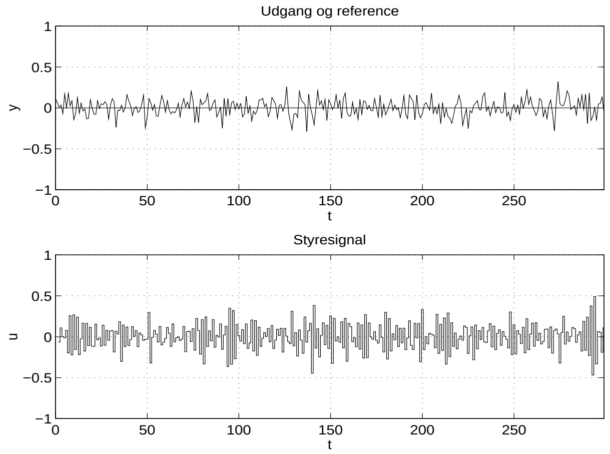

Figure 3. The output signal and the control in Example 2.1

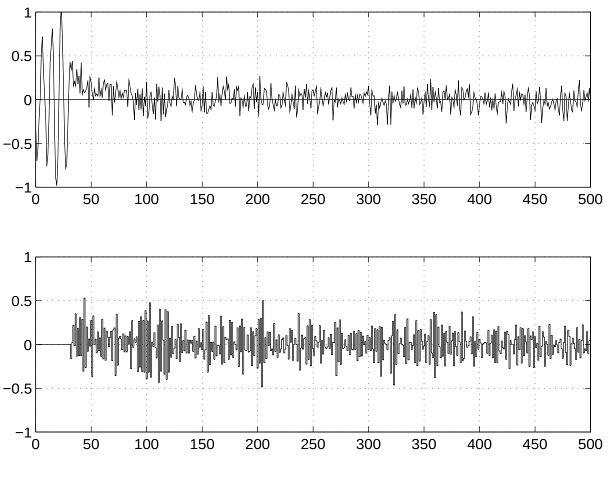

Figure 4. The output signal and the control in Example 2.1

Example: 2.3 *In this example we will study the effect of the time delay, k. Assume, that the system is the same as in example 2.1. For $k = 1$ the controller is as discussed in example 2.1. For $k = 2$ the control polynomials are:*

$$R = 1 + 1.05q^{-1} + 0.275q^{-2} \qquad S = -0.125 - 0.53q^{-1}$$

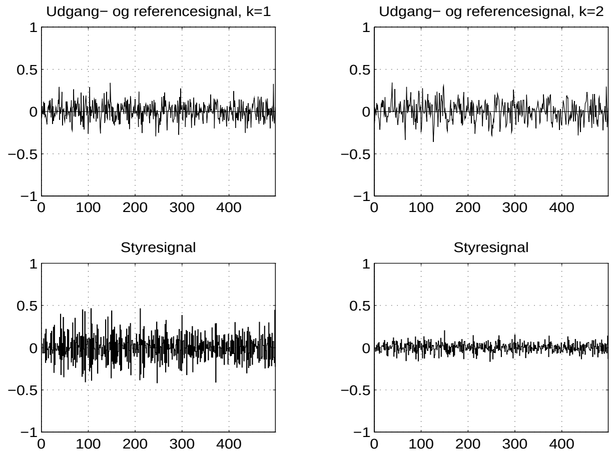

Figure 5. The output signal and the control from Example 2.2

The output signal and the control are under stationarity for $k = 1, 2$ *depicted in Figure 5. Notice, the small increment in the variance of the output signal due to the increased time delay. Also notice, the reduction in control afford.*

For $k = 1$ *and* $k = 2$ *the G polynomial is:*

$$G_1 = 1 \qquad G_2 = 1 + 0.55q^{-1}$$

That means a variance increment, which is equal $1.3025 = 1 + (0.55)^2$ *for an increased k from* 1 *to* 2. *In the table below, the empirical variance, the theoretical variance and the variance of the control is listed for 10 experiments. All numbers are in %.*

empirisk ratio	*teo. ratio*	*ratio i control variance*
149.1580	*130.2500*	*16.9856*
144.9701	*130.2500*	*15.5137*
148.4734	*130.2500*	*14.9431*
130.1090	*130.2500*	*13.7075*
142.1038	*130.2500*	*12.7749*
134.7121	*130.2500*	*12.9202*
133.3890	*130.2500*	*15.1116*
123.7364	*130.2500*	*11.5495*
140.2522	*130.2500*	*14.0588*
114.9559	*130.2500*	*12.6139*
129.8356	*130.2500*	*12.5119*
123.1263	*130.2500*	*11.3916*

□

Example: 2.4 *Let us now focus on a system as in example 2.1, just with:*

$$B = 1 + 0.95q^{-1}$$

where the system zero (in 0.95*) is close to the stability limit. For this system the minimal variance controller is:*

$$R = BG = 1 + 0.95q^{-1} \qquad S = 0.55 - 0.95q^{-1}$$

i.e.

$$u_t = -0.95u_{t-1} - 0.55y_t + 0.95y_{t-1}$$

The output and control signals are in stationarity conditions as depicted in 6. Notice, the oscillations in the control signal. □

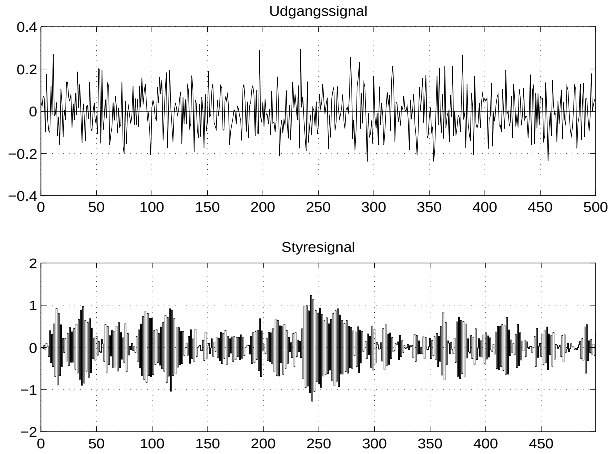

Figure 6. Output and control signal in Example 2.3

3 $\mathbf{MV}_0$ control

In the previous section we have dealt with the regulation problem without a reference signal (or the reference is zero). In this section we will extent the results in order to cope with a (non zero) reference signal or a set point. Consequently, let us focus on a control in which the cost function

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2\Big\} \tag{9}$$

is minimized.

Theorem: 3.1: Assume the system is given by (2). The MV_0, which minimize (9) is given by the control law

$$B(q^{-1})G(q^{-1})u_t = C(q^{-1})w_t - S(q^{-1})y_t \tag{10}$$

where G and S are solutions to the Diophantine equation

$$C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \tag{11}$$

with orders

$$ord(G) = k - 1 \qquad ord(S) = n - 1$$

In stationarity the control error is

$$\tilde{y}_t = y_t - w_{t-k} = G(q^{-1})e_t$$

which is a MA(k) process. □

Proof: From (32) we have

$$y_{t+k} = \frac{1}{C}\left\{BGu_t + Sy_t\right\} + Ge_{t+k}$$

and furthermore that

$$y_{t+k} - w_t = \frac{1}{C}\left[BGu_t + Sy_t - Cw_t\right] + Ge_{t+k}$$

Now

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2\Big\} = \left(\frac{1}{C}\left[BGu_t + Sy_t - Cw_t\right]\right)^2 + Var\{Ge_{t+k}\}$$

which takes its minimum for the control law given in the Theorem. □

Theorem: 3.2: Let the assumptions in Theorem 3.1 (page 7) be valid and let the situation be stationary. Then for the system in (2) the MV_0 controller will give

$$y_t = q^{-k} w_t + G e_t$$

and

$$u_t = \frac{A}{B} w_t - \frac{S}{B} e_t$$

in closed loop. □

Proof: The closed loop expression for the output comes directly from Theorem 3.1 (page 7). If this is introduced in the control law, then

$$\begin{aligned} BGu_t &= Cw_t - Sy_t \\ &= Cw_t - S(q^{-k} w_t + Ge_t) \\ &= AGw_t - SGe_t \end{aligned}$$

or as stated in the theorem. Notice, we have used the Diophantine equation (11) in the mid equation. □

4 $\mathbf{MV_1}$ control

In the previous section we saw, that the basic minimum variance controllers (MV and MV_0) indeed required too much control action. Let us then focus on a control in which the cost function has a term related to the control action, i.e. a control in which

$$J = \mathbf{E}\Big\{ (y_{t+k} - w_t)^2 + \rho u_t^2 \Big\} \tag{12}$$

is minimized.

Theorem: 4.1: Assume the system is given by (2). The MV_1, which minimize (12) is given by the control law

$$\boxed{(BG + \alpha C)\, u_t = Cw_t - Sy_t \qquad \alpha = \frac{\rho}{b_0}} \tag{13}$$

where G and S are solutions to the Diophantine equation

$$C = AG + q^{-k} S \tag{14}$$

with orders

$$ord(G) = k - 1 \qquad ord(S) = n - 1$$

□

Proof: As in Theorem 3.1 (page 7) we have from (32) that

$$y_{t+k} = \frac{1}{C} \left\{ BGu_t + Sy_t \right\} + Ge_{t+k}$$

and furthermore that

$$y_{t+k} - w_t = \frac{1}{C} \left[BGu_t + Sy_t - Cw_t \right] + Ge_{t+k}$$

Now

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2 + \rho u_t^2\Big\} = \left(\frac{1}{C}\left[BGu_t + Sy_t - Cw_t\right]\right)^2 + \rho u_t^2 + Var\{Ge_{t+k}\}$$

which takes its minimum for

$$2\frac{b_0}{C}\left[BGu_t + Sy_t - Cw_t\right] + 2\rho u_t = 0$$

or as given in the theorem. □

Theorem: 4.2: Let the assumptions in Theorem 4.1 (page 8) be valid and let the situation be stationary. Then for the system in (2) the MV_1 controller will give

$$y_t = q^{-k}\frac{B}{B + \alpha A}w_t + \frac{BG + \alpha C}{B + \alpha A}e_t$$

and

$$u_t = \frac{A}{B + \alpha A}w_t - \frac{S}{B + \alpha A}e_t$$

in closed loop. □

Proof: Firstly, focus on the output y_t. If the control law, (13), is introduced in the system description (2) then

$$y = q^{-k}\frac{B}{A}\left[\frac{C}{BG + \alpha C}w_t - \frac{S}{BG + \alpha C}y_t\right] + \frac{C}{A}e_t$$

or (when multiplying with $A[BG + \alpha C]$

$$A[BG + \alpha C]y_t = q^{-k}BCw_t - q^{-k}BSy_t + C(BG + \alpha C)e_t$$

or (after collecting terms involving y_t)

$$(ABG + q^{-k}BS + \alpha AC)y_t = q^{-k}BCw_t + C(BG + \alpha C)e_t$$

If we apply the Diophantine equation (14) we have that

$$(BC + \alpha AC)y_t = q^{-k}BCw_t + C(BG + \alpha C)e_t$$

or (after canceling C, which has all roots inside the stability area) the closed loop is as stated in the Theorem.

For the control actions we have

$$(BG + \alpha C)\,u_t = Cw_t - S\left[q^{-k}\frac{B}{A}u_t + \frac{C}{A}e_t\right]$$

or (after multiplying with A)

$$\Big[ABG + q^{-k}SB + \alpha CA\Big]\,u_t = ACw_t - SCe_t$$

or (after applying the Diophantine equation (14))

$$[BC + \alpha CA]\,u_t = ACw_t - SCe_t$$

or (after canceling C, which has all roots inside the stability area) the closed loop is as stated in the Theorem. □

5 Frequency weighted MV_1 control

The minimum variance controllers (MV and MV_0) can in some applications require a high control activity. In order to reduce the variance of the control action the MV_1 controller can be applied. Unless the system contain an integration (and then $A(1) = 0$) the MV_1 controller will for a non zero set point give a stationary error.

The standard work around is let the cost include the control move (v_t) rather than the control action (u_t) itself into the cost function.

Consequently, let us now focus on a control in which the cost function

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2 + \rho v_t^2\Big\} \qquad v_t = u_t - u_{t-1} \tag{15}$$

is minimized. Let us introduce the Δ operator as

$$\Delta = 1 - q^{-1}$$

then the cost in (15) can be written as

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2 + \rho(\Delta u_t)^2\Big\}$$

Theorem: 5.1: Assume the system is given by (2). The MV$_1a$, which minimize (15) is given by the control law

$$\boxed{(BG + \alpha C\Delta)\, u_t = Cw_t - Sy_t \qquad \alpha = \frac{\rho}{b_0}} \tag{16}$$

where G and S are solutions to the Diophantine equation

$$C = AG + q^{-k}S \tag{17}$$

with orders

$$ord(G) = k - 1 \qquad ord(S) = n - 1$$

□

Proof: As in Theorem 3.1 (page 7) we have from (32) that

$$y_{t+k} = \frac{1}{C}\{BGu_t + Sy_t\} + Ge_{t+k}$$

and furthermore that

$$y_{t+k} - w_t = \frac{1}{C}[BGu_t + Sy_t - Cw_t] + Ge_{t+k}$$

Now

$$J = \mathbf{E}\Big\{(y_{t+k} - w_t)^2 + \rho v_t^2\Big\} = \left(\frac{1}{C}[BGu_t + Sy_t - Cw_t]\right)^2 + \rho(\Delta u)_t^2 + Var\{Ge_{t+k}\}$$

which takes its minimum as given in the theorem. □

Theorem: 5.2: Let the assumptions in Theorem 5.1 (page 10) be valid and let the situation be stationary. Then for the system in (2) the MV$_1$ controller will give

$$y_t = q^{-k}\frac{B}{B + \alpha\Delta A}w_t + \frac{BG + \alpha C}{B + \alpha\Delta A}e_t$$

and

$$u_t = \frac{A}{B + \alpha\Delta A}w_t - \frac{S}{B + \alpha\Delta A}e_t$$

in closed loop. □

Proof: If the control law, (16), is introduced in the system description (2) then

$$y = q^{-k}\frac{B}{A}\left(\frac{C}{BG + \alpha\Delta C}w_t - \frac{S}{BG + \alpha\Delta C}y_t\right) + \frac{C}{A}e_t$$

or (when multiplying with $A((BG + \alpha\Delta C)$

$$A(BG + \alpha\Delta C)y_t = q^{-k}BCw_t - q^{-k}BSy_t + C(BG + \alpha C)e_t$$

After collecting terms and applying the Diophantine equation (17) we have that

$$(BC + \alpha\Delta AC)y_t = q^{-k}BCw_t + C(BG + \alpha C)e_t$$

or (after canceling C, which has all roots inside the stability area) the closed loop is as stated in the Theorem. □

6 PZ control

In the previous sections we saw, that the basic minimum variance controllers (MV and MV_0) indeed required too much control action. One way to reduce the control effort is to introduce a term in the cost function which take the control effort into considerations. Another method is to reduce the requirements to the control error. Rather than require the output should follow the reference in a close way

$$y_t = q^{-k} w_t$$

(as in the MV_0 case) we could require the output is following the reference in the following way

$$y_t = q^{-k} \frac{B_m}{A_m} w_t$$

Here the reference model, (B_m, A_m), is faster than the open loop system (the plant) but sufficient slow to reduce the control action.

Let us then focus on a control in which the cost function has a term related to the control action, i.e. a control in which

$$J = \mathbf{E}\Big\{ (A_m y_{t+k} - B_m w_t)^2 \Big\} \tag{18}$$

is minimized.

Theorem: 6.1: Assume the system is given by (2). The PZ-controller, which minimize (18) is given by the control law

$$\boxed{BGu_t = CB_m w_t - Sy_t} \tag{19}$$

where G and S are solutions to the Diophantine equation

$$A_m C = AG + q^{-k} S \tag{20}$$

with orders

$$ord(G) = k - 1 \qquad ord(S) = max(n_a - 1, n_c + n_m - k)$$

□

Proof: As in Theorem 3.1 (page 7) we have from (32) that

$$y_{t+k} = \frac{1}{C} \{BGu_t + Sy_t\} + Ge_{t+k}$$

and furthermore that

$$A_m y_{t+k} - B_m w_t = \frac{1}{C} [BGu_t + Sy_t - CB_m w_t] + Ge_{t+k}$$

Now

$$J = \mathbf{E}\Big\{ (A_m y_{t+k} - B_m w_t)^2 \Big\} = \left(\frac{1}{C} [BGu_t + Sy_t - CB_m w_t] \right)^2 + Var\{Ge_{t+k}\}$$

which takes its minimum for as given in the theorem. □

Theorem: 6.2: Let the assumptions in Theorem 6.1 (page 11) be valid and let the situation be stationary. Then for the system in (2) the PZ-controller will give

$$y_t = q^{-k}\frac{B_m}{A_m}w_t + \frac{G}{A_m}e_t$$

and

$$u_t = \frac{AB_m}{BA_m}w_t - \frac{S}{BA_m}e_t$$

in closed loop. □

Proof: Firstly, focus on the output y_t. From the proof of Theorem 6.1 (page 11) we have

$$A_m y_t - B_m w_{t-k} = Ge_t$$

or as stated in Theorem 6.2 (page 12). For the control actions we have

$$\begin{aligned} BGu_t &= B_m C w_t - S y_t \\ &= B_m C w_t - S\left(q^{-k}\frac{B_m}{A_m}w_t + \frac{G}{A_m}e_t\right) \\ &= \frac{B_m}{A_m}AGw_t - \frac{G}{A_m}e_t \end{aligned}$$

where we in the last line have used the Diophantine equation (20). From this we get the result stated in the theorem. □

7 LQG Controller

We have now seen variuous methods for detuning the minimum varince controller. In Section 4 the method consist of introducing a weight on the control action and in Section **??** we abanded the one step strategies and introduced a (finite) horizon. In this section we will extent the horizon and find a controller that in stationarity minimizing the variance of the output and (a weighted variant of) the variance of the control action. Let us first focus on a controller in which the cost

$$\bar{J}_t = \lim_{N\to\infty} E\left\{\frac{1}{N}\sum_{i=t}^{N} y_i^2 + \rho u_i^2\right\} \tag{21}$$

is minimized.

Theorem: 7.1: Assume the system is given by (2). The LQG controller, which minimize (21), is given by the control law

$$\boxed{R(q^{-1})u_t = -S(q^{-1})y_t} \tag{22}$$

where G and S are solutions to the Diophantine equation

$$P(q^{-1})C(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-k}B(q^{-1})S(q^{-1}) \tag{23}$$

with orders

$$ord(G) = n + k - 1 \qquad ord(S) = n - 1$$

The P-polynomial is the stable solution to

$$P(q^{-1})P(q) = B(q^{-1})B(q) + \rho A(q^{-1})A(q) \tag{24}$$

□

Proof: Omitted □

Theorem: 7.2: Let the assumptions in Theorem 7.1 (page 12) be valid and let the situation be stationary. Then for the system in (2) the LQG controller will give

$$y_t = \frac{R(q^{-1})}{P(q^{-1})} e_t$$

and

$$u_t = -\frac{S(q^{-1})}{P(q^{-1})} e_t$$

in closed loop. □

Proof: If the controller in (22) is introduced in the system description (2), then

$$Ay = -q^{-k} B \frac{S}{R} y_t + Ce_t$$

or

$$(AR + q^{-k} BS) y_t = RCe_t$$

If furthermore the Diophantine eqution is applied we have the closed loop (for y_t)as given in the Theorem. The closed loop description of u_t comes directly by introducing the closed loop expression for y_t into the controller. □

The controller just stated will solve the regulation problem without a setpoint. Now consider the cost function

$$\bar{J}_t = \lim_{N \to \infty} E \left\{ \frac{1}{N} \sum_{i=t}^{N} (y_i - w_t)^2 + \rho (u_i i - \bar{u})^2 \right\} \tag{25}$$

For a constant set point, w_t we have the following theorems.

Theorem: 7.3: Assume the system is given by (2). The LQG controller, which minimize (24), is given by the control law

$$\boxed{R(q^{-1}) u_t = \eta C(q^{-1}) w_t - S(q^{-1}) y_t} \tag{26}$$

where G and S are solutions to the Diophantine equation

$$P(q^{-1})C(q^{-1}) = A(q^{-1})R(q^{-1}) + q^{-k}B(q^{-1})S(q^{-1}) \quad \text{and} \quad \eta = \frac{P(1)}{B(1)} \tag{27}$$

with orders

$$ord(G) = n + k - 1 \qquad ord(S) = n - 1$$

The P-polynomial is the stable solution to

$$P(q^{-1})P(q) = B(q^{-1})B(q) + \rho A(q^{-1})A(q) \tag{28}$$

□

Proof: Omitted □

Theorem: 7.4: Let the assumptions in Theorem 7.3 (page 13) be valid and let the situation be stationary. Then for the system in (2) the LQG controller will give

$$y_t = \eta \frac{B(q^{-1})}{P(q^{-1})} + \frac{R(q^{-1})}{P(q^{-1})} e_t$$

and

$$u_t = \eta \frac{A(q^{-1})}{P(q^{-1})} w_t - \frac{S(q^{-1})}{P(q^{-1})} e_t$$

in closed loop. □

Proof: If the controller in (26) is introduced in the system description (2), then

$$Ay = q^{-k} B \left[\eta \frac{C}{R} w_t - \frac{S}{R} y_t \right] + Ce_t$$

or

$$(AR + q^{-k} BS) y_t = q^{-k} B \eta C w_t + RCe_t$$

If furthermore the Diophantine eqution is applied we have the closed loop (for y_t)as given in the Theorem. The closed loop description of u_t comes directly by introducing the closed loop expression for y_t into the controller. □

A Prediction

In this appendix it is assumed that the system is a scalar time-invariant stochastic system. The system will be given in the ARMAX, since the results can be transformed if the system description is in the BJ form or the L structure.

We will, however, first focus on a simpler problem, namely when there is no control input (or rather when there is only a stochastic input).

A.1 Prediction in the ARMA structure

Before we give the predictor for the ARMAX structure, we will handle the problem of prediction when the system is in the ARMA form.

Theorem: A.1: Let y_t be a (weakly) stationary process given by the ARMA model:

$$A(q^{-1})y_t = C(q^{-1})e_t \tag{29}$$

where $\{e_t\}$ is a white noise sequence of $\mathbb{F}(0,\sigma^2)$ distributed stochastic variable (which is independent of y_{t-i} , $i = 1,\ 2,\ ...$). Furthermore, assume that A and C have all their roots inside the stability area and are monic ($C(0) = A(0) = 1$). The minimal variance prediction is the given by:

$$\boxed{\hat{y}_{t+m|t} = \frac{S(q^{-1})}{C(q^{-1})}y_t}$$

with the error:

$$\tilde{y}_{t+m|t} = G(q^{-1})e_{t+m}$$

The polynomials, G and S, obey the Diophantine equation:

$$\boxed{C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-m}S(q^{-1})}$$

with:

$$G(0) = 1 \qquad ord(G) = m-1 \qquad \text{and} \qquad ord(S) = n-1$$

□

Proof: Let Y_t denoted the information embedded in $y_{t-i},\ i = 0,\ 1,\ ...$ (and $u_{t-i},\ i = 0,\ 1,\ ...$). The optimal solution to the prediction problem when the available data is Y_t is given by the conditional expectation, ie.:

$$\hat{y}_{t+m} = E\{y_{t+m}|Y_t\}$$

Consequently, we therefore is determine:

$$y_{t+m} = \frac{C(q^{-1})}{A(q^{-1})}e_{t+m} = h_t \star e_{t+m} = \sum_{i=0}^{\infty} h_i e_{t+m-i}$$

Since, $e_{t+m}, ...e_{t+1}$ is independent of Y_t we will split y_{t+m} into to contributions:

$$y_{t+m} = \sum_{i=0}^{m-1} h_i e_{t+m-i} + \sum_{i=m}^{\infty} h_i e_{t+m-i} \tag{30}$$

$$= \sum_{i=0}^{m-1} h_i e_{t+m-i} + \sum_{i=0}^{\infty} h_{i+m} e_{t-i} \tag{31}$$

where the first term is a moving average (MA) of the first m future contributions, i.e..

$$\sum_{i=0}^{m-1} h_i e_{t+m-i} = e_{t+m} + h_1 e_{t+m-1} + ... + h_{m-1} e_{t+1} = G(q^{-1}) e_{t+m}$$

In other words we have performed a division between the polynomials C and A:

$$\frac{C(q^{-1})}{A(q^{-1})} = G(q^{-1}) + \frac{q^{-m} S(q^{-1})}{A(q^{-1})}$$

where $G(q^{-1}$ is the result and $S(q^{-1})$ is the rest. This can also be written as:

$$C(q^{-1}) = A(q^{-1}) G(q^{-1}) + q^{-m} S(q^{-1})$$

where:

$$G(q^{-1}) = 1 + g_1 q^{-1} + ... + g_{m-1} q^{1-m}$$

Notice,

$$ord(G) = m - 1 \qquad G(0) = 1$$

Now, we have:

$$y_{t+m} = G(q^{-1}) e_{t+m} + \frac{S(q^{-1})}{A(q^{-1})} e_t$$

With the process equations we the have:

$$e_t = \frac{A(q^{-1})}{C(q^{-1})} y_t$$

or that:

$$y_{t+m} = G(q^{-1}) e_{t+m} + \frac{S(q^{-1})}{C(q^{-1})} y_t$$

The optimal prediction is then given by:

$$\hat{y}_{t+m|t} = E\{y_{t+m} | Y_t\} = \frac{S(q^{-1})}{C(q^{-1})} y_t$$

with the error:

$$\tilde{y}_{t+m|t} = G(q^{-1}) e_{t+m}$$

□

Remark: 4 Notice, the assumption that we can estimate e_t from the observation of y_t, is that we have observed y_t for $t_0 \rightarrow -\infty$ and that C is stable. When estimating e_t from y_t we are filtrating y_t through the inverse transfer function. The estimation error will tend to zero in a way determined by the roots of C. □

Remark: 5 Notice, the error is a MA process. The variance of such a process is given by:

$$Var\{\tilde{y}_{t+m|t}\} = \sigma^2 [1 + g_1^2 + ... + g_{m-1}^2]$$

and the (Auto) Covariance function (and the correlation function) is zero for for lag larger than m. □

A.2 Prediction int the ARMAX structure

Let us now return the original problem and assume the system is given by the ARMAX model in (2). If we concentrate on instant t, then u_t will effect the output at $t + k$ and forward. Let us then determine the prediction of y_{t+k}. Then in fact means a prediction horizon which equals the time delay through the system.

Theorem: A.2: Let the system be given by (2). Then the k step ahead prediction of y_t is given by:

$$\hat{y}_{t+k|t} = \frac{1}{C(q^{-1})}\left\{B(q^{-1})G(q^{-1})u_t + S(q^{-1})y_t\right\} \tag{32}$$

and the error is

$$\tilde{y}_{t+k|t} = G(q^{-1})e_{t+k}$$

where the polynomials G and S, are solutions to

$$C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \tag{33}$$

with:

$$G(0) = 1 \quad ord(G) = k-1 \quad \text{and} \quad ord(S) = n-1$$

□

Proof: If we apply the Diophantine equation

$$C = AG + q^{-k}S$$

we can for y_{t+k} write

$$y_{t+k} = \frac{1}{C}\left\{AG + q^{-k}S\right\}y_{t+k} \tag{34}$$

$$= \frac{1}{C}\{GAy_{t+k} + Sy_t\} \tag{35}$$

If we furthermore use the system equation (2), we have

$$y_{t+k} = \frac{1}{C}\{G[Bu_t + Ce_{t+k}] + Sy_t\} \tag{36}$$

$$= \frac{1}{C}\{BGu_t + Sy_t\} + Ge_{t+k} \tag{37}$$

□

It could be noticed, that for $B \equiv 0$ this predictor is equivalent to one in A.1.

B The Diophantine Equation

The Diophantine equation play a very important role in connection to stochastic control. It is a part in many design algorithms for controllers and predictors. I this appendix we will investigate the property of this equation in details.

The name comes from the fact that Diophantus of Alexandria wrote a book in the third century A.D. about the problem of finding integer solution to the equation $C = AX + BY$.

Assume, that we for 3 given polynomials, A, B and C

$$C(q^{-1}) = c_0 + c_1q^{-1} + ... + c_{n_c}q^{-n_c} \tag{38}$$

$$\bar{B}(q^{-1}) = b_1q^{-1} + ... + b_{n_b}q^{-n_b} \tag{39}$$

$$A(q^{-1}) = 1 + a_1q^{-1} + ... + a_nq^{-n} \tag{40}$$

has to determine the polynomials R and S, such that:

$$C(q^{-1}) = A(q^{-1})R(q^{-1}) + \bar{B}(q^{-1})S(q^{-1}) \tag{41}$$

Notice this set of equations are determined by A, $\bar{B}$ and C. Also notice that these polynomials are general and only in some special cases coincide with the system polynomials. It is important to notice that $\bar{B}$ obey the following

$$\bar{B}(0) = 0 \tag{42}$$

i.e. the leading coefficient in $\bar{B}$ (that is b_0) is zero.

We introduce the following basic theorem:

Theorem: B.1: The Diophantine equation (41) has a solution if and only if the common factors in A and $\bar{B}$ also is a common factor of C. □

1

Proof: See [2] □

It can also be noticed, that solutions to the Diophantine equation is not in general unique. Let R_0 and S_0 be a set of solutions to the Diophantine equation (41). Then

$$R(q^{-1}) = R_0(q^{-1}) + \bar{B}(q^{-1})F(q^{-1}) \tag{43}$$

$$S(q^{-1}) = S_0(q^{-1}) - A(q^{-1})F(q^{-1}) \tag{44}$$

is also a set of solutions. Here F is an arbitrary polynomium.

In our applications the solution can be fixed in terms of constraints on the orders of the polynomials. In order to obtain the best noise reduction we often choose

$$ord(R) = n_r = n_b - 1$$

where n_b is the order of the $\bar{B}$ polynomial. The order of S has to such that a solution exists. That means that

$$n_b + ord(S) = Max\{n_c, n + n_r\} \tag{45}$$

or that

$$n_s = ord(S) = Max\{n_a - 1, n_c - n_b\} \tag{46}$$

The Diophantine equation can be solved in various ways. One of them is the Sylvester method and the Euclidian algorithm.

C The Sylvester method

If the coefficients to the polynomials (i.e. the coefficients to q^{-i}) is identified, then the Diophantine equation is just linear set of equations. The resulting set of equations in the coefficients in R and S can be found to be:

$$\left(\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
a_1 & 1 & \ddots & \vdots & b_1 & 0 & & \vdots \\
a_2 & a_1 & & 0 & b_2 & b_1 & & 0 \\
\vdots & \vdots & & 1 & \vdots & b_2 & & b_1 \\
a_n & a_{n-1} & & a_1 & b_{n_b} & \vdots & & b_2 \\
0 & a_n & & \vdots & 0 & b_{n_b-1} & & \vdots \\
\vdots & \vdots & & a_{n-1} & \vdots & b_{n_b} & & b_{n_b-1} \\
0 & 0 & & a_n & 0 & \vdots & & b_{n_b}
\end{array}\right)
\begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n_r} \\ s_0 \\ s_1 \\ \vdots \\ s_{n_s} \end{pmatrix}
=
\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n_c} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{47}$$

where:

$$R(q^{-1}) = r_0 + r_1 q^{-1} + ... + r_{n_e} q^{-n_e} \tag{48}$$

$$S(q^{-1}) = s_0 + s_1 q^{-1} + ... + s_{n_s} q^{-n_s} \tag{49}$$

The set of equations can be expressed in a condensed form as:

$$\mathbb{S}x = z \tag{50}$$

where

$$x = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n_r} \\ s_0 \\ s_1 \\ \vdots \\ s_{n_s} \end{pmatrix} \text{ and } z = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n_c} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{51}$$

and where the Sylvester matrix, $\mathbb{S}$, is

$$\mathbb{S} = \left(\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_1 & 1 & \ddots & \vdots & b_1 & 0 & & \vdots \\ a_2 & a_1 & & 0 & b_2 & b_1 & & 0 \\ \vdots & \vdots & & 1 & \vdots & b_2 & & b_1 \\ a_n & a_{n-1} & & a_1 & b_{n_b} & \vdots & & b_2 \\ 0 & a_n & & \vdots & 0 & b_{n_b-1} & & \vdots \\ \vdots & \vdots & & a_{n-1} & \vdots & b_{n_b} & & b_{n_b-1} \\ 0 & 0 & & a_n & 0 & \vdots & & b_{n_b} \end{array}\right) \tag{52}$$

This matrix has several interesting properties in connection to system theory. It can be shown that A and $\bar{B}$ are coprime if and only if $\mathbb{S}$ are non-singular (see e.g. [1]).

C.1 Impulse response method

In certain cases the Diophantine equation degenerate to the following

$$C(q^{-1}) = A(q^{-1})G(q^{-1}) + q^{-k}S(q^{-1}) \tag{53}$$

Here

$$n_g = k - 1 \qquad n_s = max(n_a - 1, n_c - k) \tag{54}$$

In this case the solution is a simple division of polynomials.

$$\frac{C(q^{-1})}{A(q^{-1})} = G(q^{-1}) + q^{-k}\frac{S(q^{-1})}{A(q^{-1})} \tag{55}$$

and we can interpret the coefficients in G as the truncated impulse response, i.e.

$$G(q^{-1}) = \sum_{i=0}^{k-1} g_i q^{-i} \qquad \frac{C(q^{-1})}{A(q^{-1})} = \sum_{i=0}^{\infty} g_i q^{-i} \tag{56}$$

Let

$$\frac{C(q^{-1})}{A(q^{-1})} = \sum_{i=0}^{\infty} g_i q^{-i} = \sum_{i=0}^{k-1} g_i q^{-i} + \sum_{i=k}^{\infty} g_i q^{-i} \tag{57}$$

or

$$\frac{C(q^{-1})}{A(q^{-1})} = G(q^{-1}) + q^{-k} \sum_{i=0}^{\infty} g_{i+k} q^{-i} \tag{58}$$

Here S is the remainder in the division.

The algorithm can be summarized in:

1. Determine the first k coefficients in the impulse response, i.e.

$$G(q^{-1}) = \left[\frac{C(q^{-1})}{A(q^{-1})}\right]_k \tag{59}$$

2. Determine S from:

$$S(q^{-1}) = q^k (C(q^{-1}) - A(q^{-1}) G(q^{-1})) \tag{60}$$

It should be noticed that this method is only applicable when the Diophantine equation takes the simple form in (53). The method can in certain situations (in connection to predictive control design) be implemented as a recursive method (in k).

References

[1] T. Kailath. *Linear Systems.* Prentice Hall, 1980.

[2] V. Kučera. *Linear Control Systems.* Wiley-Interscience, 1979.

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