Facets for the Cardinality Constrained Quadratic Knapsack Problem and the Quadratic Selective Travelling Salesman Problem

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Abstract

This paper considers the Cardinality Constrained Quadratic Knapsack Problem (QKP) and the Quadratic Selective Travelling Salesman Problem (QSTSP). The QKP is a generalization of the Knapsack Problem and the QSTSP is a generalization of the Travelling Salesman Problem. Thus, both problems are NP hard.

The QSTSP and the QKP can be solved using branch-and-cut methods, and in doing so, good bounds can be obtained if strong constraints are used. Hence it is important to identify strong or even facet-defining constraints. This paper presents the polyhedral combinatorics of QSTSP and QKP, i.e. amongst others identify facet-defining constraints for the QSTSP and the QKP, and provide mathematical proofs that they do indeed define facets.

Keywords: Quadratic Knapsack; Quadratic Selective Travelling Salesman; Polyhedral Analysis; Facets

1 Introduction

A well-known extension of the Travelling Salesman Problem (TSP) is the Selective (or Prize-collecting) TSP: In addition to the edge-costs, each node has an associated reward (denoted the node-reward) and instead of visiting all nodes, only profitable nodes are visited. The Quadratic Selective TSP (QSTSP) has the additional complications: (1) each *pair* of nodes have an associated

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reward (denoted the edge-reward) which can be gained only if both nodes are visited; and (2) a constraint on the number of nodes selected is imposed, which we refer to as the cardinality constraint. The objective of an QSTSP is to maximize the total node-reward and edge-rewards gained minus the edge-costs incurred subject to the satisfaction of the cardinality constraint.

Conceptually the QSTSP consists of two interacting problems, a cardinality-constrained min-cost circuit problem with respect to the edge-costs and a cardinality-constrained max-reward clique problem with respect to the edge-rewards.

The cardinality constrained circuit problem (CCCP) is considered in [Bauer, 1997] where polyhedral results are presented and in [Bauer et al., 2002] where a branch and cut algorithm is discussed. The max-reward clique problem is a special case of the quadratic knapsack problem where the knapsack constraints have unit coefficients. We denote this problem the cardinality constrained quadratic knapsack problem (QKP). The quadratic knapsack problem (when coefficients are not necessarily unit) is considered in e.g. [Johnson et al., 1993], [Billionnet and Calmels, 1996] and [Caprara et al., 1999]. If edge-rewards are non-negative, the cardinality constraint will be met with equality. This is similar to the p-dispersion problem considered in [Erkut, 1990] wherein the objective is to maximize the minimum edge-reward. The p-dispersion problem is considered in [Pisinger, 1999] with an objective equivalent to the one considered here.

Various TSP-like problems are similar to QSTSP in the way that a subset of nodes has to be selected. E.g. the Prize-collecting TSP [Balas, 1989, Balas, 1995], Selective TSP [Gendreau et al., 1998, Laporte and Martello, 1990], the Orienteering problem [Fischetti et al., 1998], and the Generalized TSP [Fischetti et al., 1995, Fischetti et al., 1997]. Problems that consider the combination of a clique problem and a cycle problem has been studied in [Gendreau et al., 1995] and [Gouveia and Manuel Pires, 2001]. Gendreau, Labbe, and Laporte [Gendreau et al., 1995] study a problem where instead of imposing the cardinality constraint, an upper bound on the sum of the edge-costs are imposed. Gouveia and Manuel Pires [Gouveia and Manuel Pires, 2001] study a QSTSP-like problem with the additional requirement that some nodes must be in the ring.

In this paper we study the polyhedral combiatorics of the QKP and the QSTSP. Our interest in studying the QSTSP is due to the fact that this problem arose as a subproblem from another combinatorial optimisation problem which deals with designing hierarchical ring networks (see [Stidsen and Thomadsen]). Naturally, the faster we can solve the QSTSP, the better. The QKP, however, is an interesting problems in its own right, but we study the QKP mostly for its relevance in understanding the QSTSP. Since QKP is a generalization of the Knapsack Problem and QSTSP is a generalization of the Travelling Salesman Problem, both problems are NP hard.

A promising approach in solving these combinatorial optimisation problems is the branch-and-cut method. A significant factor in the success of the method is the use of strong constraints that at least partially describe the convex hull of the incidence vectors of all feasible solutions, in other words, the use of facet-defining cuts.

The contribution of this research is therefore the identification of some of the facet-defining cuts, the mathematical proofs that these cuts are indeed facet-defining, and the various mathematical techniques used in proving these results.

We begin with, in Section 2, giving an integer programming model for QSTSP and define the polyhedra of the QKP, CCCP, and the QSTSP. In Sections 3 and 4, we present our polyhedral results on the QKP and the QSTSP polytopes. Finally, in Section 5, we conclude our findings.

2 Integer Programming Model and the Polyhedra

We consider QSTSP defined on the undirected graph $G = (V, E)$. We assume that G is a complete undirected graph. This is not restrictive, since we can always introduce high costs for edges that do not exist.

For notational convenience, we use (U_1, U_2) to denote $\{(i, j) \in E \mid i \in U_1, j \in U_2\}$, for any $U_1, U_2 \subseteq V$; use $\delta(S)$ to denote $\{(i, j) \in E \mid i \in S, j \in V \setminus S\}$; and use $E(S)$ to denote $\{(i, j) \in E \mid i, j \in S\}$.

To give the model, we use r_i for the node-reward, w_e for the edge-reward, c_e for the edge-cost, and b for the maximum number on nodes allowed in the ring. Let x_e be the decision variable with $x_e = 1$ if $e \in E$ is chosen in the ring and 0 otherwise; y_i be the decision variable with $y_i = 1$ if $i \in V$ is on the ring, 0 otherwise; and z_e be the decision variable with $z_e = 1$, for $(i, j) \in E$ if node i and j are both on the ring, 0 otherwise. If $(i, j) = e \in E$, then z_{ij} is sometimes used in place of z_e. Given these, the QSTSP is formulated as follows.

$$\max \quad \sum_{i \in V} r_i \cdot y_i + \sum_{e \in E} w_e \cdot z_e - \sum_{e \in E} c_e \cdot x_e \tag{1}$$

$$\text{s.t.} \quad \sum_{e \in \delta(i)} x_e = 2y_i, \ \forall i \in V \tag{2}$$

$$z_e \leq y_i, \ \forall i \in V, e \in \delta(i) \tag{3}$$

$$z_{ij} \geq y_i + y_j - 1, \ \forall (i, j) \in E, i < j \tag{4}$$

$$\sum_{e \in \delta(S)} x_e \geq 2(y_k + y_l - 1), \quad \forall \emptyset \subset S \subset V, k \in S, l \notin S \tag{5}$$

$$\sum_{i \in V} y_i \leq b \tag{6}$$

$$x_e \in \{0, 1\}, \ \forall e \in E \tag{7}$$

$$y_i \in \{0, 1\}, \ \forall i \in V \tag{8}$$

$$z_e \in \{0,1\}, \forall e \in E \tag{9}$$

Constraints (2) makes sure that if and only if a node is selected, the indegree of the node is 2. Constraints (3) and (4) establishes that $z_{ij} = 1$ if and only if $y_i = y_j = 1$. Constraints (5) is the subtour elimination constraints and (6) is the cardinality constraint. Finally (7), (8) and (9) are the binary constraints. Let $n = |V|$. The Quadratic Selective Travelling Salesman(QSTS) polytope is defined to be

$$P_{QS}^{nb} = \text{ conv } \{(x,y,z) \in \mathcal{R}^{2|E|+n} | (x,y,z) \text{ satisfies } (2)-(9)\}. \tag{10}$$

We identify two related polytopes. The cardinality constrained quadratic knapsack(QK) polytope,

$$P_{QK}^{nb} = \text{ conv } \{(y,z) \in \mathcal{R}^{|E|+n} | (y,z) \text{ satisfies } (3),(4),(6),(8) \text{ and } (9)\}, \tag{11}$$

and the cardinality constrained circuit polytope,

$$P_{C}^{nb} = \text{ conv } \{(x,y) \in \mathcal{R}^{|E|+n} | (x,y) \text{ satisfies } (2),(5)-(8)\}\}. \tag{12}$$

Note that P_{QS}^{nb} is contained in the intersection of P_{QK}^{nb} and P_{C}^{nb}. Thus any valid inequality for either P_{QK}^{nb} or P_{C}^{nb} is valid for P_{QS}^{nb}. Note also that for the CCCP and QSTSP we consider in this paper, we assume that the empty cycle is considered as a feasible solution, whereas in Bauer [Bauer, 1997], it is not considered as a feasible solution. No feasible cycles with one or two nodes exists.

The contribution of this paper is to study the QK polytope and a QSTS polytope modelled without the y variables, denoted by $\tilde{P}_{QS}^{nb}$. We show that $\tilde{P}_{QS}^{nb}$ and P_{QS}^{nb} are in fact describing the same set of feasible solutions for the QSTSP, and that any facet-defining inequality defined for $\tilde{P}_{QS}^{nb}$ is also facet-defining for P_{QS}^{nb}. Then we work on $\tilde{P}_{QS}^{nb}$ and P_{QK}^{nb}: we establish the dimensions of these polytopes, and for each of them, we develop several classes of constraints and prove that they are facet-defining.

3 Polyhedral results for the QK polytope

In this section, we present our polyhedral results on the dimension of P_{QK}^{nb} and that four classes of constraints are facet-defining. The first class is the non-negativity constraint on z_e, the following two classes are generalizations of (3) and (4) respectively and the last class of constraints are obtained by modifying (6).

In what follows, we use incidence vectors $(y,z) \in \{0,1\}^{|V|+|E|}$, for $y \in \{0,1\}^{|V|}$ and $z \in \{0,1\}^{|E|}$ to represent our solutions. Each element in a vector corresponds to a node $j \in V$ or an edge $(i,j) \in E$. We also use $e_j \in \{0,1\}^{|V|+|E|}$, for any $j \in V$, to represent a vector with the value of the element corresponding to node j equals 1 and the values of all other elements equal 0; and use $e_{ij} \in \{0,1\}^{|V|+|E|}$, for any $(i,j) \in E$, to represent a vector with the value of the element corresponding to edge (i,j) equals 1 and the values of all other elements equal 0.

Theorem 3.1 *Given any $G = (V, E)$, $2 \leq b \leq |V|$, the dimension of the QK polytope, P_{QK}^{nb}, is $|E| + |V|$, i.e., it is full dimensional.*

Proof. Consider the following feasible solutions:

1. $(y, z)^0 = 0$;
2. $(y, z)^1 = e_j$, for all $j \in V$; and
3. $(y, z)^2 = e_i + e_j + e_{ij}$, for all $(i, j) \in E$.

Clearly, these give us $|E|+|V|+1$ affinely independent feasible solutions, and therefore the dimension of the QK polytope is $|E| + |V|$. □

Theorem 3.2 *Given any $G = (V, E)$, $2 \leq b \leq |V|$ the constraints, given as*

$$z_f \geq 0, \quad \forall f \in E, \tag{13}$$

are facet-defining for P_{QK}^{nb}.

Proof. We need to show that the dimension of $F = P_{QK}^{nb} \cap \{z_f = 0\}$ is $|E| + |V| - 1$. First of all, it is trivially true that F defines a proper face and therefore $\dim(F) \leq |E| + |V| - 1$. Now consider the following feasible solutions:

1. $(y, z)^0 = 0$;
2. $(y, z)^1 = e_j$, for all $j \in V$; and
3. $(y, z)^2 = e_i + e_j + e_{ij}$, for all $(i, j) \in E \setminus \{f\}$.

Clearly, these give us $|E| + |V|$ affinely independent feasible solutions, and therefore $\dim(F)$ is $|E| + |V| - 1$. □

Theorem 3.3 *Given any $G = (V, E)$, $3 \leq b \leq |V|$ the constraints, given as:*

$$\sum_{e \in (i,S)} z_e \leq y_i + \sum_{e \in E(S)} z_e, \forall i \in V, S \subseteq V \setminus \{i\}, |S| \geq 2. \tag{14}$$

are facet-defining for P_{QK}^{nb}.

Note that (3) is a special case of (14).

Proof. Let $F = P^{nb}_{QK} \cap \Big\{ \sum_{e\in(i,S)} z_e = y_i + \sum_{e\in E(S)} z_e \Big\}$. Since $(y,z) = e_j$, for any $j \in V \setminus \{i\}$, does not satisfy the constraint at equality, $\dim(F) \le \dim(P^{nb}_{QK}) - 1$. Now, we show that $\dim(F) \ge \dim(P^{nb}_{QK}) - 1$ by finding exactly $\dim(P^{nb}_{QK}) = |V| + |E|$ affinely independent feasible solutions that satisfy the constraints at equality. We do so by sequentially introducing the following vectors, each representing a feasible solution.

1. $(y,z)^1 = \{0\}$;
2. $(y,z)^2 = \{(y,z)^2_j \mid \forall j \in V \setminus \{i\}\}$ where $(y,z)^2_j = e_j$, (we have $|V| - 1$ of these solutions);
3. $(y,z)^3 = \{(y,z)^3_{ij} \mid \forall j \in S\}$ where $(y,z)^3_{ij} = e_i + e_j + e_{ij}$, for all $j \in S$, (we have $|S|$ of these solutions);
4. $(y,z)^4 = \{(y,z)^4_{jk} \mid \forall j \in S,\ k \in \bar{S} \setminus \{i\}\}$ where $(y,z)^4_{jk} = e_j + e_k + e_{jk}$, (we have $|(S, \bar{S} \setminus \{i\})|$ of these solutions);
5. $(y,z)^5 = \{(y,z)^5_{jk} \mid \forall j,k \in \bar{S} \setminus \{i\},\ j < k\}$ where $(y,z)^5_{jk} = e_j + e_k + e_{jk}$, (we have $|E(\bar{S} \setminus \{i\})|$ of these solutions);
6. $(y,z)^6 = \{(y,z)^6_{jk} \mid \forall j,k \in S,\ j < k\}$ where $(y,z)^6_{jk} = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, (we have $|E(S,S)|$ of these solutions); and
7. $(y,z)^7 = \{(y,z)^7_{jk} \mid j \in S, \forall k \in \bar{S} \setminus i\}$, where $(y,z)^7_{jk} = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, (we have $|\bar{S}| - 1$ of these solutions).

Hence, we have $|V| + |E|$ affinely independent feasible solutions in total and thus the theorem is proved. □

Theorem 3.4 *Given any $G = (V,E)$, $3 \le b \le |V|$, the constraints, given as:*

$$\sum_{e\in E(S)} z_e + 1 \ge \sum_{i\in S} y_i, \quad \forall S \subseteq V, |S| \ge 2 \tag{15}$$

are facet-defining for the QK polytope.

Note that (4) is a special case of (15).

Proof. Let $F = P^{nb}_{QK} \cap \Big\{ \sum_{e\in E(S)} z_e + 1 = \sum_{i\in S} y_i \Big\}$. Since $(y,z) = 0$ does not satisfy the constraint at equality, $\dim(F) \le \dim(P^{nb}_{QK}) - 1$. Now, we show that $\dim(F) \ge \dim(P^{nb}_{QK}) - 1$ by finding exactly $\dim(P^{nb}_{QK}) = |V| + |E|$ affinely independent feasible solutions that satisfy the constraints at equality. We do so by taking the following steps.

1. $(y,z)^1 = \{(y,z)^1_i \quad | \quad \forall i \in S\}$, where $(y,z)^1_i = e_i$ (we have $|S|$ of these solutions);

2. $(y,z)^2 = \{(y,z)^2_{ij} \quad | \quad \forall i \in S,\ j \in V\}$, where $(y,z)^2_{ij} = e_i + e_j + e_{ij}$, (we have $|(S,S)| + |(S,\bar{S})|$ of these solutions);

3. $(y,z)^3 = \{(y,z)^3_k \quad | \quad \forall k \in \bar{S}\}$, where $(y,z)^3_k = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, for a fixed $i \in S$, and a fixed $j \in S \setminus \{i\}$, (we have $|\bar{S}|$ of these solutions); and

4. $(y,z)^4 = \{(y,z)^4_{jk} \quad | \quad \forall j,k \in \bar{S}\}$, where $(y,z)^4_{jk} = e_i + e_j + e_k + e_{ij} + e_{ik} + e_{jk}$, for a fixed $i \in S$, (there are $|(\bar{S},\bar{S})|$ of these solutions).

It is obvious that the $|S| + |(S,S)| + |(S,\bar{S})|$ feasible solutions introduced in Step 1 and Step 2 are affinely independent. We now show, by contradiction, that the solutions introduced in Step 3 are in fact affinely independent to any of the previously introduced solutions. We assume that, w.l.o.g., the first solution introduced in Step 3 is $(y,z)^3_l$, for any $l \in \bar{S}$, and that $(y,z)^3_l = \sum_i \lambda_i (y,z)^1_i + \sum_{ij} \mu_{ij}(y,z)^2_{ij}$, for some $\lambda \in \mathbb{R}^{|S|}$, $\mu \in \mathbb{R}^{|(S,S)|+|(S,\bar{S})|}$, $(\lambda,\mu) \neq 0$. Now, to obtain the elements in $(y,z)^3_l$ corresponding to the z variables, we need $\mu_{ij} = \mu_{il} = \mu_{jl} = 1$, and $\mu_f = 0$ for all $f \in E \setminus \{(i,j),(i,l),(j,l)\}$. Observe that in $(y,z)^1$, the value of the elements corresponding to variable y_l is always 0, (since $l \in \bar{S}$), so y^3_l has a value of 2 instead of 1. Hence there is a contradiction. Clearly $(y,z)^3$ are independent as elements in $\bar{S}$ are all distinct, we conclude that the incidence vectors in $(y,z)^3$ are all affinely independent. Last of all, the solutions introduced in Step 4, i.e. $(y,z)^4$ are obviously affinely independent to all the previously introduced solutions. Thus the theorem is proved. $\square$

Theorem 3.5 *Given any $G = (V,E)$, $3 \leq b \leq |V| - 1$, the constraints, given as:*

$$\sum_{e \in \delta(i)} z_e \leq (b-1)y_i, \quad \forall i \in V \tag{16}$$

are facet-defining for P^{nb}_{QK}.

Constraint (16) can be obtained by multiplying (6) with y_i and noting that $y_i y_i = y_i$ and $y_i y_j = z_{ij}$.

Proof. We need to show that the dimension of $F = P^{nb}_{QK} \cap \{\sum_{e \in \delta(i)} z_e = (b-1)y_i\}$ is $|E| + |V| - 1$. Since $(y,z) = e_i$ does not satisfy constraint (16) at equality, $\dim(F) \leq |E| + |V| - 1$. Now consider the following feasible solutions:

1. $(y,z)^0 = 0$;

2. $(y,z)^1 = e_k$, for all $k \in V \setminus \{i\}$, (we have $|V| - 1$ of these solutions);

3. $(y,z)^2 = e_k + e_l + e_{kl}$, for all $\{k,l\} \subseteq V \setminus \{i\}$, (we have $|E| - (|V|-1)$ of these solutions); and

Clearly, these $|E|+1$ points are affinely independent, and satisfy (16) at equality.

Now, we are left with finding $|V|-1$ affinely independent feasible solutions. We do so by inspecting the set of all feasible solutions that selects exactly b nodes: the node i plus $b-1$ other nodes from the set $V \setminus \{i\}$. We define such a set to be $\Omega^{V,b} = \{I_1, \ldots, I_m \quad | \quad I_l = \{i\} \cup U, |I_l| = b, \quad \forall l = 1, \ldots, m\}$, where $U \subset V \setminus \{i\}$. (Note that m is finite). We denote I_l by $\{i, j_1^l, \ldots, j_{b-1}^l\}$ for each $l = 1, \ldots, m$.

Our inductive hypothesis is that $\Omega^{V,b}$ contains precisely $|V|-1$ affinely independent feasible solutions. Our proof takes the following steps: Step 1 concerns the initial case for $|V| = 4$ and $b = 3$; Step 2 concerns induction on $|V|$ while holding b constant; and Step 3 concerns induction on both b and $|V|$.

Step 1. We have found precisely 3 affinely independent feasible solutions for the case when $|V| = 4$ and $b = 3$.

Step 2. We assume that our inductive hypothesis is true for $|V| = 4, \ldots, s$ and $b = t$. We now show that it is true for $|V| = s+1$ and $b = t$. Consider the QKP defined on $\tilde{G} = (\tilde{V}, \tilde{E})$, for $\tilde{V} = V \cup \{q\}$, $\tilde{E} = (q, V) \cup E(V)$. We show that $\Omega^{\tilde{V},t}$ contains exactly s affinely independent feasible solutions. By our inductive hypothesis, there exists $\Omega^{V,t}$ that contains $s-1$ affinely independent feasible solutions, and w.l.o.g., let these $s-1$ solutions be $I_1, \ldots, I_{s-1}$. As b was held constant at t, these $s-1$ points are also feasible for $\tilde{G}$ and satisfy (16) at equality. Now consider a new solution $I_s = \{i, j_1^1, \ldots, j_{t-2}^1, q\}$. Clearly, I_s is affinely independent to any of the previously introduced solutions (wherein q is never used), and it satisfy (16) at equality.

Step 3. We assume that our inductive hypothesis holds for $|V| = 4, \ldots, s$, $b = 3, \ldots, t$, for $t \leq s-1$, and prove that it holds for $|V| = s+1$ and $b = t+1$. Recall $I_1, \ldots, I_{s-1}$ defined in Step 2. First, consider $I_s' = I_1 \cup \{k\}$, for $k \in V \setminus I_1$, which uses exactly $t+1$ nodes, and is affinely independent to $(y,z)^1$ and $(y,z)^2$ (as the node i is never selected therein). Then we define $I_l' = I_l \cup \{q\}$, for all $l = 1, \ldots, s-1$, and thus obtain $s-1$ affinely independent feasible solutions each selecting $t+1$ nodes. These are affinely independent to $(y,z)^1$, $(y,z)^2$, and I_s' due to the use of node q. Thus completes the proof. $\square$

4 Polyhedral results for the QSTS polytope

In this section, we present our polyhedral results for the QSTS polytope, $\tilde{P}_{QS}^{nb}$. (Recall that this concerns the formulation without the y variables). We first establish the dimension of $\tilde{P}_{QS}^{nb}$ and establish the links between $\tilde{P}_{QS}^{nb}$ and P_{QS}^{nb}. We then prove that five classes of constraints are facet-defining for $\tilde{P}_{QS}^{nb}$. The first class of constraints concerns the relationship between x_e and z_e; the second class of constraints is a strengthened version of the subtour elimination constraints (5); and the last three classes of constraints are also facets for the QK polytope, except that herein we use $\frac{1}{2}\sum_{e\in\delta(i)} x_e$ in place of y_i.

In what follows, we use incidence vectors $(x, z) \in \{0, 1\}^{2|E|}$, for $x, z \in \{0, 1\}^{|E|}$ to represent our solutions. We also define $(\lambda, \mu) \in \mathbb{R}^{2|E|}$, for $\lambda, \mu \in \mathbb{R}^{|E|}$, with each element in λ and μ representing an edge $e \in E$. Furthermore, when we refer to p-cycles, we refer to cycles in G that contain p nodes.

We will use the following result frequently.

Proposition 4.1 *Given an undirected graph $G = (V, E)$, $|V| = 5$, let X be the matrix generated by incident vectors of all 3- and 4-cycles in G. Under the assumption that G is complete, if $X(\lambda, \mu)^T = 0$, then $\lambda_e = \mu_e = 0$ for all $e \in (E)$.*

Proof. It can be verified that X is of rank $2|E| = 20$, hence the result follows immediately. □

Theorem 4.1 *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 5$ and $4 \leq b \leq |V|$, under the assumption that G is complete, the dimension of the QSTS polytope, $\tilde{P}^{nb}_{QS}$, is $2|E|$.*

Proof. We show, by contradiction, that the dimension of $\tilde{P}^{nb}_{QS}$ is $2|E|$. We first assume that $\tilde{P}^{nb}_{QS}$ is not full-dimensional, and hence there must be at least one equality constraint, $\lambda \cdot x + \mu \cdot z = \lambda_0$, satisfied by all feasible solutions in the polytope. Then we establish that this is true only when $\lambda_e = \mu_e = \lambda_0 = 0$, for all $e \in E$, thus implying that there is no equality constraint satisfied by all feasible solutions in the polytope and hence the polytope is full dimensional. Consider the 0-cycle defined by $(x, z) = 0$. We have $\lambda \cdot 0 + \mu \cdot 0 = \lambda_0$. Hence we get $\lambda_0 = 0$. To show that $\lambda_e = \mu_e = \lambda_0 = 0$, for all $e \in E$, consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V$, $|\tilde{V}| = 5$, and $\tilde{E} = E(\tilde{V})$. Under the assumption that G is complete, $\tilde{G}$ is also complete. Now, consider a matrix X generated by the incident vectors of all the 3-cycles and the 4-cycles in $\tilde{G}$. Since $\lambda_0 = 0$, by result of Proposition 4.1, we have $\lambda_e = \mu_e = 0$ for all $e \in \tilde{E}$. As $\tilde{G}$ is arbitrary in G, we have that $\lambda_e = \mu_e = 0$, for all $e \in E$. Hence the theorem is proved. □

Next, we discuss the relation between $\tilde{P}^{nb}_{QS}$ and P^{nb}_{QS}. Essentially, we try to establish that the two polytopes represent the same set of feasible solutions, and that facets found for one are facets for the other (with slight modifications). Hence, all facets of $\tilde{P}^{nb}_{QS}$ we propose in this paper are also facets for P^{nb}_{QS}. These results are echos of similar results of Bauer *et a.* [Bauer et al., 2002] for the CCCP.

Proposition 4.2 *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, we have that $dim(\tilde{P}^{nb}_{QS}) = dim(P^{nb}_{QS})$.*

Proof. Each incidence vector $(x, z) \in \mathbb{R}^{2|E|} \cap \tilde{P}^{nb}_{QS}$ can be represented by an incident vector $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$. For any set of $2|E| + 1$ affinely independent incident vectors that spans $\tilde{P}^{nb}_{QS}$, we can get $2|E| + 1$ affinely independent incident vectors in P^{nb}_{QS}. Thus $dim(P^{nb}_{QS}) \geq 2|E|$. As the rank of the degree constraints, (2), is $|V|$, clearly $dim(P^{nb}_{QS}) \leq 2|E| + |V| - |V|$, and thus $dim(P^{nb}_{QS}) = 2|E|$. □

Remark 4.1 *Since $dim(\tilde{P}^{nb}_{QS}) = dim(P^{nb}_{QS})$, and each incidence vector $(x, z) \in \mathbb{R}^{2|E|} \cap \tilde{P}^{nb}_{QS}$ can be represented by an incident vector $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$, the two polytopes describe the same set of feasible solutions for the QSTSP.*

Proposition 4.3 *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, if $ax + bz \leq a_0$ defines a facet for $\tilde{P}^{nb}_{QS}$, then it also defines a facet for P^{nb}_{QS}.*

Proof. The same $2|E|$ affinely independent incidence vectors $(x, z) \in \mathbb{R}^{2|E|} \cap \tilde{P}^{nb}_{QS}$ that satisfy $ax + bz \leq a_0$ at equality can be converted to $2|E|$ affinely independent incidence vectors $(x, y, z) \in \mathbb{R}^{2|E|+|V|} \cap P^{nb}_{QS}$. Hence the result. □

Proposition 4.4 *For any QSTSP defined on $G = (V, E)$ where $|V| \geq 5$, and $4 \leq b \leq |V|$, if $\alpha x + \beta y + \gamma z \leq \alpha_0$ defines a facet for P^{nb}_{QS}, then $\tilde{\alpha} x + \gamma z \leq \alpha_0$ also defines a facet for $\tilde{P}^{nb}_{QS}$, where $\tilde{\alpha}_{ij} = \alpha_{ij} + \frac{1}{2}(\beta_i + \beta_j)$.*

Proof. Suppose $\Omega = \{(x^1, y^1, z^1), \ldots, (x^{|E|}, y^{|V|}, z^{|E|})\}$ defines $2|E|$ affinely independent feasible solutions that satisfy $\alpha x + \beta y + \gamma z \leq \alpha_0$ at equality, then $\tilde{\Omega} = \{(\tilde{x}^1, 0, z^1), \ldots, (\tilde{x}^{|E|}, 0, z^{|E|})\}$, where $\tilde{x}_{ij} = x_{ij} + \frac{1}{2}(y_i + y_j)$ for all $(i, j) \in E$, (which is essentially obtained from Ω by simple linear row operations), are also affinely independent. Hence the result. □

Theorem 4.2 *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 5$ and $4 \leq b \leq |V|$, the constraints given below, are facet-defining for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

$$x_e \leq z_e, \quad \forall e \in E. \tag{17}$$

Proof. We first show that the result holds for $|V| \geq 6$ and $b \geq 4$. (For $|V| = 5$ and $5 \geq b \geq 4$, one can easily prove this by generating $2|E|$ affinely independent feasible points that satisfy (17) at equality). First we show that $\tilde{P}^{nb}_{QS} \cap \{x_e = z_e\}$ defines a proper face. This is achieved by showing that there is at least one each of a solution that satisfies the constraint at equality and a solution that does not. Let $e = (i, j)$. Consider a 4-cycle given by (l, i, m, j), for $l, m \in V \setminus \{i, j\}$, clearly $x_e = 0$ and $z_e = 1$, hence the constraint is not satisfied at equality. Now consider a 3-cycle given by (l, i, j), for $l \in V \setminus \{i, j\}$, clearly $x_e = z_e = 1$ and the constraint is satisfied at equality. Thus F defines a proper face.

Now, using Theorem 3.6 in Part I.4 of Nemhauser and Wolsey [Nemhauser and Wolsey, 1988], we need to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}^{nb}_{QS} \cap \{x_e = z_e\}$, then

$$\lambda_f = \begin{cases} \alpha, & \text{if } f = e, \\ 0, & \text{otherwise;} \end{cases} \qquad \lambda_0 = 0; \text{ and } \mu_f = \begin{cases} -\alpha, & \text{if } f = e, \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, we obtain $\lambda_0 = 0$. Now consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V \setminus \{i\}$, $e = (i, j)$, $|\tilde{V}| = 5$, and $\tilde{E} = E(\tilde{V})$. Obviously $e \notin \tilde{E}$ thus (17) holds with equality for all cycles in $\tilde{G}$. By Proposition 4.1, we thus have $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. As $\tilde{G}$ is arbitrary, we have $\lambda_f = \mu_f = 0$, for all $f \in E \setminus \{e\}$. Then, consider any arbitrary 3-cycle that contains the edge e, we get $\lambda_e + \mu_e = 0$. Let $\lambda_e = \alpha$ for some $\alpha \in \mathbb{R}$, we have $\mu_e = -\alpha$ and thus the theorem is proved. $\square$

To eliminate subtours (for the QSTSP), we propose a class of constraints which strengthens (5), given as:

$$\sum_{e \in \delta(S)} x_e \geq 2z_{kl}, \forall \emptyset \subset S \subset V, k \in S, l \notin S. \tag{18}$$

Theorem 4.3 *Given any QSTSP defined on an undirected graph $G = (V, E)$, with $|V| \geq 10$, $|V| - 5 \geq |S| \geq 5$ and $4 \leq b \leq |V|$, the constraints given by (18), are facet-defining for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

Proof. $\tilde{P}^{nb}_{QS} \cap \left\{ \sum_{e \in (S, \bar{S})} x_e = 2z_{kl} \right\}$ defines a proper face, since (18) holds with equality for the 0-cycle while it does not for the 3-cycle (k, p, q), for $p, q \in \bar{S} \setminus \{l\}$, $p \neq q$.

Now, we are left to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}^{nb}_{QS} \cap \left\{ \sum_{e \in (S, \bar{S})} x_e = 2z_{kl} \right\}$, then

$$\lambda_e = \begin{cases} \alpha, & \text{if } e \in (S, \bar{S}), \\ 0, & \text{otherwise;} \end{cases} \qquad \lambda_0 = 0; \text{ and } \mu_e = \begin{cases} -2\alpha, & \text{if } e = (k, l), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, we have $\lambda_0 = 0$. Now, consider any arbitrary subgraph $\tilde{G} = (\tilde{S}, \tilde{E})$ for $\tilde{S} \subseteq S$, $|\tilde{S}| = 5$, and $\tilde{E} = E(\tilde{S})$. Constraint (18) holds with equality for all cycles in $\tilde{G}$. Thus, by

Proposition 4.1, we have $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. As $\tilde{G}$ is arbitrary, we have $\lambda_f = \mu_f = 0$, for all $f \in E(S)$. Analogously it can be obtained that $\lambda_f = \mu_f = 0$, for all $f \in E(\bar{S})$.

Now we obtain values for all the remaining elements in (λ, μ), i.e., we find λ_e and μ_e for all $e \in (S, \bar{S})$, by comparing cycles with 3 or 4 nodes for which (18) holds with equality. In the following, we assume arbitrary i, j, m, for $i, j \in S \setminus \{k\}, i \neq j$ and $m \in \bar{S} \setminus \{l\}$.

Let (x^1, z^1) and (x^2, z^2) be the incidence vectors of the 4-cycle defined by (k, i, j, l) and the 3-cycle defined by (k, i, j) respectively. We get:

$$\lambda \cdot x^1 + \mu \cdot z^1 - (\lambda \cdot x^2 + \mu \cdot z^2) = \lambda_{jl} + \lambda_{kl} - \lambda_{jk} + \mu_{kl} + \mu_{il} + \mu_{jl} = 0. \tag{19}$$

Note that $\lambda_{jk} = 0$ since $k, j \in S$. Analogously let (x^3, z^3) be the incidence vectors of the 4-cycle defined by (k, j, i, l). We get:

$$\lambda \cdot x^3 + \mu \cdot z^3 - (\lambda \cdot x^2 + \mu \cdot z^2) = \lambda_{kl} + \lambda_{il} - \lambda_{ik} + \mu_{kl} + \mu_{il} + \mu_{jl} = 0. \tag{20}$$

Note that $\lambda_{ik} = 0$ since $k, i \in S$. By comparing (19) with (20), we get $\lambda_{il} = \lambda_{jl}$. Let $\lambda_{il} = \alpha$, by symmetry, we get $\lambda_{il} = \alpha$ for all $i \in S \setminus \{k\}$. Now by comparing the 3-cycle (k, j, l) with (19) it follows that $\mu_{il} = 0$ for all $i \in S \setminus \{k\}$.

Comparing the 4-cycle (k, i, l, j) with the 3-cycle (k, i, j), we get $\mu_{kl} = -2\alpha$ and by comparing the 3-cycle (k, j, l) with the 4-cycle (k, j, l, i), we get $\lambda_{kl} = \alpha$. Given this and by symmetry, $\lambda_{km} = \alpha$ and $\mu_{km} = 0$ for all $m \in \bar{S} \setminus \{l\}$.

By comparing the 3-cycle (i, l, k) and the 4-cycle (i, l, m, k), we get $\mu_{im} = 0$ for all $i \in S \setminus \{k\}$ and all $m \in \bar{S} \setminus \{l\}$. Last of all, by comparing the 3-cycle (k, i, l) and the 4-cycle (k, i, m, l), we obtain $\lambda_{im} = \alpha$, for all $i \in S \setminus \{k\}$, $m \in \bar{S} \setminus \{l\}$. $\square$

Theorem 4.3 does not hold for $7 \leq |V| \leq 9$, but it actually holds for $|V| = 6$, $|S| = 5$ and $4 \leq b \leq |V|$ (and for $|S| = 1$ which is the symmetric case). This can be verified by generating $2|E|$ affinely independent feasible points that satisfy (18) at equality.

Theorem 4.4 *Given any $G = (V, E)$, $|V| \geq 6$, $b \geq 4$, the constraints, given as:*

$$\sum_{e \in (i,S)} z_e \leq \frac{1}{2} \sum_{e \in \delta(i)} x_e + \sum_{e \in E(S)} z_e, \forall i \in V, S \subset V \setminus \{i\}, 1 \leq |S| \leq |V| - 5, \tag{21}$$

are facet-defining for $\tilde{P}^{nb}_{QS}$.

Constraint (21) is obtained by replacing y_i by $\frac{1}{2} \sum_{e \in \delta(i)} x_e$ in (14) and is a generalization of (3).

Proof. $\tilde{P}_{QS}^{nb} \cap \Big\{ \sum_{e\in(i,S)} z_e = \frac{1}{2} \sum_{e\in\delta(i)} x_e + \sum_{e\in E(S)} z_e \Big\}$ defines a proper face since the 0-cycle satisfies the constraint at equality whereas the 3-cycle (i,p,q), for $p,q \in \bar{S}\setminus\{i\}$, for $p \neq q$, does not. Now, we need to show that if $\lambda \cdot x + \mu \cdot z = \lambda_0$ for all $x \in \tilde{P}_{QS}^{nb} \cap \Big\{ \frac{1}{2} \sum_{e\in\delta(i)} x_e + \sum_{e\in E(S)} z_e = \sum_{e\in(i,S)} z_e \Big\}$, then

$$\lambda_e = \begin{cases} \frac{1}{2}\alpha, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases} \quad \lambda_0 = 0; \text{ and } \mu_e = \begin{cases} \alpha, & \text{if } e \in E(S), \\ -\alpha, & \text{if } e \in (i,S), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle we get $\lambda_0 = 0$. W.l.o.g. let R, k be arbitrary for $R \subseteq \bar{S}\setminus\{i\}$, $|R| = 4$ and $k \in S$. Consider the subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$, $\tilde{V} \subseteq V$, $\tilde{V} = R \cup \{k\}$ and $\tilde{E} = E(\tilde{V})$. Constraint (21) holds with equality for all cycles in $\tilde{G}$, hence by Proposition 4.1, $\lambda_e = \mu_e = 0$ for all $e \in \tilde{E}$. Since R and k are arbitrary, $\lambda_e = \mu_e = 0$ for all $e \in E(\bar{S}\setminus\{i\}) \cup (S, \bar{S}\setminus\{i\})$.

Let $k \in S$ and $p,q \in \bar{S}\setminus\{i\}$, $p \neq q$ be arbitrary. By comparing the cycles (k,i,p,q) and (k,i,p), we obtain $\lambda_{pq} + \lambda_{kq} - \lambda_{kp} + \mu_{kq} + \mu_{iq} + \mu_{pq} = 0$. Since $\lambda_{pq} = \lambda_{kq} = \lambda_{kp} = \mu_{kq} = \mu_{pq} = 0$, $\mu_{iq} = 0$. Since k, p and q are arbitrary, $\mu_{ip} = 0$ for all $p \in \bar{S}\setminus\{i\}$.

Let $k \in S$ and $p,q \in \bar{S}\setminus\{i\}$, $p \neq q$ be arbitrary. By comparing the cycles (k,p,i,q) and (k,p,i), we obtain $\lambda_{kq} + \lambda_{iq} - \lambda_{ki} + \mu_{kq} + \mu_{pq} + \mu_{iq} = 0$. Since $\lambda_{kq} = \mu_{kq} = \mu_{pq} = \mu_{iq} = 0$, $\lambda_{iq} = \lambda_{ki}$ are constant and let the constant be $\frac{1}{2}\alpha$. Since k, p and q are arbitrary, $\lambda_e = \frac{1}{2}\alpha$ for all $e \in \delta(i)$.

Let $k \in S$ and $p \in \bar{S}\setminus\{i\}$ be arbitrary. Consider the cycle (i,k,p) to obtain $\mu_{ik} = -\alpha$ for all $k \in S$.

If $|S| = 1$, we are done. Otherwise, let $k,l \in S$, $k \neq l$ and $p \in \bar{S}\setminus\{i\}$ be arbitrary. By comparing the cycles (k,l,i,p) and (k,i,l,p), we obtain $\lambda_{kl} + \lambda_{ip} = \lambda_{ki} + \lambda_{lp}$. Since $\lambda_{lp} = 0$ and $\lambda_{ip} = \lambda_{ki} = \frac{1}{2}\alpha$, $\lambda_{kl} = 0$. Since k, l and p are arbitrary, $\lambda_e = 0$ for all $e \in E(S)$.

Finally, let $k,l \in S$, $k \neq l$ be arbitrary. Consider the cycle (i,k,l) to obtain $\mu_{kl} = \alpha$. Since k and l are arbitrary, $\mu_e = \alpha$ for $e \in E(S)$. □

Theorem 4.5 *Given any $G = (V,E)$, $|V| \geq 5$, $b \geq 5$, the constraints, given as:*

$$\sum_{e\in E(S)} z_e + 1 \geq \sum_{e\in E(S)} x_e + \frac{1}{2} \sum_{e\in\delta(S)} x_e, \forall S \subset V, 2 \leq |S| \leq |V| - 3, \tag{22}$$

are facet-defining for $\tilde{P}_{QS}^{nb}$.

Constraint (22) is obtained by replacing y_i by $\frac{1}{2} \sum_{e\in\delta(i)} x_e$ in (15). Note that (4) is a special case of (22).

Proof. $\tilde{P}^{nb}_{QS} \cap \Big\{ \sum_{e\in E(S)} z_e + 1 = \sum_{e\in E(S)} x_e + \frac{1}{2} \sum_{e\in\delta(S)} x_e \Big\}$ defines a proper face since the 3-cycle (i,j,k), $i \in S$, $j,k \in \bar{S}$ satisfies the constraint at equality and the 0-cycle does not.

Now, we need to show that if $\lambda\cdot x + \mu\cdot z = \lambda_0$ for all $x \in \tilde{P}^{nb}_{QS} \cap \Big\{ \sum_{e\in E(S)} z_e + 1 = \sum_{e\in E(S)} x_e + \frac{1}{2} \sum_{e\in\delta(S)} x_e \Big\}$, then

$$\lambda_e = \begin{cases} -\alpha, & \text{if } e \in E(S), \\ -\frac{1}{2}\alpha, & \text{if } e \in \delta(S), \\ 0, & \text{otherwise;} \end{cases} \qquad \lambda_0 = \alpha; \text{ and } \mu_e = \begin{cases} \alpha, & \text{if } e \in E(S), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

W.l.o.g. let $R \subseteq S$, $|R| = 2$ and $T \subseteq \bar{S}$, $|T| = 3$ be arbitrary. Consider the subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$, $\tilde{V} \subseteq V$, $\tilde{V} = R \cup T$, (so $|\tilde{V}| = 5$) and $\tilde{E} = E(\tilde{S})$. Let $\lambda_0 = \alpha$. Let the matrix X be generated by the incident vectors of all the cycles in $\tilde{G}$ for which (22) holds with equality. X is found to be of rank $2|\tilde{E}| = 20$, thus $X(\lambda,\mu)^T = \alpha$ has an unique solution. The solution is $\lambda_e = -\alpha$ for all $e \in E(R)$, $\lambda_e = -\frac{1}{2}\alpha$ for all $e \in \delta(R)$, and $\lambda_e = 0$ for all $e \in E(T)$; $\mu_e = \alpha$ for all $e \in E(R)$ and $\mu_e = 0$ for all $e \in \delta(R) \cup E(T)$. Since R is arbitrary in S, T is arbitrary in $\bar{S}$, and each $e \in E$ is in this arbitrarily chosen $\tilde{G}$, $\lambda_e = -\alpha$ for all $e \in E(S)$, $\lambda_e = -\frac{1}{2}\alpha$ for all $e \in \delta(S)$ and $\lambda_e = 0$ for all $e \in E(\bar{S})$, $\mu_e = \alpha$ for all $e \in E(S)$ and $\mu_e = 0$ for all $e \in \delta(S) \cup E(\bar{S})$. □

The following constraints are found to be very effective in practise when solving QSTSPs using a branch-and-cut method (see [Stidsen and Thomadsen]):

$$\sum_{e\in\delta(i)} z_e \le \frac{b-1}{2} \sum_{e\in\delta(i)} x_e, \; \forall i \in V. \tag{23}$$

Constraint (23) is obtained by replacing y_i with $\frac{1}{2} \sum_{e\in\delta(i)} x_e$ in constraint (16).

Theorem 4.6 *Given any QSTSP defined on an undirected graph $G = (V,E)$, with $|V| \ge 6$ and $4 \le b \le |V|-1$, the constraints given by (23) are facet-defining for the QSTS polytope, $\tilde{P}^{nb}_{QS}$.*

Proof. $\tilde{P}^{nb}_{QS} \cap \Big\{ \sum_{e\in\delta(i)} z_e = \frac{b-1}{2} \sum_{e\in\delta(i)} x_e \Big\}$ defines a proper face, since any 3-cycle (i,j,k), $j,k \in V \setminus \{i\}$, $j \ne k$ does not satisfy the constraint at equality whereas the 0-cycle does.

Now, we are left to show that if $\lambda\cdot x + \mu\cdot z = \lambda_0$ for all $x \in \tilde{P}^{nb}_{QS} \cap \Big\{ \sum_{e\in\delta(i)} z_e = \frac{b-1}{2} \sum_{e\in\delta(i)} x_e \Big\}$, then

$$\lambda_e = \begin{cases} \frac{\alpha(b-1)}{2}, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases} \qquad \lambda_0 = 0; \text{ and } \mu_e = \begin{cases} -\alpha, & \text{if } e \in \delta(i), \\ 0, & \text{otherwise;} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

By considering the 0-cycle, it can be obtained that $\lambda_0 = 0$. Now, consider any arbitrary subgraph $\tilde{G} = (\tilde{V}, \tilde{E})$ for $\tilde{V} \subseteq V \setminus \{i\}$, $|\tilde{V}| = 5$, and $\tilde{E} = E(\tilde{V})$. Since all cycles in $\tilde{G}$ satisfies constraint (23) at equality, it follows from Proposition 4.1 that $\lambda_f = \mu_f = 0$, for all $f \in \tilde{E}$. As $\tilde{G}$ is arbitrary, we have $\lambda_f = \mu_f = 0$, for all $f \in E(V \setminus \{i\})$.

Let $\{i_1, \ldots, i_{b-1}\} \subseteq V \setminus \{i\}$ be arbitrary. Now compare the two b-cycles $(i, i_1, i_2, i_3, \ldots, i_{b-1})$ and $(i, i_2, i_1, i_3, \ldots, i_{b-1})$. This gives $\lambda_{ii_1} + \lambda_{i_2i_3} = \lambda_{ii_2} + \lambda_{i_1i_3}$. Since $\lambda_{i_2i_3} = \lambda_{i_1i_3} = 0$, λ_{ii_a} is constant for $a = 1, \ldots, b-1$ and let the constant be $\frac{\alpha(b-1)}{2}$. Since $\{i_1, \ldots, i_{b-1}\} \subseteq V \setminus \{i\}$ is arbitrary, $\lambda_{ij} = \frac{\alpha(b-1)}{2}$ for all $j \in V \setminus \{i\}$. Finally compare the b-cycle $(i, i_1, i_2, i_3, \ldots, i_{b-1})$ with the $(b-1)$-cycle $(i_1, i_2, i_3, \ldots, i_{b-1})$ to obtain $\lambda_{ii_1} + \lambda_{ii_{b-1}} - \lambda_{i_1i_{b-1}} + \sum_{k=1}^{b-1} \mu_{ik} = 0$. Since $\lambda_{i_1i_{b-1}} = 0$ and $\lambda_{ii_1} = \lambda_{ii_{b-1}} = \frac{\alpha(b-1)}{2}$, $\mu_{ii_a} = -\alpha$ for $a = 1, \ldots, b-1$. Since $\{i_1, \ldots, i_{b-1}\} \subseteq V \setminus \{i\}$ is arbitrary, $\mu_{ij} = -\alpha$ for all $j \in V \setminus \{i\}$ and the theorem is proved. □

5 Conclusion

In this paper, we studied the polyhedra of the Quadratic Knapsack Problem and the Quadratic Selective Travelling Salesman Problem. For each of these polytopes, we established its dimension, identified a number of strong constraints, and proved that these constraints are indeed facet-defining cuts. Various mathematical techniques were used in proving these results.

These results are of great significance in the implementation of a branch-and-cut method for obtaining exact solutions. The benefit of using such facet-defining cuts is that it improves the quality of the linear programming relaxation bounds.

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