

# Gaussian Integrals

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Version 2, January 5, 2005

## 1 Product of Gaussians

Suppose that  $\mathbf{x} \in \mathcal{R}^d$  and

$$p_i(\mathbf{x}) \sim \mathcal{N}(\mathbf{a}_i, \mathbf{A}_i) = \frac{1}{|2\pi \mathbf{A}_i|^{\frac{1}{2}}} e^{-\frac{1}{2} \mathbf{x}_i^\top \mathbf{A}_i^{-1} \mathbf{x}_i} \quad (1)$$

where  $\mathbf{x}_i = \mathbf{x} - \mathbf{a}_i$ ,  $i = 1, 2, \dots, M$ .

Consider the evaluation of integrals like

$$\int |\mathbf{x}|^q \prod_i p_i(\mathbf{x}) d\mathbf{x}, \quad \int |\mathbf{x}|^q \prod_i p_i(\mathbf{x}) d\mathbf{x} \quad (2)$$

The solutions can be obtained by using the fundamental proposition Eq. (6).

## 2 Fundamental Proposition

Define a common Gaussian density  $p_c(\mathbf{x}) \sim \mathcal{N}(\mathbf{c}, \mathbf{C})$  with

$$\mathbf{x}_c = \mathbf{x} - \mathbf{c} \quad (3)$$

$$\mathbf{x}_c = \mathbf{C} \left( \sum_i \mathbf{A}_i^{-1} \mathbf{x}_i \right) \quad (4)$$

$$\mathbf{C} = \left( \sum_i \mathbf{A}_i^{-1} \right)^{-1} \quad (5)$$

then

$$p_c(\mathbf{x}) = \frac{\prod_\ell |2\pi \mathbf{A}_\ell|^{\frac{1}{2}}}{|2\pi \mathbf{C}|^{\frac{1}{2}}} \cdot \prod_k p_k(\mathbf{x}) \cdot \prod_{i>j} \exp \left[ \frac{1}{2} (\mathbf{a}_i - \mathbf{a}_j)^\top \mathbf{B}_{ij} (\mathbf{a}_i - \mathbf{a}_j) \right] \quad (6)$$

where

$$\mathbf{B}_{ij} = \mathbf{B}_{ji} = \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} \quad (7)$$

**Proof** Consider writing

$$\mathbf{x}_c^\top \mathbf{C}^{-1} \mathbf{x}_c = \sum_i \mathbf{x}_i^\top \mathbf{A}_i^{-1} \mathbf{x}_i - \sum_{i>j} (\mathbf{x}_i - \mathbf{x}_j)^\top \mathbf{B}_{ij} (\mathbf{x}_i - \mathbf{x}_j) \quad (8)$$

The left hand side of (8) can cf. (4) and (5) be expressed as:

$$\mathbf{x}_c^\top \mathbf{C}^{-1} \mathbf{x}_c = \sum_{ij} \mathbf{x}_i^\top \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} \mathbf{x}_j \quad (9)$$

The right hand side of (8) can be rewritten as

$$\sum_i \mathbf{x}_i^\top \left( \mathbf{A}_i^{-1} - \sum_{j \neq i} \mathbf{B}_{ij} \right) \mathbf{x}_i + \sum_{j \neq i} \mathbf{x}_i^\top \mathbf{B}_{ij} \mathbf{x}_j \quad (10)$$

Comparison of (9) and (10) provides the relations:

$$\mathbf{B}_{ij} = \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} \quad (11)$$

$$\mathbf{A}_i^{-1} - \sum_{j \neq i} \mathbf{B}_{ij} = \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_i^{-1} \quad (12)$$

Equation (12) is true since

$$\mathbf{A}_i^{-1} - \sum_{j \neq i} \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} = \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_i^{-1} \quad (13)$$

$\Leftrightarrow$

$$\mathbf{A}_i^{-1} - \sum_j \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} = \mathbf{0} \quad (14)$$

$\Leftrightarrow$

$$\sum_j \mathbf{A}_i^{-1} \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} = \mathbf{A}_i^{-1} \quad (15)$$

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$$\sum_j \left( \sum_k \mathbf{A}_k^{-1} \right)^{-1} \mathbf{A}_j^{-1} = \mathbf{I} \quad (16)$$

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$$\sum_j \mathbf{A}_j^{-1} = \sum_k \mathbf{A}_k^{-1} \quad (17)$$

### 3 General Cases

#### 3.1 Integral of products

Since  $\int p_c(\mathbf{x}) d\mathbf{x} = 1$  then by using (6)

$$I = \int \prod_k p_k(\mathbf{x}) d\mathbf{x} = \frac{|2\pi\mathbf{C}|^{\frac{1}{2}}}{\prod_\ell |2\pi\mathbf{A}_\ell|^{\frac{1}{2}}} \cdot \prod_{i>j} \exp \left[ -\frac{1}{2}(\mathbf{a}_i - \mathbf{a}_j)^\top \mathbf{B}_{ij}(\mathbf{a}_i - \mathbf{a}_j) \right] \quad (18)$$

### 4 Special Cases

#### 4.1 Propagation of mean value

Consider the evaluation of  $\int \phi_j(\mathbf{x})p(\mathbf{x}) d\mathbf{x}$  where  $p(\mathbf{x}) = \mathcal{N}(\mathbf{u}, \mathbf{S})$  and

$$\phi_j(\mathbf{x}) = \exp[-(\mathbf{x} - \mathbf{x}_j)^\top \mathbf{\Lambda}^{-1}(\mathbf{x} - \mathbf{x}_j)/2] = |2\pi\mathbf{\Lambda}|^{\frac{1}{2}} \cdot \mathcal{N}(\mathbf{x}_j, \mathbf{\Lambda}) \quad (19)$$

In the case  $M = 2$ , cf. (5) and (7),

$$\mathbf{B}_{12} = \mathbf{A}_1^{-1} (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1} \mathbf{A}_2^{-1} = (\mathbf{A}_1(\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})\mathbf{A}_2)^{-1} = (\mathbf{A}_1 + \mathbf{A}_2)^{-1} \quad (20)$$

$$\mathbf{C} = (\mathbf{A}_1^{-1} + \mathbf{A}_2^{-1})^{-1} \quad (21)$$

According to (18) with  $\mathbf{a}_1 = \mathbf{x}_j$ ,  $\mathbf{A}_1 = \mathbf{\Lambda}$ ,  $\mathbf{a}_2 = \mathbf{u}$ , and  $\mathbf{A}_2 = \mathbf{S}$ .

$$\begin{aligned} I &= \frac{|\mathbf{C}|^{\frac{1}{2}}}{|2\pi\mathbf{A}_1|^{\frac{1}{2}}|2\pi\mathbf{A}_2|^{\frac{1}{2}}} \cdot \exp \left[ -\frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2)^\top (\mathbf{A}_1 + \mathbf{A}_2)^{-1} (\mathbf{a}_1 - \mathbf{a}_2) \right] \\ &= \frac{1}{|2\pi\mathbf{A}_1|^{\frac{1}{2}} \sqrt{|\mathbf{C}^{-1}\mathbf{A}_2|}} \cdot \exp \left[ -\frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2)^\top (\mathbf{A}_1 + \mathbf{A}_2)^{-1} (\mathbf{a}_1 - \mathbf{a}_2) \right] \\ &= \frac{1}{|2\pi\mathbf{A}_1|^{\frac{1}{2}} |\mathbf{A}_1^{-1}\mathbf{A}_2 + \mathbf{I}|^{\frac{1}{2}}} \cdot \exp \left[ -\frac{1}{2}(\mathbf{a}_1 - \mathbf{a}_2)^\top (\mathbf{A}_1 + \mathbf{A}_2)^{-1} (\mathbf{a}_1 - \mathbf{a}_2) \right] \end{aligned} \quad (22)$$

That is,

$$\begin{aligned} I_j &= \int \phi_j(\mathbf{x})p(\mathbf{x}) d\mathbf{x} = |2\pi\mathbf{\Lambda}|^{\frac{1}{2}} \int \mathcal{N}(\mathbf{x}_j, \mathbf{\Lambda})\mathcal{N}(\mathbf{u}, \mathbf{S}) d\mathbf{x} \\ &= |\mathbf{\Lambda}^{-1}\mathbf{S} + \mathbf{I}|^{-\frac{1}{2}} \cdot \exp \left[ -\frac{1}{2}(\mathbf{x}_j - \mathbf{u})^\top (\mathbf{\Lambda} + \mathbf{S})^{-1} (\mathbf{x}_j - \mathbf{u}) \right] \end{aligned} \quad (23)$$