Frequency Dependent Stability for Inverse Boundary Value Problems

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Kongens Lyngby 2014
Master of Science

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Summary

The purpose of this thesis is to investigate how the stability of an inverse boundary value problem depends on the frequency of the underlying partial differential equations model.

The thesis consists of an introduction to the inverse boundary value problem of the Helmholtz equation with a potential followed by an investigation of the linearised problem and the stability hereof. The main work is a detailed study of the proofs in [9] and how the result translates when considering the Dirichlét-to-Neumann map. We establish a stability estimate for a general frequency and conclude that the stability increases with frequency. Furthermore, for a particular problem investigation of the optimality of the estimate is performed and it is concluded that a sharper estimate holds in this particular case.

Preface

This thesis was prepared at the department of Applied Mathematics and Computer Science at the Technical University of Denmark in fulfilment of the requirements for acquiring a M.Sc. in Mathematical Modelling and Computing. It represents the completion of my honors master program at DTU and the workload corresponds to 30 ECTS points. The thesis was conducted in the spring semester of 2014 under the supervision of Associate Professor Kim Knudsen.

Primarily the thesis examines the relationship between the stability of an inverse boundary value problem for the Helmholtz equation with a potential and the frequency of the problem. The main result is based on the article [9] by Isakov, Nagasayu and Wang. It furthermore treats the subjects of linearisation of the inverse problem and investigation of the optimality of the stability estimate derived in a particular setting.

The prerequisites for reading this thesis is a basic understanding of the theory of partial differential equations, functional analysis and the concept of weak solutions of second order elliptic partial differential equations including familiarity with Sobolev spaces. Appendix A includes some useful definitions and results regarding Sobolev spaces used in the thesis.

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June, 2014

Acknowledgements

First and foremost, I would like to thank my supervisor and honors mentor Associate Professor Kim Knudsen from DTU Compute for presenting me with the subject of this thesis and for taking the time to discuss the issues of the problem with me.

Kim has been actively involved in my studies during the entire master program and his supervision and our discussions have been invaluable in general and for this project in particular. Also, the special courses Kim has held for my benefit have helped me to develope a more intuitive understanding of some very theoretical subjects.

I would also like to thank Professor Ole Christensen for many academic discussions had particularly during my time as a teaching assistant, and my fellow student Berit M. Martensen for a good working relationship during many projects and courses.

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Chapter 1

Introduction and motivation

The study of inverse boundary value problems is the branch of applied mathematics, that arise from the need to find information about quantities in the interior of a domain from measurements made at the boundary. A classic example is the Calderón problem, also known as the inverse conductivity problem. Calderón asked the question, if it is possible to determine the electrical conductivity in a body from measurements of the relationship between voltage and current at the boundary of the body. The inverse conductivity problem turns out to be a non-linear problem. Linear inverse problems are in many ways alike and can be described using singular value expansion, see for example [10]. Non-linear inverse problems on the other hand are almost all unique and must be investigated separately. A good starting point, when investigating a non-linear problem is to linearise the problem around an educated guess and use the methods of linear inverse problems. However, if the problem is in some sense *very* non-linear and a good guess is not available, there is no way to be sure that the conclusions from the linearisation will give any useful information.

In general many inverse problems are known to be have poor stability, meaning they are very sensitive to measurements errors. Put in another way, poor stability means that two sets of boundary data close to each other can give rise to two very different solutions. The inverse conductivity problem is known to have very poor stability.

We will use the inverse conductivity problem as a motivation to a wider range of problems, where a parameter k is added. The conductivity equation will correspond to the case of zero k.

The main goal of this thesis is to investigate how the stability of a non-linear inverse boundary value problem depends on the frequency of the underlying partial differential equations model in the case of the Helmholtz equation with a potential by use of [9]. As a starting point we wish to examine the linearised problem and the stability hereof. Also when a stability estimate is derived for a general frequency, the question of optimality in some sense is interesting to ask and examine for specific problems.

In section 1.1 the inverse conductivity problem is defined and the non-linearity of the problem observed. Section 1.2 transforms the problem into an inverse boundary value problem for the Schrödinger equation and defines an inverse problem in this setting. In section 1.3 the parameter k is finally introduced and the inverse boundary value problem, which is the root of the discussion of the rest of this work, is defined.

We will throughout this report work on a bounded subset of $\mathbb{R}^n$ denoted Ω with smooth boundary $\partial\Omega$ and assume that all functions are real-valued, unless otherwise stated. Also note the notation used in the proofs of stability estimates, where a general constant denotes different constants depending on the same quantities.

1.1 The inverse conductivity problem

Consider the elliptic partial differential equation called the conductivity equation

$$\nabla \cdot (\sigma(x)\nabla u(x)) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary. The equation describes the propagation of electromagnetic fields or waves in a body. (1.1) can be derived from Maxwell's equations, see for example [14]. u is the electric potential and σ the conductivity inside the body Ω. We assume that $\sigma = 1$ in a neighbourhood of the boundary $\partial\Omega$.

The inverse conductivity problem is also called electrical impedance tomography (EIT). It is defined and investigated in for example [14] and the definitions here follow this book. As mentioned, the inverse conductivity problem is the problem of determining the conductivity σ inside a body from electrical boundary measurements. The measurements in EIT can be performed by applying a voltage distribution through a set of electrodes on the boundary of the body and then measuring the corresponding current distribution at the boundary. The mapping of voltage to current is called the Dirichlét-to-Neumann map, since the voltage at the boundary is the Dirichlét data and the current is the natural Neumann data of the problem. The inverse problem of EIT is non-linear and the conductivity does not depend continuously on the data as shown by an example in [14], meaning that the problem lacks stability and is ill-posed.

Now the weak formulation of the problem (1.1) is

$$0 = \int_\Omega \nabla \cdot (\sigma(x)\nabla u(x))\, \phi \, dx = -\int_\Omega \sigma(x)\nabla u(x) \cdot \nabla\phi \, dx$$

for any test function $\phi \in C_c^\infty(\Omega)$.

For (1.1) to be a second order *elliptic* partial differential equation, there must exist a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} \sigma(x)\delta_{ij}\xi_i\xi_j = \sigma(x)\sum_{i=1}^{n}\xi_i\xi_i = \sigma(x)|\xi|^2 \geq \theta|\xi|^2$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$. This means that if we assume that

$$\sigma > 0,$$

then (1.1) is an elliptic equation. We also assume that $\sigma \in L^\infty(\Omega)$.

For the Dirichlét problem

$$\begin{aligned} \nabla \cdot (\sigma(x)\nabla u(x)) &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

to have a unique solution $u \in H^1(\Omega)$, when $f \in H^{1/2}(\partial\Omega)$ and $\sigma \in L^\infty(\Omega)$, we assume that zero is not an eigenvalue of (1.1) ([5] p. 323). It then holds that $\|u\|_{H^1(\Omega)} \leq C\,\|f\|_{H^{1/2}(\partial\Omega)}$, where C depends on σ and Ω ([7] p. 183).

Now the current through the boundary is the normal derivative of the solution multiplied with σ and can be defined as an element in $H^{-1/2}(\partial\Omega)$. To do this let us define the Dirichlét-to-Neumann map

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_\sigma\left(u\Big|_{\partial\Omega}\right) = \sigma\frac{\partial u}{\partial\eta}\Big|_{\partial\Omega}. \tag{1.3}$$

The map Λ_σ is the data of the inverse problem of EIT. To see that $\Lambda_\sigma f \in H^{-1/2}(\partial\Omega)$ for $f \in H^{1/2}(\partial\Omega)$, define the dual pairing

$$\langle \Lambda_\sigma f, g\rangle = \int_\Omega \sigma\nabla u \cdot \nabla e_g \, dx, \tag{1.4}$$

where $u \in H^1(\Omega)$ solves (1.2) and $e_g \in H^1(\Omega)$ is an extension of g, so $e_g|_{\partial\Omega} = g$. If $\Lambda_\sigma f$ and g are elements of $L^2(\partial\Omega)$ this definition is identical to the inner product on $L^2(\partial\Omega)$, since by subtracting zero in the sense of (1.1) and using Green's theorem gives

$$\begin{aligned}
(\Lambda_\sigma f, g)_{L^2(\partial\Omega)} &= \int_{\partial\Omega} \Lambda_\sigma f g \, ds \\
&= \int_{\partial\Omega} \sigma \frac{\partial u}{\partial\eta}\Big|_{\partial\Omega} e_g|_{\partial\Omega} \, ds \\
&= \int_{\partial\Omega} \sigma \frac{\partial u}{\partial\eta} e_g \, ds - \int_\Omega (\nabla\sigma\nabla u)\, e_g \, dx \\
&= \int_\Omega \sigma\nabla u \cdot \nabla e_g \, dx.
\end{aligned}$$

First note that (1.4) is independent of the choice of extension e_g. Let e_g^1 and e_g^2 be two extensions of g to Ω. Then $\phi = e_g^1 - e_g^2 \in H_0^1(\Omega)$ and since u is a weak solution to (1.1) we find that

$$\begin{aligned}
\int_\Omega \sigma\nabla u \cdot \nabla e_g^1 \, dx - \left(\int_\Omega \sigma\nabla u \cdot \nabla e_g^2 \, dx\right) &= \int_\Omega \sigma\nabla u \cdot \nabla\phi \, dx \\
&= -\int_\Omega \nabla\sigma\nabla u\phi \, dx \\
&= 0,
\end{aligned}$$

so the definition (1.4) of $\Lambda_\sigma f$ is independent of the choice of extension. To see that is a linear bounded functional on $H^{1/2}(\partial\Omega)$, let $\alpha, \beta \in \mathbb{C}$, $g_1, g_2 \in H^{1/2}(\partial\Omega)$ and e_{g_1}, e_{g_2} be extensions of g_1 and g_2 respectively. This means that $(\alpha e_{g_1} + \beta e_{g_2})|_{\partial\Omega} = \alpha g_1 + \beta g_2$, so $\alpha e_{g_1} + \beta e_{g_2}$ is an extension of $\alpha g_1 + \beta g_2$ and it holds that

$$\begin{aligned}
\langle \Lambda_\sigma f, \alpha g_1 + \beta g_2\rangle &= \int_\Omega \sigma\nabla u \cdot \nabla\left(\alpha e_{g_1} + \beta e_{g_2}\right) \, dx \\
&= \alpha \int_\Omega \sigma\nabla u \cdot \nabla e_{g_1} \, dx + \beta \int_\Omega \sigma\nabla u \cdot \nabla e_{g_2} \, dx \\
&= \alpha\langle \Lambda_\sigma f, g_1\rangle + \beta\langle \Lambda_\sigma f, g_2\rangle.
\end{aligned}$$

Also

$$\begin{aligned}|\langle \Lambda_\sigma f, g\rangle| &\le \int_\Omega |\sigma \nabla u \cdot \nabla e_g| \ dx \\ &\le \|\sigma\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla e_g\|_{L^2(\Omega)} \\ &\le \|\sigma\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|e_g\|_{H^1(\Omega)} \\ &\le C \|\sigma\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|g\|_{H^{1/2}(\partial\Omega)},\end{aligned}$$

by Hölder's inequality and the assumption $\sigma \in L^\infty(\Omega)$. This means that Λ_σ is a linear and bounded functional on $H^{1/2}(\partial\Omega)$, hence an element of the dual space $H^{-1/2}(\partial\Omega)$ and the definition of the Dirichlét-to-Neumann map (1.3)-(1.4) is well-defined.

Knowing the Dirichlét-to-Neumann map is equivalent to knowing the relationship between the applied voltage and the measured current at the boundary and is, as mentioned, the data for the inverse problem of EIT. The forward problem of EIT can be described by the operator

$$\Lambda : \sigma \to \Lambda_\sigma.$$

Even though we have just shown that the Dirichlét-to-Neumann map is a linear and bounded operator the operator Λ is a non-linear operator. This can be seen from the weak definition of the Dirichlét-to-Neumann map (1.4). The solution u is a function of σ and in (1.4) ∇u is multiplied with σ, so the Dirichlét-to-Neumann map is a non-linear function of σ.

1.2 Transformation to Shrödinger's equation

It is possible to transform the partial differential equation (1.1) into a time-independent Schrödinger equation

$$(-\Delta + q)v = 0 \quad \text{in} \quad \Omega,$$

where $q \in L^\infty(\Omega)$ is a function of σ only. In general a partial differential equation of this form is a type of wave equation that describes the wave function u in a system with potential q.

The reduction to a Schrödinger equation is done since the primary part of the Schrödinger equation is the Laplacian and this makes it easier to work with. By doing this we will have moved the first order term acting on σ to a second order term acting on u and a zeroth order term q. It is necessarily to assume that $\sigma \in C^2(\Omega)$ to perform this transformation. Define

$$q(x) = \frac{\Delta\sqrt{\sigma(x)}}{\sqrt{\sigma(x)}} \quad \text{and} \quad v(x) = \sqrt{\sigma(x)}u(x). \tag{1.5}$$

Note that due to the positivity of σ we can also write $u = \sigma^{-1/2}v$. By the assumption $\sigma = 1$ near the boundary, now means that $q = 0$ near the boundary, hence q is compactly supported in Ω. Plugging the above expression for v (or u) into the conductivity equation (1.1) reduces it to a partial differential equation for v.

$$\begin{aligned}
0 &= \nabla \cdot (\sigma \nabla u) \\
&= \nabla \cdot (\sigma \nabla (\sigma^{-1/2} v)) \\
&= \nabla \cdot \left(-\frac{1}{2}\sigma^{-1/2} (\nabla\sigma)\, v + \sigma^{1/2}\nabla v \right) \\
&= \frac{1}{4}\sigma^{-3/2}\nabla\sigma \cdot \nabla\sigma v - \frac{1}{2}\sigma^{-1/2}\left((\Delta\sigma)\, v + \nabla\sigma\cdot\nabla v\right) + \frac{1}{2}\sigma^{-1/2}\nabla\sigma\cdot\nabla v + \sigma^{1/2}\Delta v \\
&= \left(\frac{1}{4}\sigma^{-3/2}\nabla\sigma\cdot\nabla\sigma - \frac{1}{2}\sigma^{-1/2}(\Delta\sigma)\right) v + \sigma^{1/2}\Delta v \\
&= -\Delta\left(\sigma^{1/2}\right) v + \sigma^{1/2}\Delta v,
\end{aligned}$$

since

$$\Delta\left(\sigma^{1/2}\right) = \nabla\cdot\left(\frac{1}{2}\sigma^{-1/2}\nabla\sigma\right) = -\frac{1}{4}\sigma^{-3/2}\nabla\sigma\cdot\nabla\sigma + \frac{1}{2}\sigma^{-1/2}\Delta\sigma.$$

Multiplying the above partial differential equation for v with $-\sigma^{-1/2}$ then gives the Schrödinger equation

$$0 = \left(-\Delta + \frac{\Delta\left(\sigma^{1/2}\right)}{\sigma^{1/2}}\right) v = \left(-\Delta + q\right) v.$$

We then arrive at the Dirichlét boundary value problem for the Schrödinger equation

$$\begin{aligned}
\left(-\Delta + q\right) v &= 0 \quad \text{in} \quad \Omega \\
v &= g \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{1.6}$$

Again we have a second order elliptic Dirichlét problem and if we assume that zero is not an eigenvalue of (1.6) there exists a unique solution ([5]) and it holds that $\|u\|_{H^1(\Omega)} \leq C\,\|g\|_{H^{1/2}(\partial\Omega)}$, where C depends on q and Ω ([7] p. 183).

Again we define the Dirichlét-to-Neumann map.

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_q g = \left.\frac{\partial v}{\partial \eta}\right|_{\partial\Omega},$$

where v and g satisfies (1.6). Note the notation here, where it is important to know that $\Lambda_q \neq \Lambda_\sigma$ for $q = \sigma$. In the following it will be clear from the context which definition is used. The dual pairing is defined by

$$\langle \Lambda_q g, h \rangle = \int_\Omega \nabla v \cdot \nabla e_h + q v e_h \, dx,$$

where e_h is any function in $H^1(\Omega)$ with $e_h \Big|_{\partial\Omega} = h$. Again this boils down to the $L^2(\partial\Omega)$ inner product when $\Lambda_q g$ and h lies in $L^2(\partial\Omega)$ by similar arguments as used for the conductivity equation. Also when $q \in L^\infty(\Omega)$ the Dirichlét-to-Neumann map is a bounded map from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, independent of the choice of extension e_h and symmetric. See [18] Lemma 3.4 for proof that is very similar to the one used for the conductivity equation. We will in section 1.3 prove similar results for a problem, where this is a special case, so it will be skipped here.

As mentioned the inverse problem of EIT has very poor stability and it is shown (for example in [18] Theorem 4.2) that when the potential q_l for $l = 1, 2$ is bounded as $\|q_l\|_{L^\infty(\Omega)} \leq M$ for some $M > 0$, it holds that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C \left|\log\left(\|\Lambda_{q_1} - \Lambda_{q_1}\|\right)\right|^{-\frac{2}{n+2}}, \tag{1.7}$$

for small enough operator norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$, where C depends on n, Ω, M and $\text{supp}(q_2 - q_1)$. This means that we have logarithmic type stability, which is a very poor kind of stability, since the function $|\log(x)|^{-a}$ for $0 < a < 1$ grows exponentially for very small x. It means that even a very small change in Λ_q can result in a relatively much larger change in q. Mandache proved in [12] that for a general q, this logarithmic type of stability is optimal in some sense, that is, it is not possible to get for example Hölder type stability like $\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C \|\Lambda_{q_1} - \Lambda_{q_1}\|^a$ for a constant C and for some $a \in (0, 1)$.

1.3 Helmholtz equation with a potential

We will now introduce the parameter $k \geq 0$ and consider the more general boundary value problem defined by the Helmholtz equation with a potential q

$$\begin{aligned} (\Delta + k^2 + q(x))u(x) &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n \\ u &= f \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.8}$$

where $q \in L^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded set with smooth boundary $\partial\Omega$ as before. We are interested in the possible change of stability after adding the parameter k^2 and how this change depends on the size of k.

We can think of (1.8) as the propagation of acoustic waves in an inhomogeneous body with wave number, or frequency, k. The case $q = 0$ everywhere in Ω then corresponds to a homogeneous body.

Like in the previous sections define the Dirichlét-to-Neumann map Λ_q by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_q f = \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega}, \tag{1.9}$$

where u solves (1.8). Note again the notation. Here Λ_q is not equivalent to Λ_q defined in the previous section if $k \neq 0$. If we were to stay with the definition from the previous section, we could write Λ_{q+k^2} here, but we will from now on only consider the problem (1.8) and thus Λ_q refers to (1.9) unless otherwise clearly stated.

We again assume that zero is not an eigenvalue of (1.8), since then there exists a unique solution $u \in H^1(\Omega)$ of (1.8) ([5]). If this is not the case the Dirichlét-to-Neumann operator would not be well-defined, since there could exist more than one solution u.

The map (1.9) is well-defined and bounded by the following arguments. The weak definition of the Dirichlét-to-Neumann map for $f, g \in H^{1/2}(\partial\Omega)$ is

$$\langle \Lambda_q f, g \rangle = \int_\Omega \nabla u \cdot \nabla e_g \, dx - \int_\Omega (k^2 + q) u e_g \, dx, \tag{1.10}$$

where u solves (1.8) and $e_g \in H^1(\Omega)$ is any extension of g, that is $e_g|_{\partial\Omega} = g$.

Like with the Schrödinger equation in the previous section, if $\Lambda_q f$ and g are elements in $L^2(\partial\Omega)$ the dual pairing of the two elements must coincide with the $L^2(\partial\Omega)$-inner product. This is true since

$$\begin{aligned}
(\Lambda_q f, g)_{L^2(\partial\Omega)} &= \int_{\partial\Omega} \Lambda_q f \; g \, ds \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} e_g|_{\partial\Omega} \, ds \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \eta} e_g \, ds - \int_\Omega \left(\left(\Delta + k^2 + q\right) u \right) e_g \, dx \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \eta} e_g \, ds - \int_\Omega \Delta u e_g \, dx - \int_\Omega (k^2 + q) u e_g \, dx \\
&= \int_{\partial\Omega} \frac{\partial u}{\partial \eta} e_g \, ds - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} e_g \, ds + \int_\Omega \nabla u \cdot \nabla e_g \, dx - \int_\Omega (k^2 + q) u e_g \, dx \\
&= \int_\Omega \nabla u \cdot \nabla e_g \, dx - \int_\Omega (k^2 + q) u e_g \, dx,
\end{aligned}$$

which is equivalent to (1.10).

Now let $f \in H^{1/2}(\partial\Omega)$. As before we need to show that (1.10) is uniquely defined, well-defined, linear and bounded to see that it defines an element of the dual space $H^{-1/2}(\partial\Omega)$.

First, to show uniqueness, we need to show that (1.10) does not depend of the particular choice of the extension e_g of g. To do this, let e_g^1 and e_g^2 be two extensions of g in $H^1(\Omega)$. Denote the element $\phi \in H_0^1(\Omega)$ by

$$\phi = e_g^1 - e_g^2.$$

Since u is a weak solution of (1.8) it holds that

$$0 = \int_\Omega (\Delta + k^2 + q)u\phi \, dx = -\int_\Omega \nabla u \cdot \nabla\phi \, dx + \int_\Omega (k^2+q)u\phi \, dx,$$

so the difference of (1.10) when using two different extensions is

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla e_g^1 \, dx - \int_\Omega (k^2+q)ue_g^1 \, dx - \left(\int_\Omega \nabla u \cdot \nabla e_g^2 \, dx - \int_\Omega (k^2+q)ue_g^2 \, dx\right) \\ = \int_\Omega \nabla u \cdot \nabla\phi \, dx - \int_\Omega (k^2+q)u\phi \, dx \, dx \\ = 0.\end{aligned}$$

Hence (1.10) is independent of the choice of extension of g.

Second, let $\alpha, \beta \in \mathbb{C}$, $g_1, g_2 \in H^{1/2}(\partial\Omega)$ and let e_{g_1}, e_{g_2} be extensions of g_1 and g_2 respectively, so $(\alpha e_{g_1} + \beta e_{g_2})|_{\partial\Omega} = \alpha g_1 + \beta g_2$ and

$$\begin{aligned}\langle \Lambda_q f, (\alpha g_1 + \beta g_1)\rangle &= \int_\Omega \nabla u \cdot \nabla(\alpha e_{g_1} + \beta e_{g_2}) \, dx - \int_\Omega (k^2+q)u(\alpha e_{g_1} + \beta e_{g_2}) \, dx \\ &= \alpha\langle \Lambda_q f, g_1\rangle + \beta\langle \Lambda_q f, g_1\rangle.\end{aligned}$$

(1.10) is well-defined (and hence bounded) since $q \in L^\infty(\Omega)$ and by use of Hölder's inequality it holds that

$$\begin{aligned}\langle \Lambda_q f, g\rangle &= \left|\int_\Omega \nabla u \cdot \nabla e_g \, dx - \int_\Omega (k^2+q)ue_g \, dx\right| \\ &\le \|\nabla u\|_{L^2(\Omega)} \|\nabla e_g\|_{L^2(\Omega)} + \left(k^2 + \|q\|_{L^\infty(\Omega)}\right) \|u\|_{L^2(\Omega)} \|e_g\|_{L^2(\Omega)} \\ &\le \left(\|\nabla u\|_{L^2(\Omega)} + \left(k^2 + \|q\|_{L^\infty(\Omega)}\right) \|u\|_{L^2(\Omega)}\right) \|e_g\|_{H^1(\Omega)} \\ &\le C\left(\|\nabla u\|_{L^2(\Omega)} + \left(k^2 + \|q\|_{L^\infty(\Omega)}\right) \|u\|_{L^2(\Omega)}\right) \|g\|_{H^{1/2}(\partial\Omega)},\end{aligned}$$

where C is a constant depending on k and Ω. This means that $\Lambda_q f(g) = \langle \Lambda_q f, g\rangle$ is a bounded linear functional on $H^{1/2}(\partial\Omega)$, hence and element of $H^{-1/2}(\partial\Omega)$. Also note that

$$\begin{aligned}
\|\Lambda_q f\| &\leq C\left(\|\nabla u\|_{L^2(\Omega)} + \left(k^2 + \|q\|_{L^\infty(\Omega)}\right)\|u\|_{L^2(\Omega)}\right) \\
&\leq C\max\{1, k^2 + \|q\|_{L^\infty(\Omega)}\}\left(\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \\
&\leq C\max\{1, k^2 + \|q\|_{L^\infty(\Omega)}\}\sqrt{2}\,\|u\|_{H^1(\Omega)} \\
&\leq C_k\max\{1, k^2 + \|q\|_{L^\infty(\Omega)}\}\sqrt{2}\,\|f\|_{H^{1/2}(\Omega)} \\
&\leq C_k\,\|f\|_{H^{1/2}(\Omega)},
\end{aligned}$$

where C_k depends on k and Ω.

In conclusion (1.10) defines a dual pairing on $\left(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)\right)$ and

$$|\langle \Lambda_q f, g\rangle| \leq \|\Lambda_q f\|_{H^{-1/2}(\partial\Omega)}\,\|g\|_{H^{1/2}(\partial\Omega)}, \tag{1.11}$$

where

$$\|\Lambda_q f\|_{H^{-1/2}(\partial\Omega)} \leq C_k\,\|f\|_{H^{1/2}(\partial\Omega)}, \tag{1.12}$$

so the Dirichlét-to-Neumann map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is well-defined and bounded.

As mentioned (1.8) is the subject of the rest of this work. Let us define the define the forward operator of our problem by

$$\Lambda : q \to \Lambda_q. \tag{1.13}$$

The inverse problem is then to determine the potential q from knowledge of the Dirichlét-to-Neumann operator (1.9), that is, our inverse operator is

$$\Lambda^{-1} : \Lambda_q \to q, \tag{1.14}$$

where Λ_q is defined weakly by (1.10). The stability of the inverse problem depends on the choice of spaces. By now, it is assumed that $q \in L^\infty$, but we will later on assume more regularity on q, namely that $q \in H^s(\Omega)$ for some non-negative integer s.

As it was the case with the inverse problem of EIT discussed in the section 1.1, this problem is too non-linear by similar arguments. The solution u of (1.8) depends on q and the second integral in (1.10) contains a multiplication of u and q, meaning that the Dirichlét-to-Neumann map Λ_q is a non-linear function of q.

The stability estimate (1.7) tells us that in the case $k = 0$ this inverse problem has logarithmic type stability, which is a very poor type of stability. The rest of this work is dedicated to the investigation of how the stability of the inverse problem behaves when k increases,, so we are looking to derive a stability estimate of the inverse problem (1.14) for a general k. This will be done mainly based on the article [9] by Isakov, Nagayasu, Uhlmann and Wang. Furthermore, it is interesting to numerically see how the operator norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ behaves with k for specific choices of q_1, q_2 and Ω. This would give an indication of how an optimal stability estimate looks, at least in the specific chosen case.

First, since our problem is non-linear a good starting point is to look into the stability of the linearised problem. As mentioned, for this to give a good indication of the real non-linear problem, a good starting guess of q is needed. Practically, this depends on the specific application of the problem.

Chapter 2

Linearisation

In [2] Calderón considered the inverse conductivity problem. He assumes that the conductivity is constant one plus a perturbation function, that is, $\sigma = 1 + \delta$. He then linearises the problem around $\sigma = 1$ and shows uniqueness of this linearised problem, where the aim now is to find the perturbation function δ. Inspired by this and the demonstration of Calderón's results in [14], this chapter treats the linearisation of the inverse boundary value problem of the Helmholtz equation with potential (1.14). Section 2.1 deals with finding the Fréchet derivative and hence the linearisation. Furthermore, the question of uniqueness of the linearised problem is examined.

As mentioned, a linearised version of non-linear inverse problem are often investigated, since it is an easier problem to deal with and it may give an indication of how the non-linear problem behaves. In section 2.2 a stability estimate is derived for the linearised problem. The derivation is build on the article [9], but is simplified greatly since this article deals with the non-linear problem. The proof will also serve as a way to understand where the difficulties arise in chapter 3, where the stability of the non-linear problem is investigated based on the same article. A discussion of the assumption on the dimension is also included here.

2.1 Fréchet derivative

Recall the non-linear forward problem (1.13)

$$\Lambda : q \mapsto \Lambda_q,$$

where $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the Dirichlét-to-Neumann map defined weakly by (1.10). Inspired by Calderón we will linearise around $q = 0$, since $q = 0$ in the zero frequency case corresponds to $\sigma = 1$ in the conductivity equation.

Definition 2.1 *The Fréchet derivative of $F : U \subseteq B_1 \to B_2$, where B_1, B_2 are Banach spaces, is the linear and bounded map $dF : B_1 \to B_2$ satisfying*

$$\frac{\|F(u+h) - F(u) - dF(h)\|}{\|h\|} \to 0 \quad as \quad h \to 0. \tag{2.1}$$

Note that an equivalent way of writing Definition 2.1 is

$$F(u+h) = F(h) + dFh + o(\|h\|), \tag{2.2}$$

since $F(u+h) - F(u) - dFh \in o(\|h\|)$ means exactly (2.1).

Using this definition means that the Fréchet derivative of the weak definition of Λ satisfies

$$\langle \Lambda[q]f, g\rangle = \langle \Lambda[0]f, g\rangle + \left\langle d\Lambda\Big|_{q=0}[q]f, g \right\rangle + o(\|q\|),$$

where f is the boundary function of the problem (1.8) and g of (2.6) below. Rearranging gives

$$\langle (\Lambda_q - \Lambda_0)\, f, g\rangle = \left\langle d\Lambda\Big|_{q=0}[q]f, g \right\rangle + o(\|q\|), \tag{2.3}$$

Now our candidate to the Fréchet derivative can be described using the dual pairing

$$\left\langle d\Lambda\Big|_{q=0}[q]f, g \right\rangle = \int_\Omega q u_0 v \, dx, \tag{2.4}$$

where u_0 and v satisfy the linearised boundary value problems

$$\begin{aligned} (\Delta + k^2)u_0 &= 0 \quad \text{in} \quad \Omega \\ u_0 &= f \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} (\Delta + k^2)v = 0 &\quad \text{in} \quad \Omega \\ v = g &\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.6}$$

We write the potential is zero plus a perturbation function δ, that is,

$$q = 0 + \delta.$$

We now wish to show that (2.4) is in fact the Fréchet derivative, in the sense of (2.1). The proof below is inspired by [14].

Let $f, g \in H^{1/2}(\partial\Omega)$, u satisfy (1.8) and v satisfy (2.6). Then

$$\langle(\Lambda_q - \Lambda_0)f, g\rangle = \langle\Lambda_q f, g\rangle - \langle f, \Lambda_0 g\rangle = \int_\Omega (q-0)uv\, dx = \int_\Omega \delta uv\, dx.$$

Write $u = u_0 + \tilde{u}$, where u_0 satisfies (2.5). This means that $\tilde{u} \in H_0^1(\Omega)$ and satisfies

$$(\Delta + k^2 + q)\tilde{u} = (\Delta + k^2 + q)u_0 = (q-0)u_0 = \delta u_0 \qquad \text{in} \quad \Omega.$$

Plugging $u = u_0 + \tilde{u}$ into the expression for $\langle(\Lambda_q - \Lambda_0)f, g\rangle$ above gives

$$\langle(\Lambda_q - \Lambda_0)f, g\rangle = \int_\Omega \delta u_0 v\, dx + \int_\Omega \delta\tilde{u}v\, dx.$$

The first integral is exactly our candidate for the Fréchet derivative (2.4), so to show that it satisfies (2.3), we must show that the second integral is $o(\|\delta\|)$, meaning that

$$\frac{\left|\int_\Omega \delta\tilde{u}v\, dx\right|}{\|\delta\|} \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.7}$$

Since $\tilde{u} \in H_0^1(\Omega)$ solves the second order elliptic partial differential equation derived above we have ([7] p. 183)

$$\|\tilde{u}\|_{H^1(\Omega)} \leq C\|\delta u_0\|_{H^1(\Omega)} \leq C\|\delta\|_{L^\infty(\Omega)}\|u_0\|_{H^1(\Omega)} \leq C\|\delta\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)},$$

where C is a constant depending on k, q and Ω. Using this and Hölder's inequality gives the desired result

$$\begin{aligned} \frac{\left|\int_\Omega \delta\tilde{u}v\, dx\right|}{\|\delta\|} &\leq \frac{\|\delta\|_{L^\infty(\Omega)}\|\tilde{u}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}}{\|\delta\|} \\ &\leq \frac{\|\delta\|_{L^\infty(\Omega)} C\|\delta\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^1(\Omega)}}{\|\delta\|} \\ &\leq C\|\delta\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)}\|g\|_{H^{1/2}(\partial\Omega)} \\ &\to 0 \quad \text{as} \quad \delta \to 0, \end{aligned}$$

so (2.4)

$$\left\langle d\Lambda\Big|_{q=0}[q]f, g\right\rangle = \int_\Omega q u_0 v \, dx,$$

is the Fréchet derivative. Let us check that this in fact is an element of $H^{-1/2}(\partial\Omega)$ and see how the operator norm is bounded.

$$\begin{aligned}\left|\left\langle d\Lambda\Big|_{q=0}[q]f, g\right\rangle\right| &= \left|\int_\Omega q u_0 v \, dx\right| \\ &\leq \|q\|_{L^\infty(\Omega)} \|u_0\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq \|q\|_{L^\infty(\Omega)} \|u_0\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\leq C_k \|q\|_{L^\infty(\Omega)} \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}, \end{aligned} \tag{2.8}$$

where C_k depends on k and Ω. This means that

$$\left\|d\Lambda\Big|_{q=0}[q]f\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq \|q\|_{L^\infty(\Omega)} \|u_0\|_{L^2(\Omega)} \leq C_k \|q\|_{L^\infty(\Omega)} \|f\|_{H^{1/2}(\partial\Omega)}$$

for all f, so the operator is bounded by

$$\left\|d\Lambda\Big|_{q=0}[q]\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq C_k \|q\|_{L^\infty(\Omega)}. \tag{2.9}$$

In conclusion, we can compare the linearised and non-linear problems. If we think of the non-linear inverse problem as given

$$\langle(\Lambda_q - \Lambda_0) f, g\rangle = \int_\Omega q u v \, dx,$$

find q. The linearised inverse problem is then, given

$$\left\langle d\Lambda\Big|_{q=0}[q]f, g\right\rangle = \int_\Omega q u_0 v \, dx,$$

find q.

Before continuing with the stability of the linearised inverse problem, let us consider the question of uniqueness. This is again build on [14]. To do this note that writing $u_0 = u_1$, $f = f_1$ and $v = u_2$, $g = f_2$ (2.5) and (2.6) can be combined as

$$\begin{aligned}(\Delta + k^2)u_l(x) &= 0 \quad \text{in} \quad \Omega \\ u_l &= f_l \quad \text{in} \quad \partial\Omega,\end{aligned} \tag{2.10}$$

for $l = 1, 2$. There are solutions u_l of then form

$$u_l(x) = e^{i\xi_l \cdot x}, \tag{2.11}$$

where $\xi \in \mathbb{R}^n$ and $\xi \cdot \xi = k^2$. To see this we plug (2.11) into (2.10)

$$(\Delta + k^2)e^{i\xi\cdot x} = (i\xi)\cdot(i\xi)e^{i\xi\cdot x} + k^2 e^{i\xi\cdot x} = \left(-k^2 + k^2\right)e^{i\xi\cdot x} = 0.$$

Note that the requirement $\xi^2 = k^2$ is equivalent to

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \quad \text{and} \quad Re(\xi)\cdot Im(\xi) = 0. \tag{2.12}$$

Now let $\xi_l = \pi\alpha \pm i\pi\beta$, where $\alpha, \beta \in \mathbb{R}^n$ satisfies (2.12), that is,

$$\xi_1 = \pi\alpha + i\pi\beta, \quad \xi_2 = \pi\alpha - i\pi\beta.$$

Injectivity of the linearised problem then follows from assuming that

$$0 = \int_\Omega q u_1 u_2 \, dx = \int_\Omega q e^{i\pi(\alpha+i\beta)\cdot x} e^{i\pi(\alpha-i\beta)\cdot x} \, dx = \int_\Omega q e^{2\pi i\alpha\cdot x} \, dx,$$

since this means that the Fourier transform of the zero extension of q to $\mathbb{R}^n$ is zero for all α. Hence $q = 0$ in Ω and we can conclude that the inverse problem is unique.

2.2 Stability

We now wish to investigate the stability of the inverse linearised problem, where (2.4) represents the data and the goal is to find q. We are looking to find a function Φ, such that for two potentials q_1, q_2 we have a bound

$$\|q_1 - q_2\| \le \Phi\left(d\Lambda\Big|_{q=0}[q_1] - d\Lambda\Big|_{q=0}[q_2]\right).$$

Now since the problem is linear, when q_1 and q_2 are solutions to the inverse problem, so is $q_1 - q_2$. This means that the above is equivalent to

$$\|q\| \le \Phi\left(d\Lambda\Big|_{q=0}[q]\right).$$

This can be seen as an estimate of the inverse operator and the boundedness hereof. If a problem is stable q depends continuously on $d\Lambda|_{q=0}[q]$. Put in another way, if two sets of data are *close* the corresponding two solutions will also be *close*. A third way of thinking, is if the noise or error of the data is small, the corresponding solution will be close to the true solution.

To derive a stability estimate of the linearised problem, we first need to find a bound on the Fourier transform of the zero extension of q. Note that in the following C stands for a general constant depending on n, s, Ω, M and $\operatorname{supp}(q_1 - q_2)$. C can appear more that once in an equation without necessarily being the same constant. If specific constants are needed it will be clear from the notation.

Theorem 2.1 *Let $n \geq 3$ and let $d\Lambda|_{q_l=0}[q_l]$ be given by (2.4) for $l = 1, 2$. Let $s > n/2$ be an integer and assume that $q \in H^s(\Omega)$ has compact support in Ω. Denote the zero extension of $q_1 - q_2$ from Ω to $\mathbb{R}^n$ by $\tilde{q}$. Then for $r \geq 0$, $\eta \in \mathbb{R}^n$, $|\eta| = 1$, $k^2 + a^2 > \frac{r^2}{4}$ and for $k \geq 1$ it holds that*

$$|\mathcal{F}q(r\eta)| \leq Ck^2 e^{aC} \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)},$$

where C depends only on Ω and n.

Proof. Let C be a general constant depending only on n and Ω. We will use the complex exponential solutions (2.11) with ξ_1 and ξ_2 chosen such that

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \tag{2.13a}$$
$$0 = Re(\xi) \cdot Im(\xi), \tag{2.13b}$$

and

$$\xi_1 + \xi_2 = -r\eta$$

since then it holds for $u_1 = e^{i\xi_1 \cdot x}$ and $u_2 = e^{i\xi_2 \cdot x}$ are solutions, but also

$$u_1 u_2 = e^{-ir\eta\cdot x}. \tag{2.14}$$

This will be useful later, when computing the Fourier transform. These requirements means that ξ_1, ξ_2 must be chosen such that

$$Re(\xi_2) = -r\eta - Re(\xi_1) \quad \text{and} \quad Im(\xi_2) = -Im(\xi_1) \tag{2.15}$$

and (2.13) hold. Let $r \geq 0$, $\eta \in \mathbb{R}^n$ such that $|\eta| = 1$. Choose $\eta^\perp$ and $\zeta \in \mathbb{R}^n$ such that they are orthogonal to each other and to η, that is,

$$\eta \cdot \eta^\perp = \eta \cdot \zeta = \zeta \cdot \eta^\perp = 0 \tag{2.16}$$

and such that they have lengths

$$|\eta^\perp| = 1 \quad \text{and} \quad |\zeta| = a. \tag{2.17}$$

Define

$$\xi_1 = -\frac{r}{2}\eta + \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp + i\zeta \tag{2.18a}$$

$$\xi_2 = -\frac{r}{2}\eta - \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp - i\zeta. \tag{2.18b}$$

Let us now first check that the assumptions (2.13a) and (2.13b) holds.

$$\begin{aligned} |Re(\xi_l)|^2 &= \left(-\frac{r}{2}\eta \pm \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp\right)^2 \\ &= \frac{r^2}{4} + k^2 + a^2 - \frac{r^2}{4}k^2 + a^2 \\ &= k^2 + |Im(\xi_l)|^2 \end{aligned}$$

and

$$Re(\xi) \cdot Im(\xi) = \left(-\frac{r}{2}\eta \pm \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp\right) \cdot (\pm\zeta) = 0.$$

Finally it holds that

$$i(\xi_1 + \xi_2) = -ir\eta. \tag{2.19}$$

This means that when given k, r and η satisfying the assumptions given on them in the theorem, we can choose $\eta^\perp$ and then ζ in a way such that $a^2 > \frac{r^2}{4} - k^2$. Note that this is only possible, since we are in dimension $n \geq 3$, due to the fact that we split the real part of ξ_l into two orthogonal parts, which are also orthogonal to the imaginary part. This is only possible if there are at least three dimensions.

Lastly note that

$$\begin{aligned} |\xi_l|^2 &= \left|-\frac{r}{2}\eta \pm \sqrt{a^2 + k^2 - \frac{r^2}{4}}\alpha_n \pm i\zeta\right|^2 \\ &= \left(-\frac{r}{2}\eta \pm \sqrt{a^2 + k^2 - \frac{r^2}{4}}\alpha_n \pm i\zeta\right) \cdot \left(-\frac{r}{2}\eta \pm \sqrt{a^2 + k^2 - \frac{r^2}{4}}\alpha_n \mp i\zeta\right) \\ &= \frac{r^2}{4} + a^2 + k^2 - \frac{r^2}{4} + a^2 \\ &= 2a^2 + k^2. \end{aligned}$$

By (2.8), the relation (2.14) and since $\tilde{q}$ is the zero extension of q the following holds for u_1, u_2 satisfying (2.5) and (2.6) respectively.

$$\begin{aligned}
|\mathcal{F}\tilde{q}(r\eta)| &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} q e^{-ir\eta\cdot x}\, dx \right| \\
&= (2\pi)^{-n/2} \left| \int_{\Omega} q u_1 u_2 \, dx \right| \\
&= (2\pi)^{-n/2} \left| \left\langle d\Lambda\Big|_{q=0}[q] f_1, f_2 \right\rangle \right| \\
&\leq (2\pi)^{-n/2} \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \|f_1\|_{H^{1/2}(\partial\Omega)} \|f_2\|_{H^{1/2}(\partial\Omega)} .
\end{aligned}$$

This means that we have to bound $\|f_l\|_{H^{1/2}(\partial\Omega)}$ for $l = 1, 2$. Now

$$\|f_l\|_{H^{1/2}(\partial\Omega)} = \|u_l|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} \leq C \|u_l\|_{H^1(\Omega)} .$$

Now choose $R > 0$ large enough such that $\Omega \subseteq B_0(R)$. This means that we have the estimate

$$|u_l(x)| = \left|e^{i\xi_l\cdot x}\right| = \left|e^{i(Re(\xi_l)+iIm(\xi_l))\cdot x}\right| \leq \left|e^{-Im(\xi_l)\cdot x}\right| \leq e^{|\zeta\cdot x|} \leq e^{aR},$$

so

$$\|u_l\|_{L^2(\Omega)} = \left(\int_\Omega |u_l(x)|^2 \, dx\right)^{1/2} \leq |\Omega|^{1/2} e^{aR}.$$

Using this also means that

$$\begin{aligned}
\|\nabla u_l\|_{L^2(\Omega)} &= \|i\xi_l u_l\|_{L^2(\Omega)} = |\xi_l| \, \|u_l\|_{L^2(\Omega)} \leq |\Omega|^{1/2} |\xi_l| e^{aR} \\
&= C(2a^2 + k^2)^{1/2} e^{aR} \leq Cke^{aC},
\end{aligned}$$

since there exists a C such that $ae^{aR} \leq e^{CaR}$. Combining these two bounds gives

$$\begin{aligned}
\|f_l\|_{H^{1/2}(\partial\Omega)} &\leq C \|u_l\|_{H^1(\Omega)} \\
&= C \left(\|u_l\|^2_{L^2(\Omega)} + \|\nabla u_l\|^2_{L^2(\Omega)}\right)^{1/2} \\
&\leq C \left(e^{2aR} + k^2 e^{2aR}\right)^{1/2} \\
&\leq C \left(k^2 e^{2aR} + k^2 e^{2aR}\right)^{1/2} \\
&\leq Cke^{aR}.
\end{aligned}$$

This means that we have the estimate

$$|\mathcal{F}\tilde{q}(r\eta)| \leq Ck^2 e^{Ca} \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$$

as desired. □

We can now prove the main result of this section.

Theorem 2.2 *Let $n \geq 3$ and let $d\Lambda|_{q_l=0}[q_l]$ be given by (2.4) for $l = 1, 2$. Let $s > \frac{n}{2}$ be an integer and assume $q_l \in H^s(\Omega)$ has compact support. Denote the zero extension of $q_1 - q_2$ from Ω to $\mathbb{R}^n$ by $\tilde{q}$. Then for $k \geq 1$ the following stability estimate holds*

$$\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \leq Ck^2 \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}, \tag{2.20}$$

where C depends only on n, s and Ω.

Proof. Now let C be a general constant depending only on n, s, and Ω. Write $\gamma \in \mathbb{R}^n$ in polar coordinates, $\gamma = r\eta$, where $r \geq 0$ and η lies on the unit sphere in $\mathbb{R}^n$. That means that $\eta \in S^{n-1}$ and letting $A(n-1)$ be the area of $S(n-1)$, $A(n-1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, the area of a sphere of radius r is $A(n-1)r^{n-1}$. This means we have the change of variables

$$d\gamma = \frac{(2\pi)^{n/2}}{\Gamma(n/2)} r^{n-1}\, dr\, d\eta. \tag{2.21}$$

Now changing to these polar coordinates in $\mathbb{R}^n$ gives the following estimate for the zero extension of $q_2 - q_1$

$$
\begin{aligned}
\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |1+|\gamma|^2|^{-s} |\mathcal{F}\widetilde{q}(\gamma)|^2 \, d\gamma\right)^{1/2} \\
&= \left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 \, \frac{(2\pi)^{n/2}}{\Gamma(n/2)} r^{n-1} \, dr \, d\eta\right)^{1/2} \\
&= C\left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta\right)^{1/2} \\
&\leq C\Bigg(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} r^{n-1} \\
&\quad \cdot \left| Ck^2 e^{Ca} \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \right|^2 \, dr \Bigg)^{1/2} d\eta \\
&= Ck^2 \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\quad \cdot \left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} r^{n-1} e^{2Ca} \, dr \, d\eta.\right)^{1/2}
\end{aligned}
$$

Note that there is no assumptions on $a = |Im(\xi_l)|$ except for the relationship it must have the $Re(\xi_l)$. This means that no matter how we choose a we can construct x_l, $l = 1, 2$ so Theorem 2.1 holds for any a. This means that we can pick a to be any number independent of r (this is not going to be the case in the non-linear case). Using this means that

$$
\begin{aligned}
\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)} &\leq Ck^2 \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\quad \cdot \left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} e^{2Ca} r^{n-1} \, dr \, d\eta.\right)^{1/2} \\
&= Ck^2 e^{Ca} \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\quad \cdot \left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} r^{n-1} \, dr \, d\eta.\right)^{1/2} \\
&\leq Ck^2 \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)},
\end{aligned}
$$

since $s > n/2$ implies that

$$\begin{aligned}
\int_0^\infty |1+r^2|^{-s} r^{n-1}\, dr &\leq \int_0^\infty (1+r)^{-2s}(1+r)^{n-1}\, dr \\
&= \int_0^\infty (1+r)^{-2s+n-1}\, dr \\
&= \frac{1}{n-2s} \left[(1+r)^{-2s+n}\right]_{r=0}^{\infty}\, dr \\
&= -\frac{1}{n-2s} \\
&= C
\end{aligned} \tag{2.22}$$

and we can pick any positive value of a, so $e^{Ca} \leq C$.

□

The stability estimate in Theorem 2.2 shows that the linear inverse problem has a Lipschitz type stability. This means that the inverse operator

$$\Lambda_L^{-1} : \left(H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)\right) \to H^{-s}(\mathbb{R}^n)$$

is bounded with operator norm $\|\Lambda_L^{-1}\| \leq Ck^2$. If noise or error ϵ were added to the exact data $d\Lambda\Big|_{q=0}[q]$ we would get a solution bounded as

$$\left\|\Lambda_L^{-1}\left(d\Lambda\Big|_{q=0}[q] + \epsilon\right)\right\| = \|\widetilde{q}\| + \left\|\Lambda_L^{-1}\epsilon\right\| \leq \|\widetilde{q}\| + Ck^2 \|\epsilon\|.$$

For a fixed frequency k, this means that the noisy solution converges towards the true solution when the noise $\|\epsilon\|$ goes to zero with a rate of Ck^2. This is much better stability, than for example the logarithmic type observed in section 1.2 and a stable problem in the sense of Hadamard.

As mentioned, the linearised problem can give us an intuition of how the non-linear problem behaves if we linearise around a good guess in some sense. In general, looking at the Helmholtz equation with potential $(\Delta + q + k^2)u = 0$ for a fixed potential it seems reasonable that $q = 0$ is more likely to be a good choice when k is large. When k is small q dominates the term $q + k^2$, but when k increases a fixed q becomes less important. This way of thinking indicates that the stability result obtained in Theorem 2.2 may be a better estimate for the non-linear case when k is large compared to small k. Of course this is only a supposition and to justify it a thorough investigation of the linearisation error and its frequency dependence should be performed.

Note that in Theorem 2.1 we could have relaxed the assumption $n \geq 3$ to $n \geq 2$, by simply setting $\eta^{\perp} = 0$ in the construction of ξ_1 and ξ_2. This would mean that $|Im(\xi)|$ would be dependent of r and must satisfy $|Im(\xi_l)|^2 = \frac{r^2}{4} - k^2$ (so we must also require $2k < r$), but this is not a problem in this linear case, since we have no assumptions on $|Im(\xi)|$ to gain existence of the solutions of the form (2.11). However, this is not going to be the case in the non-linear problem as will be seen in chapter 3, since we here need $|Im(\xi_l)|$ to be big enough to have existence of so called complex geometric optics solutions. If $|Im(\xi_l)|$ was a function of r in the non-linear case, we would only be able to estimate the Fourier Transform of $\widetilde{q}$ for low Fourier frequencies r. The reason for this will be clear in chapter 3. If we wanted an estimate for the low-frequencies only, it would be possible to stay in 2 dimensions.

However, the reason for keeping the assumption that $n \geq 3$ in the linear case here, is the use of $|Im(\xi)|$ being independent of r in the proof of Theorem 2.2. In two dimensions $|Im(\xi)| = \frac{r^2}{4} - k^2$ and we would get a divergent integral in the estimate of $\|\tilde{q}\|_{H^{-s}(\Omega)}$

$$\|\tilde{q}\|_{H^{-s}(\Omega)} \leq Ce^{-Ck^2}k^2 \left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \cdot \left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} e^{Cr^2} r^{n-1}\, dr\, d\eta. \right)^{1/2}.$$

The exponential part e^{Cr^2} dominates the polynomial part $(1+r^2)^{-s}r^{n-1}$ hence the integral diverges. This illustrates that if we were looking for an estimate for the low Fourier frequencies only, that is for $r \leq T$, we would get a finite integral instead and a stability estimate could be derived. In [2] Calderón discussed the stability of the linearised version of the conductivity equation (1.1) in dimension $n \geq 2$. Here the idea is to estimate $\phi * \tilde{q}$, where $\mathcal{F}\phi \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function, so it has compact support in $\mathbb{R}^n$

$$\text{supp } \mathcal{F}\phi(r\eta) = \{r\eta \mid 0 \leq r \leq T\})$$

and it holds that

$$0 \leq \mathcal{F}\phi(r\eta) \leq 1 \quad \text{and} \quad \mathcal{F}\eta(r\eta) = 1 \quad \text{near supp } \mathcal{F}\phi(r\eta).$$

Then

$$\begin{aligned}\|\phi * \tilde{q}\|_{H^{-s}(\Omega)} &= C\left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}(\phi * \tilde{q})(r\eta)|^2 r^{n-1}\, dr\, d\eta\right)^{1/2} \\ &= C\left(\int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\phi(r\eta)|^2 |\mathcal{F}\tilde{q}(r\eta)|^2 r^{n-1}\, dr\, d\eta\right)^{1/2} \\ &\leq C\left(\int_0^T \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\tilde{q}(r\eta)|^2 r^{n-1}\, dr\, d\eta\right)^{1/2},\end{aligned}$$

by use of the properties of $\mathcal{F}\phi(r\eta)$. An estimate of the Fourier transform could then be used to find a bound of $\|\phi * \tilde{q}\|_{H^{-s}(\Omega)}$ by $\left\| d\Lambda\Big|_{q=0}[q] \right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ without arriving at a divergent integral. Note that the constant C here also depends on the support of $\mathcal{F}\phi$. In general considering all r, if the Fourier coefficients for $\tilde{q}$ are equal or close to zero for large frequencies this seems reasonable as a good estimate, but in general we cannot use an estimate for only the low frequencies and this is the reason we assume $n \geq 3$ in Theorem 2.2.

Chapter 3

Increasing Stability

We will now move on to the main focus of this work, the stability of the non-linear inverse boundary value problem (1.14) of the Helmholtz equation with a potential. The result from the investigation of the linearised problem in chapter 2 suggests that we might expect Lipschitz type stability when k increases. To see if this is really the case we will start out in section 3.1 by introducing the concept of complex geometric optics solutions, which is an essential tool in the derivation of an stability estimate. In section 3.2 useful and important results are derived. In particular an identity involving the difference of two potentials will prove crucial in the discussion of type of stability. Furthermore, uniqueness of the inverse problem is proved using CGO solutions and similar arguments as in the linearised case. The results from section 3.1 and section 3.2 are used in section 3.3 to derive a bound on the Fourier transform of the difference of potentials following the same procedure as in section 2.2. It will here be clear, where the difficulties arise in the non-linear case compared to the linear. The last section 3.4 then finally gives a detailed proof of the type of stability we can expect of the non-linear inverse problem.

3.1 Complex Geometric Optics Solutions

Consider the Helmholtz equation (1.8). We are looking for solutions called complex geometric optics (CGO) solutions. They have the form

$$u = e^{i\xi\cdot x}\left(1+\psi\right), \tag{3.1}$$

where $\xi \in \mathbb{C}^n$ satisfies $\xi\cdot\xi = k^2$ like in the linear case, but now the perturbation function ψ is added. A way of thinking of CGO solutions is that the complex exponential $e^{i\xi\cdot x}$ is *almost* in the kernel of (1.8), but since the Laplace operator is perturbed with k^2+q the complex exponential is also perturbed with ψ.

The procedure to show existence of solutions (3.1) is to derive a partial differential equation for the perturbation function ψ and show existence of a solution to this problem. Plugging (3.1) into the partial differential equation (1.8) gives an equation for the perturbation function ψ.

$$\begin{aligned}
0 &= (\Delta + k^2 + q)e^{i\xi\cdot x}(1+\psi)\\
&= \Delta\left(e^{i\xi\cdot x}(1+\psi)\right) + (k^2+q)e^{i\xi\cdot x}(1+\psi)\\
&= \nabla\cdot\left(i\xi e^{i\xi\cdot x}(1+\psi) + e^{i\xi\cdot x}\nabla\psi\right) + (k^2+q)e^{i\xi\cdot x}(1+\psi)\\
&= -\xi^2 e^{i\xi\cdot x}(1+\psi) + 2i\xi e^{i\xi\cdot x}\nabla\psi + e^{i\xi\cdot x}\Delta\psi + (k^2+q)e^{i\xi\cdot x}(1+\psi)\\
&= 2i\xi e^{i\xi\cdot x}\nabla\psi + e^{i\xi\cdot x}\Delta\psi + +qe^{i\xi\cdot x}(1+\psi)
\end{aligned}$$

The reason for chosen ξ such that $\xi^2 = k^2$ is now clear. Multiplying with $e^{-i\xi\cdot x}$ gives

$$\left(\Delta + 2i\xi\cdot\nabla + q(x)\right)\psi = -q(x). \tag{3.2}$$

This means that if we can show that there exists a solution $\psi \in H^s(\Omega)$ to (3.2), we have shown that there exists a solution of the form (3.1) in $H^s(\Omega)$ to the problem (1.8).

Sylvester and Uhlmann showed existence and uniqueness of a solution to (1.1) in $\mathbb{R}^n$ of the form (3.1) in [20]. In this section we will show existence of CGO solutions of the Helmholtz equation with a potential (1.8). The proofs are build on [8] and [18], which are works that among other things discuss part of the result in [20]. Here existence of CGO solutions of the Schrödinger equation (1.6) is shown and an estimate of ψ in $L^2(\Omega)$ is given. We will here generalize these results to solutions of the Helmholtz equation with a potential (1.8) and also derive a bound of ψ in $H^s(\Omega)$.

Let us first consider the case where $q = 0$ on the left-hand side of (3.2) and call this the case of zero potential. Also denote the right-hand side by f, so we get the partial differential equation

$$(\Delta + 2i\xi \cdot \nabla)\,\psi = f. \tag{3.3}$$

As mentioned Sylvester and Uhlmann considered the case $\Omega = \mathbb{R}^n$ in [20]. This case gives a good indication of where the problem of the proof lies since the Fourier transform is defined and Fourier transforming derivatives turns into multiplication.

$$\begin{aligned}\mathcal{F}f(m) &= \left(\mathcal{F}\left(\Delta + 2i\xi \cdot \nabla\right)\psi\right)(m)\\ &= \mathcal{F}\left(\Delta\psi\right)(m) + 2\xi \cdot \mathcal{F}\left(i\nabla\psi\right)(m)\\ &= -(|m|^2 + 2\xi \cdot m)\mathcal{F}\psi(m).\end{aligned}$$

Considering the formal case where $|m|^2 + 2\xi \cdot m \neq 0$ this means that

$$\mathcal{F}\psi(m) = -\frac{\mathcal{F}f(m)}{|m|^2 + 2\xi \cdot m}. \tag{3.4}$$

We would then get a solution

$$\psi = (\Delta + 2i\xi \cdot \nabla)^{-1} f = \mathcal{F}^{-1}\left(-\frac{\widehat{f}(m)}{|m|^2 + 2\xi \cdot m}\right),$$

where

$$\|\psi\|_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |1 + |m|^2|^s |\mathcal{F}\psi(m)|^2 \, dm = \int_{\mathbb{R}^n} \frac{|1 + |m|^2|^s}{||m|^2 + 2\xi \cdot m|^2} |\mathcal{F}f(m)|^2 \, dm.$$

If we could then find a lower bound of $||m|^2 + 2\xi \cdot m|^2$ greater than zero and independent of m we would be able to bound $\|\psi\|_{H^s(\mathbb{R}^n)}$ by $\|f\|_{H^s(\mathbb{R}^n)}$. The problem here is that for a given ξ there are m that give rise to zeros of $||m|^2 + 2\xi \cdot m|^2$, so it is not always well-defined to write (3.4).

However, $\Omega \neq \mathbb{R}^n$ so let us now come back to the case where $\Omega \subset \mathbb{R}^n$ is a bounded domain, where the Fourier transform is not defined.

Theorem 3.1 *Let $s \geq 0$ be an integer. Assume that $\xi \in \mathbb{C}^n$ satisfies $|Im(\xi)| \geq \epsilon$ for some $\epsilon > 0$ and*

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \quad \text{and} \quad Re(\xi) \cdot Im(\xi) = 0,$$

since this is equivalent to $\xi \cdot \xi = k^2$. Then there exists a constant C_0 depending only on Ω such that for any $f \in H^s(\Omega)$ with compact support in Ω (3.3) has a solution $\psi \in H^s(\Omega)$ satisfying

$$\|\psi\|_{H^s(\Omega)} \leq \frac{C_0}{|Im(\xi)|} \|f\|_{H^s(\Omega)}. \tag{3.5}$$

Proof.

Consider the cube $Q = [-R, R]^n$, where R is large enough for $\Omega \subseteq Q$. We will solve (3.3) in Q by using an appropriate orthonormal basis and the idea is to use something similar to the Fourier basis. We want a basis (w_m) of $L^2(Q)$ such that $||m|^2 + 2\xi \cdot m| \neq 0$, so we first define the lattice

$$L = \left\{ m = (m_1, m_2, \ldots, m_n)^T \in \mathbb{R}^n \;\middle|\; \frac{R}{\pi} m_j \in \mathbb{Z},\ j \neq 2 \quad \text{and} \quad \frac{R}{\pi} m_2 + \frac{1}{2} \in \mathbb{Z} \right\} \tag{3.6}$$

and the functions

$$w_m = \frac{1}{(2R)^{n/2}} e^{im \cdot x}, \quad x \in Q,\ m \in L. \tag{3.7}$$

The reason for chosen (3.6) in this way will be clear later. Note that if instead we had chosen the lattice as $\left\{ k \in \mathbb{R}^n \;\middle|\; \frac{R}{\pi} k_j \in \mathbb{Z},\ j \in \mathbb{N} \right\}$ then (w_k) would be the usual orthonormal Fourier basis for $L^2(Q)$. To see that $(w_m)_{m \in L}$ is also an orthonormal basis, note that $k_j = m_j$ except $k_2 = m_2 - 1/2$. (3.7) is obviously an orthonormal set. To see that (3.7) is also complete, we use that (w_k) is complete. Assume $\frac{1}{(2R)^{n/2}} \int_\Omega g e^{im \cdot x}\, dx = 0$ for $g \in L^2(Q)$, so

$$\begin{aligned} 0 &= \frac{1}{(2R)^{n/2}} \int_\Omega g\ \Pi_{j=1}^n e^{im_j x_j}\ dx = \frac{1}{(2R)^{n/2}} \int_\Omega g\ e^{i\left(k_2 - \frac{1}{2}\right) x_2}\ \Pi_{j \neq 2} e^{ik_j x_j}\ dx \\ &= \int_\Omega g\ e^{-i\frac{1}{2} x_2}\ w_k\ dx. \end{aligned}$$

Since (w_k) is complete, it holds that $g(x) e^{-i\frac{1}{2}x_2} = 0 \Rightarrow g = 0$ showing that $(w_m)_{m \in L}$ is also complete. To be able to expand the right hand side of (3.3), f, in this orthonormal basis we extend to Q. f is compactly supported in Ω, so we can extend it by zero. Denote this zero extension $\tilde{f} \in H^s(Q)$. Then we expand in the orthonormal basis $(w_m)_{m \in L}$

$$\tilde{f} = \sum_{m \in L} \tilde{f}_m w_m,$$

where $\tilde{f}_m = \left(\tilde{f}, w_m \right)_{L^2(Q)}$. Consider now (3.3) in Q and denote the solution $\tilde{\psi} \in H^1(Q)$. Again expand in $(w_m)_{m \in L}$

$$\tilde{\psi} = \sum_{m \in L} \tilde{\psi}_m w_m,$$

where $\tilde{\psi}_m = \left(\tilde{\psi}, w_m \right)_{L^2(Q)}$.

Now assume that $\tilde{\psi}$ is periodic on Q and multiply the left-hand side of (3.3) with $\overline{w_m}$, integrate over Q and perform integration by parts. This gives the following (corresponding to the Fourier transform in Q)

$$\begin{aligned}
\left(\tilde{f}, w_m\right)_{L^2(Q)} &= \frac{1}{(2R)^{n/2}} \int_Q \tilde{f}(x) e^{-im\cdot x}\, dx \\
&= \frac{1}{(2R)^{n/2}} \int_Q (\Delta + 2i\xi\cdot\nabla)\, \tilde{\psi}(x) e^{-im\cdot x}\, dx \\
&= \frac{1}{(2R)^{n/2}} \left(\int_Q \nabla\cdot\nabla\tilde{\psi}(x) e^{-im\cdot x}\, dx + \int_Q 2i\xi\cdot\nabla\tilde{\psi}(x) e^{-im\cdot x}\, dx\right) \\
&= \frac{1}{(2R)^{n/2}} \left(\int_{\partial Q} \nabla\tilde{\psi}(x) e^{-im\cdot x}\cdot\eta\, ds + i\int_Q \nabla\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx\right) \\
&\quad + \frac{2i}{(2R)^{n/2}} \left(\int_{\partial Q} \xi\cdot\tilde{\psi}(x) e^{-im\cdot x}\cdot\eta\, ds + i\int_Q \xi\cdot\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx\right) \\
&= \frac{1}{(2R)^{n/2}} \left(i\int_Q \nabla\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx - 2\int_Q \xi\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx\right) \\
&= \frac{1}{(2R)^{n/2}} \left(i\int_{\partial Q} \tilde{\psi}(x) m e^{-im\cdot x}\cdot\eta\, ds - \int_Q \tilde{\psi}(x)|m|^2 e^{-im\cdot x}\, dx\right) \\
&\quad - \frac{2}{(2R)^{n/2}} \left(\int_Q \xi\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx\right) \\
&= -\frac{1}{(2R)^{n/2}} \left(\int_Q \tilde{\psi}(x)|m|^2 e^{-im\cdot x}\, dx + 2\int_Q \xi\tilde{\psi}(x)\cdot m e^{-im\cdot x}\, dx\right) \\
&= -\frac{1}{(2R)^{n/2}} \left(|m|^2\int_Q \tilde{\psi}(x) e^{-im\cdot x}\, dx + 2\xi\cdot m\int_Q \tilde{\psi}(x) e^{-im\cdot x}\, dx\right) \\
&= -\left(|m|^2 + 2\xi\cdot m\right)\left(\tilde{\psi}, w_m\right)_{L^2(Q)}.
\end{aligned}$$

Hence we have the following relationship between the coefficients of $\tilde{f}$ and $\tilde{\psi}$

$$\tilde{f}_m = -\left(|m|^2 + 2\xi\cdot m\right)\tilde{\psi}_m. \tag{3.8}$$

Inspired by this we define the operator $R_\xi : H^s(Q) \to H^s(Q)$ such that $R_\xi\Delta_\xi = I$, where Δ_ξ is the partial differential operator (3.3) $\Delta_\xi\psi = (\Delta + 2i\xi\cdot\nabla)\,\psi.$, so

$$R_\xi\tilde{f} = -\sum_{m\in L} \frac{\tilde{f}_m}{|m|^2 + 2\xi\cdot m} w_m = \tilde{\psi}.$$

This is a formal definition, since we do not know if $|m|^2 + 2\xi\cdot m = 0$ for some m and we have not shown that $R_\xi\tilde{f} \in H^s(Q)$. The choice of the shifted lattice L deals with the first problem. First note that since

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \quad \text{and} \quad Re(\xi)\cdot Im(\xi) = 0$$

we can write

$$\xi = t\omega + ia\omega^{\perp},$$

where $|Re(\xi)| = t \geq 0$, $|Im(\xi)| = a > 0$, $t^2 = k^2 + a^2$, $\omega, \omega_{\perp}^{k} \in \mathbb{R}^n$ such that $\omega \cdot \omega^{\perp} = 0$ and $|\omega| = |\omega^{\perp}| = 1$. By rotating coordinates in the right way we can assume that $\omega = e_1$ and $\omega^{\perp} = e_2$, where $e_1 = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^n$ and $e_2 = (0, 1, 0, \ldots, 0)^T \in \mathbb{R}^n$. This choice turns (3.3) in Q into

$$(\Delta + 2i(te_1 + iae_2) \cdot \nabla))\tilde{\psi} = \left(\Delta + 2i\left(t\frac{\partial}{\partial x_1} + ia\frac{\partial}{\partial x_2}\right)\right))\tilde{\psi} = \tilde{f} \quad \text{in } Q$$

and it turns (3.8) into

$$\tilde{f}_m = -\left(|m|^2 + 2(tm_1 + iam_2)\right)\tilde{\psi}_m. \tag{3.9}$$

Now the following bound holds

$$||m|^2 + 2(tm_1 + iam_2)| \geq |Im(|m|^2 + 2(tm_1 + iam_2))| = 2a|m_2|,$$

so the problem $||m|^2 + 2\xi \cdot m| = 0$ can be avoided if $|m_2| > 0$. This is why $\frac{R}{\pi}m_2$ is shifted away from integers in (3.6), since then $|m_2| \geq \frac{\pi}{2R}$. Thus

$$||m|^2 + 2(tm_1 + iam_2)| \geq \frac{a\pi}{R}$$

and the division above is justified. It is assumed that $|Im(\xi)| = a \geq \epsilon > 0$, so

$$\begin{aligned}\left\|R_\xi \tilde{f}\right\|_{L^2(Q)}^2 = \left\|\tilde{\psi}\right\|_{L^2(Q)}^2 = \sum_{m\in L} |\tilde{\psi}_m|^2 = \sum_{m\in L} \frac{\left|\tilde{f}_m\right|^2}{\left||m|^2 + 2(tm_1 + iam_2)\right|^2} \\ \leq \frac{R^2}{a^2\pi^2} \sum_{m\in L} \left|\tilde{f}_m\right|^2,\end{aligned}$$

hence

$$\left\|\tilde{\psi}\right\|_{L^2(Q)} \leq \frac{C_0}{|Im(\xi)|}\left\|\tilde{f}\right\|_{L^2(Q)}$$

where $C_0 = \frac{R}{\pi}$.

Now to see that $R_\xi \tilde{f} = \tilde{\psi} \in H^s(Q)$, that is, $D^\alpha R_\xi \tilde{f} = D^\alpha \tilde{\psi} \in L^2(Q)$ for all $|\alpha| \leq s$, use that $\tilde{f} \in H^s(Q)$.

$$\begin{aligned}\left(D^\alpha \tilde{f}\right)_m &= \left(D^\alpha \tilde{f}, w_m\right)_{L^2(Q)} \\ &= \int_Q w_m \, D^\alpha \tilde{f} \, dx \\ &= (-1)^{|\alpha|} \int_Q D^\alpha w_m \, \tilde{f} \, dx \\ &= (-1)^{|\alpha|}(im)^\alpha \tilde{f}_m,\end{aligned}$$

where $(im)^\alpha = \sum_{j=1}^{n}(im_j)^{\alpha_j}$. This shows that

$$D^\alpha \tilde{f} = \sum_{m\in L}(-1)^{|\alpha|}(im)^\alpha \tilde{f}_m w_m.$$

We wish to show that $D^\alpha R_\xi \tilde{f} = D^\alpha \tilde{\psi}$ exists and that (3.1) holds. We have that

$$\begin{aligned}
\left(D^\alpha \tilde{\psi}\right)_m &= \left(D^\alpha \tilde{\psi}, w_m\right)_{L^2(Q)} \\
&= \int_Q w_m D^\alpha \tilde{\psi}\, dx \\
&= (-1)^{|\alpha|}\int_Q \tilde{\psi}\, D^\alpha w_m\, dx \\
&= (-1)^{|\alpha|}(im)^\alpha \tilde{\psi}_m \\
&= (-1)^{|\alpha|}(im)^\alpha \frac{\tilde{f}_m}{m\cdot m + 2\xi\cdot m} \\
&= \frac{\left(D^\alpha \tilde{f}\right)_m}{m\cdot m + 2\xi\cdot m}.
\end{aligned}$$

Computing the L^2-norm of $D^\alpha \tilde{\psi}$ for any α, where $|\alpha| \leq s$ by use of Parseval's equation gives

$$\left\|D^\alpha \tilde{\psi}\right\|_{L^2(Q)}^2 = \sum_{m\in L}\frac{\left|\left(D^\alpha \tilde{f}\right)_m\right|^2}{|m\cdot m + 2\xi\cdot m|^2} \leq \frac{R^2}{t^2\pi^2}\sum_{m\in L}\left|(D^\alpha \tilde{f})_m\right|^2 = \frac{R^2}{t^2\pi^2}\left\|D^\alpha \tilde{f}\right\|_{L^2(Q)}^2.$$

Hence $D^\alpha \tilde{\psi} \in L^2(Q)$ for all $|\alpha| \leq s$. Summing then gives

$$\left\|\tilde{\psi}\right\|_{H^s(Q)} = \left\|R_\xi \tilde{f}\right\|_{H^s(Q)} = \left(\sum_{|\alpha|\leq s}\left\|D^\alpha R_\xi \tilde{f}\right\|_{L^2(Q)}^2\right)^{1/2} \leq \frac{R}{\pi|Im(\xi)|}\left\|\tilde{f}\right\|_{H^s(Q)}.$$

Since f was extended by zero outside Ω into Q it holds that $\left\|\tilde{f}\right\|_{H^s(Q)} = \|f\|_{H^s(\Omega)}$ and $\left\|\tilde{\psi}\right\|_{H^s(Q)} \geq \|\psi\|_{H^s(\Omega)}$ so we get the desired estimate in Ω.

□

Using this theorem we can find a bound on the solution to (3.2), when $|Im(\xi)|$ is large enough. We first need the following theorem about a perturbation of the identity and operator norm of its inverse. The proof is build on [11].

Theorem 3.2 *Let $A : X \to X$, where $A = I + B$, be a bounded operator from X onto X, where also $B : X \to X$ is bounded. If $\|B\| < 1$, then the perturbation of the identity operator $I + B$ is invertible and it holds that*

$$\left\|A^{-1}\right\| = \left\|(I+B)^{-1}\right\| \leq \frac{1}{1-\|B\|}.$$

Proof.
Let $x \in X$. It then holds for a $K > 0$ that

$$\|Ax\| = \|(I+B)x\| = \|Ix + Bx\| \geq \|x\| - \|Bx\| \geq \|x\| - \|B\| \|x\| = (1 - \|B\|) \|x\|.$$

Hence there exists a constant $K_1 = 1 - \|B\| > 0$ such that $\|A\| \geq K_1 \|x\|$ for all $x \in X$. This means that $K_1 \|x\| \leq \|Ax\| \leq K_2 \|x\|$, hence A is injective, since assuming $Ax = 0$ implies that

$$K_1 \|x\| \leq 0 \leq K_2 \|x\|,$$

so $x = 0$. This means that A in invertible (since it is already assumed to be surjective).

Now

$$1 = \left\|AA^{-1}\right\| = \left\|(1+B)A^{-1}\right\| \geq (1 - \|B\|) \left\|A^{-1}\right\|,$$

so $\left\|A^{-1}\right\| \leq \dfrac{1}{1-\|B\|}$.

□

Using this result we can show existence of solutions of the (3.1) to the Helmholtz equation with a potential.

Theorem 3.3 *Let $s > \frac{n}{2}$ be an integer and assume that $\xi \in \mathbb{C}^n$ satisfies*

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \quad \text{and} \quad Re(\xi) \cdot Im(\xi) = 0,$$

since this means that $\xi \cdot \xi = k^2$. Then there is a constant C_1 depending on Ω, such that if

$$|Im(\xi)| \geq C_1 \|q\|_{H^s(\Omega)} \tag{3.10}$$

then there exists a solution $u \in H^s(\Omega)$ of (1.8) of the form (3.1), where $\psi \in H^s(\Omega)$ satisfies

$$\|\psi\|_{H^s(\Omega)} \leq \frac{2C_0}{|Im(\xi)|} \|q\|_{H^s(\Omega)}, \tag{3.11}$$

where C_0 is the constant from Theorem 3.1.

PROOF.

In the case with no potential, that is, when $q = 0$ on the left-hand side of (3.2) we know from Theorem 3.1 that a solution is $\tilde{\psi} = R_\xi \tilde{q}$ in $H^s(Q)$, where $Q = [-R, R]^n$ and R is chosen large enough for $\Omega \subseteq Q$ to hold. In general q might not be zero, so we try applying R_ξ to (3.2) in Q

$$R_\xi \left(\Delta_\xi + \tilde{q}(x)\right) \tilde{\psi} = \left(I + R_\xi \tilde{q}\right) \tilde{\psi} = R_\xi \tilde{q}.$$

Now if $\|R_\xi \tilde{q}\| < 1$ we have that $I + R_\xi \tilde{q}$ is invertible and $\left\| (I + R_\xi \tilde{q})^{-1} \right\| \leq \frac{1}{1 - \|R_\xi \tilde{q}\|}$ by Theorem 3.2. Set $C_1 = 2C_0$, where C_0 is the constant from Theorem 3.1 and use that $R_\xi : H^s(Q) \to H^s(Q)$ is bounded

$$\begin{aligned}
\left\| R_\xi \tilde{q} \tilde{\psi} \right\|_{H^s(Q)} &\leq \frac{C_0}{|Im(\xi)|} \left\| \tilde{q} \tilde{\psi} \right\|_{H^s(Q)} \\
&\leq \frac{C_0}{|Im(\xi)|} \|\tilde{q}\|_{H^s(Q)} \left\| \tilde{\psi} \right\|_{H^s(Q)} \quad \text{by} \quad \textit{Theorem A.3} \\
&\leq \frac{C_0}{C_1 \|q\|_{H^s(Q)}} \|\tilde{q}\|_{H^s(Q)} \left\| \tilde{\psi} \right\|_{H^s(Q)} \\
&= \frac{1}{2} \left\| \tilde{\psi} \right\|_{H^s(Q)},
\end{aligned}$$

hence $\|R_\xi \tilde{q}\| \leq \frac{1}{2} < 1$. By Theorem 3.2 $(I + R_\xi \tilde{q})^{-1}$ exists and

$$\left\| (I + R_\xi \tilde{q})^{-1} \right\| \leq \frac{1}{1 - \|R_\xi \tilde{q}\|} \leq \frac{1}{1 - \frac{1}{2}} = 2.$$

This means that

$$\tilde{\psi} = (I + R_\xi \tilde{q})^{-1} R_\xi \tilde{q}$$

solves (3.2) and

$$\begin{aligned}
\left\| \tilde{\psi} \right\|_{H^s(Q)} &= \left\| (I + R_\xi q)^{-1} R_\xi \tilde{q} \right\|_{H^s(Q)} \leq \left\| (I + R_\xi \tilde{q})^{-1} \right\| \|R_\xi \tilde{q}\|_{H^s(Q)} \\
&\leq \frac{2C_0}{|Im(\xi)|} \|\tilde{q}\|_{H^s(Q)}.
\end{aligned}$$

Using that $\tilde{q}$ is a zero extension of q to Q means that $\|\tilde{q}\|_{H^s(Q)} = \|q\|_{H^s(\Omega)}$ and we also have that $\|\psi\|_{H^s(\Omega)} \leq \left\| \tilde{\psi} \right\|_{H^s(Q)}$, so

$$\|\psi\|_{H^s(\Omega)} \leq \frac{2C_0}{|Im(\xi)|} \|q\|_{H^s(\Omega)}. \tag{3.12}$$

Now since we have shown that (3.2) has a solution $\tilde{\psi} \in H^s(Q)$, it means that (1.8) has a solution in $H^s(Q)$ of form (3.1), where $\tilde{\psi}$ satisfies the bound given in (3.12) under the given assumptions.

□

Note that C_1 is chosen explicit in the proof $C_1 = 2C_0 = \frac{2R}{\pi}$, since C_0 is also given explicitly in Theorem 3.1. Theorem 3.3 means that the perturbation function ψ goes to zero in $H^s(\Omega)$ when $|Im(\xi)| \to \infty$. This means that the perturbation term in the CGO solution (3.1) will be small for large $Im(\xi)$ and the solution to (1.8) will look like a complex exponential $e^{i\xi\cdot x}$.

3.2 Important identities

Before moving on to the actual investigation of the stability of the inverse problem (1.14) we need a few results, that will be shown in this section and used in the next.

First we define the dual pairing on $(H^{-s}(\Omega), H^s(\Omega))$ for $s \geq 1$ like done in [1]. For $u \in H^s(\Omega)$ let $v \in L^2(\Omega)$ and define ϕ by

$$\phi(u) = \langle \phi, u \rangle = (v, u)_{L^2(\Omega)} = \int_\Omega vu \, dx. \tag{3.13}$$

By this definition if $\phi \in H^s(\Omega)$ then $\langle \phi, u \rangle = (\phi, u)_{L^2(\Omega)}$. Let us check that this defines a dual pairing, that is, that $\phi \in H^{-s}(\Omega)$. First, we must make sure that (3.13) is independent of the choice of v. Let v_1 and v_2 be two $L^2(\Omega)$ functions. Then

$$0 = \langle \phi, u \rangle - \langle \phi, u \rangle = \int_\Omega v_1 u \, dx - \int_\Omega v_2 u \, dx = \int_\Omega (v_1 - v_2) u \, dx.$$

Now this is true for all compactly supported smooth functions u, so $v_1 = v_2$ a.e. in Ω.

Second, we can check that $\phi \in H^{-s}(\Omega)$. It is easy to see that (3.13) is linear. Let $\alpha, \beta \in \mathbb{C}$ and $u_1, u_2 \in H_0^s(\Omega)$

$$\langle \phi, \alpha u_1 + \beta u_2 \rangle = \int_\Omega v(\alpha u_1 + \beta u_2) \, dx = \alpha \langle \phi, u_1 \rangle + \beta \langle \phi, u_2 \rangle.$$

Boundedness follows from Hölder's inequality

$$|\langle \phi, u\rangle| = \left|\int_\Omega vu\ dx\right| \leq \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \|u\|_{H^s(\Omega)}.$$

Hence $\phi \in H^{-s}(\Omega)$ and

$$\left|\int_\Omega \phi u\ dx\right| \leq \|\phi\|_{H^{-s}(\Omega)} \|u\|_{H^s(\Omega)}. \tag{3.14}$$

The following theorem uses this result and will prove useful later.

Theorem 3.4 *Let $u \in H_0^s(\Omega)$. For all $\epsilon > 0$ it holds that*

$$\|u\|^2_{L^2(\Omega)} \leq \epsilon \|u\|^2_{H^s(\Omega)} + \frac{1}{\epsilon} \|u\|^2_{H^{-s}(\Omega)}.$$

Proof. Let $\alpha = \epsilon \frac{\|u\|_{H^s(\Omega)}}{\|u\|_{H^{-s}(\Omega)}}$. Then $\alpha \geq 0$ and $\alpha + \frac{1}{\alpha} \geq 2 > 1$, since $g(\alpha) = \alpha + \frac{1}{\alpha}$ has global minimum at $(1, 2)$. This means that

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= \left|(f, f)_{L^2(\Omega)}\right| \\
&= \left|\langle u, u\rangle_{H^s(\Omega), H^{-s}(\Omega)}\right| \\
&\leq \|u\|_{H^{-s}(\Omega)} \|u\|_{H^s(\Omega)} \\
&\leq \|u\|_{H^{-s}(\Omega)} \|u\|_{H^s(\Omega)} \left(\epsilon \frac{\|u\|_{H^s(\Omega)}}{\|u\|_{H^{-s}(\Omega)}} + \frac{1}{\epsilon} \frac{\|u\|_{H^{-s}(\Omega)}}{\|u\|_{H^s(\Omega)}}\right) \\
&= \epsilon \|u\|^2_{H^s(\Omega)} + \frac{1}{\epsilon} \|u\|^2_{H^{-s}(\Omega)}.
\end{aligned}$$

□

The next theorem is based on [14] (Theorem 15.1) and is important in the next section.

Theorem 3.5 *For any two solutions $u_l \in H^1(\Omega)$, $l = 1, 2$ to*

$$\begin{aligned}
(\Delta + k^2 + q_l) u_l &= 0, \\
u_l\Big|_{\partial\Omega} &= f_l,
\end{aligned} \tag{3.15}$$

where $f_l \in H^{1/2}(\partial\Omega)$ the following identity holds

$$\int_\Omega (q_2 - q_1) u_1 u_2\ dx = \int_{\partial\Omega} f_1 (\Lambda_{q_1} - \Lambda_{q_2}) f_2\ ds. \tag{3.16}$$

Proof. Using that u_1 and u_2 are weak solutions of (3.15) and performing integration by part gives

$$\begin{aligned}
0 &= \int_\Omega u_1(\Delta + k^2 + q_2)u_2 - u_2(\Delta + k^2 + q_1)u_1 \, dx \\
&= -\int_\Omega \nabla u_1 \cdot \nabla u_2 \, dx + \int_{\partial\Omega} f_1 \frac{\partial u_2}{\partial \eta}\Big|_{\partial\Omega} ds + \int_\Omega \nabla u_2 \cdot \nabla u_1 \, dx - \int_{\partial\Omega} f_2 \frac{\partial u_1}{\partial \eta}\Big|_{\partial\Omega} ds \\
&\quad + \int_\Omega u_1 q_2 u_2 - u_2 q_1 u_1 \, dx \\
&= \int_{\partial\Omega} f_1 \Lambda_{q_2} f_2 \, ds - \int_{\partial\Omega} f_2 \Lambda_{q_1} f_1 \, ds + \int_\Omega u_2(q_2 - q_1)u_1 \, dx.
\end{aligned}$$

Rearranging gives

$$\int_\Omega (q_2 - q_1)u_1 u_2 \, dx = \int_{\partial\Omega} f_2 \Lambda_{q_1} f_1 \, ds - \int_{\partial\Omega} f_1 \Lambda_{q_2} f_2 \, ds. \tag{3.17}$$

Now let v solve $(\Delta + k^2 + q_1)v = 0$ in Ω and $v = f_2$ on $\partial\Omega$. Then using similar arguments as above

$$\begin{aligned}
0 &= \int_\Omega u_1(\Delta + k^2 + q_1)v - v(\Delta + k^2 + q_1)u_1 \, dx \\
&= \int_{\partial\Omega} f_1 \frac{\partial v}{\partial \eta}\Big|_{\partial\Omega} ds - \int_{\partial\Omega} f_2 \frac{\partial u_1}{\partial \eta}\Big|_{\partial\Omega} ds \\
&= \int_{\partial\Omega} f_1 \Lambda_{q_1} f_2 \, ds - \int_{\partial\Omega} f_2 \Lambda_{q_1} f_1 \, ds,
\end{aligned}$$

hence

$$\int_{\partial\Omega} f_2 \Lambda_{q_1} f_1 \, ds = \int_{\partial\Omega} f_1 \Lambda_{q_1} f_2 \, ds. \tag{3.18}$$

Plugging this result into (3.17) gives the desired result

$$\int_\Omega (q_2 - q_1)u_1 u_2 \, dx = \int_{\partial\Omega} f_1 \left(\Lambda_{q_1} - \Lambda_{q_2}\right) f_2 \, ds.$$

□

Note that (3.17) means that the Dirichlét-to-Neumann map Λ_q is symmetric, that is,

$$\langle \Lambda_q f, g \rangle = \langle f, \Lambda_q g \rangle. \tag{3.19}$$

Now Theorem 3.5 means that

$$\int_\Omega (q_2 - q_1)u_1u_2\,dx = \langle f_1, (\Lambda_{q_1} - \Lambda_{q_2})f_2\rangle = \langle (\Lambda_{q_1} - \Lambda_{q_2})\,f_1, f_2\rangle,$$

so using (1.11) and (1.12) we get the following inequality

$$|\langle (\Lambda_{q_1} - \Lambda_{q_2})\,f_1, f_2\rangle| \le \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\,\|f_1\|_{H^{1/2}(\partial\Omega)}\,\|f_2\|_{H^{1/2}(\partial\Omega)}\,,$$

which means that

$$\left|\int_\Omega (q_2 - q_1)u_1u_2\,dx\right| \le \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\,\|f_1\|_{H^{1/2}(\partial\Omega)}\,\|f_2\|_{H^{1/2}(\partial\Omega)}\,. \tag{3.20}$$

This is the estimate used as an important step in the next section.

Note that Theorem 3.5 also implies injectivity, that is, uniqueness of the inverse problem (1.14). The following proof of uniqueness is based on [18], but we will use CGO solutions of the form $e^{i\xi\cdot x}(1+\psi)$ instead of $e^{i\xi\cdot x}(a+\psi)$ as used in [18].

Let $\Lambda_{q_1}, \Lambda_{q_2}$ be two Dirichlét-to-Neumann maps defined as usual by (1.9). Assume that

$$\Lambda_{q_1} = \Lambda_{q_2}. \tag{3.21}$$

We wish to show injectivity, meaning that (3.21) implies $q_1 = q_2$. Plugging (3.21) into the result of Theorem 3.5 implies that

$$\int_\Omega (q_2 - q_1)u_1u_2\,dx = 0 \tag{3.22}$$

for any solutions $u_l \in H^1(\Omega)$ to (3.15), where $l = 1, 2$. Let $\xi_l \in \mathbb{C}^n$ and choose ξ_1 and ξ_2 like we did in (2.18) such that $\xi_1 + \xi_2 = -r\eta$, where $r \ge 0$ and $\eta \in \mathbb{R}^n$, $|\eta| = 1$. Then we are looking for CGO solutions such that the product u_1u_2 is close to $e^{i(\xi_1+\xi_2)\cdot x}$, since then (3.22) is close to the Fourier transform of $(q_2 - q_1)\chi_\Omega$, where χ is the characteristic function. Recall that this was also the argument in the case of the linearised problem. We know that if the Fourier transform is zero, so is the function itself by Plancherel's equation. By the choice of ξ_l (3.22) becomes

$$\begin{aligned} 0 &= \int_\Omega (q_2 - q_1)e^{i\xi_1\cdot x}(1+\psi_1)e^{i\xi_1\cdot x}(1+\psi_2)\,dx \\ &= \int_\Omega (q_2 - q_1)e^{-ir\eta\cdot x}\,dx + \int_\Omega (q_2 - q_1)(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx. \end{aligned}$$

Recall from Theorem 3.3 that for large enough $|Im(\xi)|$ it holds that $\|\psi_l\|_{H^s(\Omega)} \leq \frac{2C_0}{|Im(\xi)|}\|q\|_{H^s(\Omega)}$. So under the assumptions of Theorem 3.1 and Theorem 3.3 by using Hölder's inequality and letting $|Im(\xi)| \to \infty$ the second integral above go to zero and we are left with

$$0 = \int_\Omega (q_2 - q_1)e^{-ir\eta\cdot x}\, dx = \int_{\mathbb{R}^n} \widetilde{q}e^{-ir\eta\cdot x}\, dx = \mathcal{F}\widetilde{q}(r\eta),$$

where $\widetilde{q}$ is the zero extension of $q_2 - q_1$. This implies that $q_1 = q_2$ on Ω as desired.

3.3 Fourier transform of potentials

We are now almost ready to investigate the stability of the non-linear inverse problem (1.14), but like in the linearised case we first need an estimate of the Fourier transform of the zero extension of $q_1 - q_2$. The rest of this chapter relies on the work [9] by Isakov, Nagayasu and Wang and is also inspired by the similar article [15] by Nagayasu, Uhlmann and Wang. The article [9] provides a stability estimate for the inverse boundary value problem for the Helmholtz equation with potential (1.8) (called the Schrödinger equation in [9]), whereas [15] does the same for an acoustic equation $(\Delta + k^2 q)u = 0$ using very similar arguments. Both articles work with Cauchy data, $C_q = \left(u|_{\partial\Omega}, \frac{\partial u}{\partial\eta}|_{\partial\Omega}\right)$, as the boundary measurements. This is done to assure well-definedness of the problem for all $k \geq 1$. We will here derive a similar stability result, but use the Dirichlét-to-Neumann map, which gives a sharper estimate, but also means that we have to assume that zero is not an eigenvalue of (1.8).

Note that in the following C stands for a general constant depending on n, s, Ω, M and $\text{supp}(q_1 - q_2)$. C can appear more that once in an equation without necessarily being the same constant. When explicit constants are needed it will be clear from the notation.

The following proof starts like that of Theorem 2.1 in the linearised case. Since we no longer have the simple complex exponential, but CGO solutions, the main difficulties arise from the arrival of the perturbation function ψ.

Theorem 3.6 *Let $n \geq 3$ and let Λ_{q_l} be the Dirichlét-to-Neumann operator (1.9) for $l = 1, 2$. Let $s \geq \frac{n}{2}$ be an integer and $M > 0$. Assume $\|q_l\|_{H^s(\Omega)} \leq M$ and supp$(q_1 - q_2) \subset \Omega$. Denote the zero extension of $q_1 - q_2$ from Ω to $\mathbb{R}^n$ by $\tilde{q}$. Then for $r \geq 0$, $\eta \in \mathbb{R}^n$ satisfying $|\eta| = 1$, $a \geq C_1 M$ where C_1 is the constant* *from Theorem 3.3,* $k^2 + a^2 > \frac{r^2}{4}$*, and for* $k \geq 1$ *it holds*

$$|\mathcal{F}\tilde{q}(r\eta)| \leq Ck^2 e^{Ca} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} + \frac{C}{a} \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}, \tag{3.23}$$

where C *depends only on* M, s, n, Ω *and* $supp(q_1 - q_2)$*.*

Proof. By Theorem 3.3 we can construct solutions $u_l(x)$ to (1.8) under given assumptions. The theorem tells us that if $\xi_l \in \mathbb{C}^n$ satisfies

$$|Re(\xi)|^2 = k^2 + |Im(\xi)|^2 \tag{3.24a}$$

$$Re(\xi) \cdot Im(\xi) = 0 \tag{3.24b}$$

and

$$|Im(\xi)| \geq C_1 \|q\|_{H^s(\Omega)}, \tag{3.25}$$

then there exists a solution $u_l(x)$ to (1.8), where $q = q_l$, of the form

$$u_l(x) = e^{i\xi_l \cdot x}(1 + \psi_l(x))$$

and it holds that

$$\|\psi_l\|_{H^s(\Omega)} \leq \frac{C}{|Im(\xi_l)|} \|q\|_{H^s(\Omega)}. \tag{3.26}$$

Note that under these assumptions (3.25) and (3.26) implies that

$$\|\psi\|_{H^s(\Omega)} \leq \frac{C}{|Im(\xi_l)|} \|q\|_{H^s(\Omega)} \leq \frac{C}{C_1 \|q\|_{H^s(\Omega)}} \|q\|_{H^s(\Omega)} = C. \tag{3.27}$$

We will now choose ξ_l the same way as in the linear case (2.18). For this proof to be able to stand alone we will repeat the arguments here. Let $r \geq 0$, $\eta \in \mathbb{R}^n$ such that $|\eta| = 1$. Choose $\eta^\perp$ and $\zeta \in \mathbb{R}^n$ such that they are orthogonal to each other and to η

$$\eta \cdot \eta^\perp = \eta \cdot \zeta = \zeta \cdot \eta^\perp = 0 \tag{3.28}$$

and such that they have lengths

$$|\eta^\perp| = 1 \quad \text{and} \quad |\zeta| = a. \tag{3.29}$$

We now wish to define ξ_1 and ξ_2, such that the assumptions in Theorem 3.3 is satisfied as well as $\xi_1 + \xi_2 = -r\eta$. Define

$$\xi_1 = -\frac{r}{2}\eta + \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp + i\zeta \tag{3.30a}$$

$$\xi_2 = -\frac{r}{2}\eta - \sqrt{k^2 + a^2 - \frac{r^2}{4}}\eta^\perp - i\zeta. \tag{3.30b}$$

Let us now first check that the assumptions (3.24) and (3.25) holds.

$$\begin{aligned}|Re(\xi_l)|^2 &= \left(-\frac{r}{2}\eta \pm \sqrt{k^2+a^2-\frac{r^2}{4}}\eta^\perp\right)^2 \\ &= \frac{r^2}{4} + k^2 + a^2 - \frac{r^2}{4}k^2 + a^2 \\ &= k^2 + |Im(\xi_l)|^2\end{aligned}$$

and

$$Re(\xi)\cdot Im(\xi) = \left(-\frac{r}{2}\eta \pm \sqrt{k^2+a^2-\frac{r^2}{4}}\eta^\perp\right)\cdot(\pm\zeta) = 0.$$

Also $|Im(\xi_l)| = a \geq C_1 M \geq C_1 \|q\|_{H^s(\Omega)}$ by assumption. Finally it holds that

$$i(\xi_1+\xi_2) = -ir\eta. \tag{3.31}$$

This means that when given k, r and η satisfies the assumptions given on them in the theorem, we can choose $\eta^\perp$ and then ζ in a way such that $a^2 > \frac{r^2}{4} - k^2$. Note that this is only possible, since we are in dimension $n \geq 3$, due to the fact that we split the real part of ξ_l into two orthogonal parts, which are also orthogonal to the imaginary part. This is only possible if there are at least three dimensions.

Letting $r\eta \in \mathbb{R}^n$ be the Fourier frequency we have

$$|\mathcal{F}\widetilde{q}(r\eta)| = \frac{1}{(2\pi)^{n/2}}\left|\int_{\mathbb{R}^n} \widetilde{q}(x)e^{-ir\eta\cdot x}\,dx\right|.$$

Recall that $i(\xi_1+\xi_2) = -ir\eta$, so using that $u_l = e^{i\xi_l\cdot x}(1+\psi_l)$ we have that $u_1u_2 = e^{-ir\eta\cdot x}(1+\psi_1)(1+\psi_2)$. Let us first use this in the following integral over Ω.

$$\begin{aligned}\int_\Omega (q_2-q_1)u_1u_2\,dx &= \int_\Omega (q_2-q_1)e^{i(\xi_1+\xi_2)\cdot x}(1+\psi_1(x)+\psi_2(x)+\psi_1(x)\psi_2(x))\,dx \\ &= \int_\Omega (q_2-q_1)e^{-ir\eta\cdot x}(1+\psi_1(x)+\psi_2(x)+\psi_1(x)\psi_2(x))\,dx.\end{aligned}$$

This means

$$\begin{aligned}&\left|\int_\Omega (q_2-q_1)e^{-ir\eta\cdot x}\,dx\right| = \\ &\left|\int_\Omega (q_2-q_1)u_1u_2\,dx - \int_\Omega (q_2-q_1)e^{-ir\eta\cdot x}(\psi_1(x)+\psi_2(x)+\psi_1(x)\psi_2(x))\,dx\right| \leq \\ &\left|\int_\Omega (q_2-q_1)u_1u_2\,dx\right| + \left|\int_\Omega (q_2-q_1)e^{-ir\eta\cdot x}(\psi_1(x)+\psi_2(x)+\psi_1(x)\psi_2(x))\,dx\right|.\end{aligned} \tag{3.32}$$

Let us start out by considering the first integral. Again the arguments are similar to those of Theorem 2.1. By (3.20) we have that

$$\begin{aligned}\left|\int_\Omega (q_2-q_1)u_1u_2\,dx\right| &= |\langle(\Lambda_{q_1}-\Lambda_{q_2})\,f_1,f_2\rangle| \\ &\leq \|\Lambda_{q_1}-\Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\|f_1\|_{H^{1/2}(\partial\Omega)}\|f_2\|_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

Now

$$\begin{aligned}\|f_l\|_{H^{1/2}(\partial\Omega)} &= \|u_l|_{\partial\Omega}\|_{H^{1/2}(\partial\Omega)} \\ &\leq C\,\|u_l\|_{H^1(\Omega)} \\ &= C\left(\|u_l\|^2_{L^2(\Omega)}+\|\nabla u_l\|^2_{L^2(\Omega)}\right)^{1/2}.\end{aligned}$$

Let $R>0$ be large enough for $\Omega\subset B_R(0)$. Then for $x\in\Omega$

$$\begin{aligned}|u_l(x)| &= \left|e^{i\xi_l\cdot x}(1+\psi_l)\right| \\ &= \left|e^{i(-\frac{r}{2}\eta\pm\sqrt{k^2+a^2-\frac{r^2}{4}}\eta^\perp\pm i\zeta)\cdot x}(1+\psi_l)\right| \\ &\leq e^{|-\zeta\cdot x|}\,(1+|\psi_l|) \\ &\leq e^{|\zeta||x|}\left(1+\|\psi_l\|_{L^\infty(\Omega)}\right) \\ &\leq e^{aR}\left(1+\|\psi_l\|_{L^\infty(\Omega)}\right) \\ &\leq e^{aR}\left(1+C\,\|\psi_l\|_{H^s(\Omega)}\right) \quad \text{by} \quad \textit{Theorem A.4} \\ &\leq Ce^{aR}.\end{aligned}$$

Using this gives

$$\|u_l\|_{L^2(\Omega)}=\left(\int_\Omega |u_l|^2\;dx\right)^{1/2}\leq\left(\int_\Omega \left|Ce^{aR}\right|^2\;dx\right)^{1/2}\leq Ce^{aR}.$$

To find a bound on ∇u_l, we use that $\|\nabla\psi\|_{L^2(\Omega)}\leq\|\psi\|_{H^s(\Omega)}\leq C$, since $s>n/2\geq 3/2>1$.

$$\begin{aligned}\|\nabla u_l\|_{L^2(\Omega)} &= \left\|(1+\psi_l)\nabla e^{i\xi_l\cdot x}+e^{i\xi_l\cdot x}\nabla\psi_l\right\|_{L^2(\Omega)} \\ &\leq \left\|(1+\psi_l)i\xi_l e^{i\xi_l\cdot x}\right\|_{L^2(\Omega)}+\left\|e^{i\xi_l\cdot x}\nabla\psi_l\right\|_{L^2(\Omega)} \\ &= |\xi_l|\,\|u_l\|_{L^2(\Omega)}+e^{aR}\,\|\nabla\psi_l\|_{L^2(\Omega)} \\ &\leq \sqrt{k^2+2a^2}Ce^{aR}+e^{aR}C \\ &\leq Cke^{Ca},\end{aligned}$$

since there exists a constant C such that $a \leq e^{Ca}$. Combining these results gives an upper bound of the norm of f

$$\|f_l\|_{H^{1/2}(\partial\Omega)} \leq Cke^{Ca}.$$

Coming back to the integral, this implies that

$$\left|\int_\Omega (q_2 - q_1)u_1u_2 \, dx\right| \leq Ck^2e^{Ca} \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}. \tag{3.33}$$

Consider now the second integral in (3.32). This is where the proof differs from the linearised case. Let $\phi \in C_0^\infty(\Omega)$ satisfy $\phi = 1$ near $\text{supp}(q_1 - q_2)$, so it represents the $\text{supp}(q_1 - q_2)$. Since $q_1 - q_2$ has compact support in Ω it holds that $\phi(\psi_1 + \psi_2 + \psi_1\psi_2) \in H_0^s(\Omega)$ and we have

$$\begin{aligned}
&\left|\int_\Omega (q_2 - q_1)e^{-ir\eta\cdot x}(\psi_1 + \psi_2 + \psi_1\psi_2) \, dx\right| \\
&\qquad = \left|\int_\Omega (q_2 - q_1)e^{-ir\eta\cdot x}\phi(\psi_1 + \psi_2 + \psi_1\psi_2) \, dx\right| \\
&\qquad = \left|\left\langle (q_2 - q_1)e^{-ir\eta\cdot x}, \phi(\psi_1 + \psi_2 + \psi_1\right\rangle_{H^s(\Omega)\to H^{-s}(\Omega)}\right| \\
&\qquad \leq \|q_2 - q_1\|_{H^{-s}(\Omega)} \|\phi(\psi_1 + \psi_2 + \psi_1\psi_2)\|_{H^s(\Omega)}
\end{aligned}$$

by (3.14). Using that $H^s(\Omega)$ is an algebra (Theorem A.3) and the bounds (3.26) and (3.27) we get

$$\begin{aligned}
\|\phi(\psi_1 + \psi_2 + \psi_1\psi_2)\|_{H^s(\Omega)} &\leq \|\phi\|_{H^s(\Omega)} \left(\|\psi_1\|_{H^s(\Omega)} + \|\psi_2\|_{H^s(\Omega)} + \|\psi_1\|_{H^s(\Omega)} \|\psi_2\|_{H^s(\Omega)}\right) \\
&\leq \|\phi\|_{H^s(\Omega)} \left(\frac{CM}{a} + \frac{CM}{a} + C\frac{CM}{a}\right) \\
&\leq \frac{C}{a}.
\end{aligned}$$

Combining the two results for the two integrals in (3.32) means that

$$\begin{aligned}
\left|\int_\Omega (q_2 - q_1)e^{-ir\eta\cdot x} \, dx\right| &\leq Ck^2e^{Ca} \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\qquad + \frac{C}{a} \|q_2 - q_1\|_{H^{-s}(\Omega)}.
\end{aligned}$$

Extending $q_1 - q_2$ by zero to $\mathbb{R}^n$, it then holds that the Fourier Transform of the zero extension $\widetilde{q}$ is bounded as

$$\begin{aligned}
|\mathcal{F}\widetilde{q}(r\eta)| &= \left|\frac{1}{(2\pi)^{n/2}} \int_\Omega \widetilde{q}e^{-ir\eta\cdot x} \, dx\right| \\
&\leq Ck^2e^{Ca} \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{a} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}.
\end{aligned} \tag{3.34}$$

□ When moving on to estimating $\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}$ in the next section it turns out

to be an advantage to consider two different ranges of r. This is due to the fact that the above estimate depends on r. Doing this gives the following lemma as an easy consequence of Theorem 3.6.

Lemma 3.1 *Let $n \geq 3$ and let Λ_{q_l} be the Dirichlét-to-Neumann operator (1.9) for $l = 1, 2$. Let $s > \frac{n}{2}$ be an integer and $M > 0$. Assume $\|q_l\|_{H^s(\Omega)} \leq M$ and supp$(q_1 - q_2) \subset \Omega$. Denote the zero extension of $q_1 - q_2$ from Ω to $\mathbb{R}^n$ by $\tilde{q}$. Let*

$$R > C_1 M,$$

where C_1 is the constant from Theorem 3.3. Then for $r \geq 0$, $\eta \in \mathbb{R}^n$ satisfying $|\eta| = 1$, $a \geq C_1 M$ with $k^2 + a^2 > \frac{r^2}{4}$ and for $k \geq 1$ the following holds.

For $0 \leq r \leq k + R$

$$|\mathcal{F}\widetilde{q}(r\eta)| \leq Ck^2 e^{CR} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{R} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \tag{3.35}$$

and for $r > k + R$

$$|\mathcal{F}\widetilde{q}(r\eta)| \leq Ck^2 e^{Cr} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{r} \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}, \tag{3.36}$$

where C depends only on M, s, n, Ω and supp$(q_1 - q_2)$ and C_1 is the constant from Theorem 3.3.

Proof. n the first case, assuming that $0 \leq r \leq k + R$, choose

$$a = R$$

in Theorem 3.6. This means that $a = R > C_1 M \geq C_1 \|q_l\|_{H^s(\Omega)}$ and $r \leq k + a$, so $r^2 \leq 2(k^2 + a^2) \leq 4(k^2 + a^2)$ and it follows

$$|\mathcal{F}\widetilde{q}(r\eta)| \leq Ck^2 e^{CR} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{R} \|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)}.$$

In the second case, assuming that $r > k + R$, letting

$$a = r$$

means that $a = r > k + R > k + C_1 M \geq C_1 \|q_l\|_{H^s(\Omega)}$ and also $k^2 + a^2 = k^2 + r^2 > r^2 \geq r^2/4$. It then follows

$$|\mathcal{F}\widetilde{q}(r\eta)| \leq Ck^2 e^{Cr} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{r} \|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}.$$

□

3.4 Proof of main stability result

Now we will prove the main stability result. This is not a constructive proof, since it is based on the result we are trying to show, so it does not offer much intuition. The reason for the difficulties here, compared to the linear case, is that the use of CGO solutions in Theorem 3.6 requires assumptions on $|Im(\xi)| = a$.

Theorem 3.7 *Let $n \geq 3$ and let Λ_{q_l} be the Dirichlét-to-Neumann operator (1.9) for $l = 1, 2$. Let $s > \frac{n}{2}$ and $M > 0$. Assume $\|q_l\|_{H^s(\Omega)} \leq M$ and supp$(q_1 - q_2) \subset \Omega$ and that u_l solves (3.15) with $q = q_l$. Denote the zero extension of $q_1 - q_2$ from Ω to $\mathbb{R}^n$ by $\tilde{q}$. Then for $k \geq 1$ and $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq \frac{1}{e}$ the following estimate holds*

$$\begin{aligned}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \leq{}& Ck^2 \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ &+ C\left(k + \log \frac{1}{\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}}\right)^{-(2s-n)}, \end{aligned} \tag{3.37}$$

where C depends only on n, s, Ω, M and supp$(q_1 - q_2)$.

Proof.

Again let C be a general constant depending only on n, s, M and Ω. Write $\gamma \in \mathbb{R}^n$ in polar coordinates, $\gamma = r\eta$, where $r \geq 0$ and η lies on the unit sphere in $\mathbb{R}^n$ we use the change of variables (2.21) as in the proof of Theorem 2.2

$$d\gamma = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}\, dr\, d\eta.$$

Performing this change of variables gives the following estimate for the zero extension of $q_2 - q_1$

$$
\begin{aligned}
\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |1+|\gamma|^2|^{-s} |\mathcal{F}\widetilde{q}(\gamma)|^2 \, d\gamma \\
&= \int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 \, \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} \, dr \, d\eta \\
&= C \int_0^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta \\
&= C \int_0^{k+R} \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta \\
&\quad + C \int_{k+R}^{T} \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta \\
&\quad + C \int_{T}^{\infty} \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta \\
&= C(I_1 + I_2 + I_3),
\end{aligned}
$$

where $R > C_1 M$ and $T \geq k+R$ are constants that will be chosen later. Comparing to the linear case, the estimate of the Fourier transform depends on r, which is why we here need to split the integral into several parts and investigate then separately. Splitting up the integrals in this way allows us to use Theorem 3.6 on the first two integrals I_1 and I_2. Let us first consider the last integral I_3. Note that

$$
|\mathcal{F}\widetilde{q}(\gamma)| \leq \int_{\mathbb{R}^n} |\widetilde{q}(x)| \, dx = \int_{\Omega} |q_1(x) - q_2(x)| \, dx \leq |\Omega|^{1/2} \, \|q_1 - q_2\|_{L^2(\Omega)} \, ,
$$

so

$$
\begin{aligned}
I_3 &= C \int_T^\infty \int_{S(n-1)} |1+r^2|^{-s} |\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1} \, dr \, d\eta \\
&\leq C|\Omega| \, \|q_1 - q_2\|^2_{L^2(\Omega)} \int_{S(n-1)} d\eta \int_T^\infty |1+r^2|^{-s} r^{n-1} \, dr \\
&\leq C \, \|q_2 - q_1\|^2_{L^2(\Omega)} \int_T^\infty \frac{r^{n-1}}{r^{2s}} \, dr.
\end{aligned}
$$

Letting $m = 2s - n$ and remembering that $-m = n - 2s < 0$ we get

$$
\begin{aligned}
I_3 &\leq C \, \|q_2 - q_1\|^2_{L^2(\Omega)} \int_T^\infty r^{n-1-2s} \, dr \\
&= C \, \|q_2 - q_1\|^2_{L^2(\Omega)} \, \frac{1}{n-2s} \left[r^{n-2s} \right]_{r=T}^{\infty} \\
&= C \, \|q_2 - q_1\|^2_{L^2(\Omega)} \, T^{-m}.
\end{aligned}
$$

Since $q_2 - q_1 \in H_0^s(\Omega)$ Theorem 3.4 implies that for any $\epsilon > 0$ it holds that

$$\begin{aligned} I_3 &\leq CT^{-m}\left(\epsilon \left\|q_1 - q_2\right\|^2_{H^{-s}(\Omega)} + \frac{1}{\epsilon}\left\|q_1 - q_2\right\|^2_{H^s(\Omega)}\right) \\ &\leq CT^{-m}\left(\epsilon \left\|\widetilde{q}\right\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{1}{\epsilon}M^2\right). \end{aligned} \tag{3.38}$$

Note that $s > n/2$ means that (2.22) holds, so coming back to the first integral I_1, Theorem 3.6 and (2.22) implies that

$$\begin{aligned} I_1 &= \int_0^{k+R}\int_{S(n-1)} |1+r^2|^{-s}|\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1}\,dr\,d\eta \\ &\leq C\int_{S(n-1)} d\eta \int_0^{k+R} |1+r^2|^{-s}\Big(Ck^2e^{CR}\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ &\quad + \frac{C}{R}\left\|\tilde{q}\right\|_{H^{-s}(\mathbb{R}^n)}\Big)^2 r^{n-1}\,dr \\ &\leq Ck^4e^{2CR}\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{R^2}\left\|\tilde{q}\right\|^2_{H^{-s}(\mathbb{R}^n)} \\ &\quad \cdot \int_0^\infty |1+r^2|^{-s} r^{n-1}\,dr \\ &= Ck^4e^{2CR}\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{C}{R^2}\left\|\tilde{q}\right\|^2_{H^{-s}(\mathbb{R}^n)}. \end{aligned} \tag{3.39}$$

Finally we need the following two estimates

$$\begin{aligned} \int_{k+R}^T e^{Cr}|1+r^2|^{-s}r^{n-1}\,dr &\leq e^{CT}\int_{k+R}^T |1+r^2|^{-s}r^{n-1}\,dr \\ &\leq e^{CT}\int_0^\infty |1+r^2|^{-s}r^{n-1}\,dr \\ &\leq Ce^{CT}, \end{aligned}$$

and

$$\begin{aligned}
\int_{k+R}^{T}(1+r^2)^{-s}r^{n-3}\,dr &\le \int_{k+R}^{T} r^{n-3-2s}\,dr \\
&= \frac{1}{n-2-2s}\left(T^{n-2-2s}-(k+R)^{n-2-2s}\right) \\
&\le \frac{1}{n-2-2s}(k+R)^{n-2-2s} \\
&\le C\frac{1}{(k+R)^{2+m}} \\
&\le C\frac{1}{(k+R)^{2}} \\
&\le \frac{C}{R^2} \quad \text{since} \quad k\ge 1.
\end{aligned}$$

Using these we get the following for the second integral, I_2, we get

$$\begin{aligned}
I_2 &= \int_{k+R}^{T}\int_{S(n-1)} |1+r^2|^{-s}|\mathcal{F}\widetilde{q}(r\eta)|^2 r^{n-1}\,d\eta\,dr \\
&\le A(n-1)\int_{k+R}^{T}|1+r^2|^{-s}\Big(Ck^2e^{Cr}\left\|\Lambda_{q_1}-\Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\quad + \frac{C}{r}\left\|\tilde{q}\right\|_{H^{-s}(\mathbb{R}^n)}\Big)^2 r^{n-1}\,dr \\
&\le k^4C\left\|\Lambda_{q_1}-\Lambda_{q_2}\right\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\int_{k+R}^{T}|1+r^2|^{-s}e^{Cr}r^{n-1}\,dr \\
&\quad + C\left\|\tilde{q}\right\|^2_{H^{-s}(\mathbb{R}^n)}\int_{k+R}^{T}|1+r^2|^{-s}r^{n-3}\,dr \\
&\le Ck^4e^{CT}\left\|\Lambda_{q_1}-\Lambda_{q_2}\right\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + C\frac{\left\|\tilde{q}\right\|^2_{H^{-s}(\mathbb{R}^n)}}{R^2}. \qquad (3.40)
\end{aligned}$$

We can now combine these three results and get a joined bound of $\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}$.

$$\begin{aligned}
\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &\leq C(I_1 + I_2 + I_3) \\
&\leq C\left(k^4 e^{2CR}\|\Lambda_{q_1} - \Lambda_{q_2}\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \frac{\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}}{R^2}\right) \\
&\quad + Ck^4 e^{CT}\|\Lambda_{q_1} - \Lambda_{q_2}\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + C\frac{\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}}{R^2} \\
&\quad + CT^{-m}\left(\epsilon\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + \frac{1}{\epsilon}M^2\right) \\
&= \left(Ck^4 e^{2CR} + Ck^4 e^{CT}\right)\|\Lambda_{q_1} - \Lambda_{q_2}\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\
&\quad + C\left(\frac{2}{R^2} + T^{-m}\epsilon\right)\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + C\frac{T^{-m}}{\epsilon}
\end{aligned}$$

The rest of the proof is rather technical, where it is investigated and shown how to choose the parameters R, ϵ and T to get to the final estimate (3.37). It turns out that it is desirable to divide the remaining proof into two cases. Define the cases

$$(i) \quad k + R \leq p\log\left(\frac{1}{D}\right) \tag{3.41}$$

and

$$(ii) \quad k + R \geq p\log\left(\frac{1}{D}\right), \tag{3.42}$$

where $R > C_1 M$ and $p > 0$ will be chosen later and we denote denote $D = \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ for simplicity.

- Case *(i)*

Denoting some specific constants will help us choose R and ϵ in a desirable way. Let C_2, C_3 be positive constants depending only in n, s, Ω, M and $\operatorname{supp}(q_2 - q_1)$ such that

$$\begin{aligned}
\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} &\leq C\left(k^4 e^{2CR} + k^4 e^{CT}\right)D^2 \\
&\quad + \left(\frac{C_2}{R^2} + C_3 T^{-m}\epsilon\right)\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + C\frac{T^{-m}}{\epsilon}
\end{aligned}$$

We now pick ϵ and R such that the second term is a constant 1/2 times $\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)}$. To do this let

$$R > 2\sqrt{C_2} \quad \text{and} \quad \epsilon = \frac{T^m}{4C_3}, \tag{3.43}$$

so

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq C\left(k^4 e^{2CR} + k^4 e^{CT}\right) D^2 + \left(\frac{1}{4} + \frac{1}{4}\right) \|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + CT^{-2m}.$$

Since R is simply a large enough constant we get

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4 D^2 + Ck^4 e^{CT} D^2 + CT^{-2m}. \tag{3.44}$$

We now choose

$$T = p \log\left(\frac{1}{D}\right),$$

which is greater than or equal to $k + R$ by the condition (3.41). We wish to show that there exists a $C_4 > 0$ such that

$$k^4 e^{CT} D^2 \leq C_4 \left(k + \log\left(\frac{1}{D}\right)\right)^{-2m} \tag{3.45}$$

and

$$T^{-2m} \leq C_4 \left(k + \log\left(\frac{1}{D}\right)\right)^{-2m}, \tag{3.46}$$

since then by (3.44)

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4 D^2 + 2C_4 \left(k + \log\left(\frac{1}{D}\right)\right)^{-2m}$$

which is our desired result. Let us now examine what it requires to satisfy (3.45) and (3.46).

By the choice of T (3.46) is equivalent to

$$C_4^{-\frac{1}{2m}} \left(k + \log\left(\frac{1}{D}\right)\right) \leq p \log\left(\frac{1}{D}\right). \tag{3.47}$$

Now by the case *(i)* condition (3.41) we have

$$k + \log\left(\frac{1}{D}\right) \leq k + R + \log\left(\frac{1}{D}\right) \leq (p+1)\log\left(\frac{1}{D}\right), \tag{3.48}$$

which means that (3.47) (i.e. (3.46) holds whenever

$$C_4^{-\frac{1}{2m}} \leq \frac{p}{p+1}. \tag{3.49}$$

Now, looking at (3.45) and using that the logarithmic function is increasing and the choice of T, we have that it is equivalent to

$$\log\left(k^4 e^{CT} D^2\right) \leq \log\left(C_4\left(k+\log\left(\frac{1}{D}\right)\right)^{-2m}\right) \Leftrightarrow$$

$$4\log(k) + CT + 2\log(D) \leq \log(C_4) - 2m\log\left(k+\log\left(\frac{1}{D}\right)\right) \Leftrightarrow$$

$$4\log(k) + (Cp-2)\log\left(\frac{1}{D}\right) + 2m\log\left(k+\log\left(\frac{1}{D}\right)\right) \leq \log(C_4), \quad (3.50)$$

since $\log(D) = -\log\left(\frac{1}{D}\right)$. Using (3.48), which comes from being in case *(i)*, that is, and that $k \leq k+R \leq p\log\left(\frac{1}{D}\right)$, we can bound the left-hand side of (3.50) by

$$\begin{aligned}\text{LHS of (3.50)} &\leq 4\log\left(p\log\left(\frac{1}{D}\right)\right) + (Cp-2)\log\left(\frac{1}{D}\right)\\ &\quad + 2m\log\left((p+1)\log\left(\frac{1}{D}\right)\right)\\ &\leq 4\log(p) + (Cp-2)\log\left(\frac{1}{D}\right) + 2m\log(p+1)\\ &\quad + 2(m+2)\log\left(\log\left(\frac{1}{D}\right)\right).\end{aligned}$$

Now choosing

$$p \leq \frac{3}{2C}, \quad (3.51)$$

we get

$$\begin{aligned}\text{LHS of (3.50)} &\leq 4\log\left(\frac{3}{2C}\right) + \left(\frac{3}{2}-2\right)\log\left(\frac{1}{D}\right) + 2m\log\left(\frac{3}{2C}+1\right)\\ &\quad + 2(m+2)\log\left(\log\left(\frac{1}{D}\right)\right)\\ &= 4\log\left(\frac{3}{2C}\right) + 2m\log\left(\frac{3}{2C}+1\right) + 2(m+2)\log\left(\log\left(\frac{1}{D}\right)\right)\\ &\quad - \frac{1}{2}\log\left(\frac{1}{D}\right)\\ &\leq 4\log\left(\frac{3}{2C}\right) + 2m\log\left(\frac{3}{2C}+1\right) + \max_{z\geq 1}\left\{-\frac{1}{2}z + 2(m+2)z\right\}\end{aligned}$$

Note that

$$\begin{aligned}\max_{0<D\leq 1/e}\left(2(m+2)\log\left(\log\left(\frac{1}{D}\right)\right) - \frac{1}{2}\log\left(\frac{1}{D}\right)\right) &= \max_{z\geq 1}\left(2(m+2)\log(z) - \frac{1}{2}z\right)\\ &= 2(m+2)\left(\log(4m+8)-1\right),\end{aligned}$$

since $-\frac{1}{2}z + 2(m+2)\log(z)$ has global maximum at $z = 4m+8$. This means that

$$\text{LHS of (3.50)} \leq 4\log\left(\frac{3}{2C}\right) + 2m\log\left(\frac{3}{2C}+1\right) + 2(m+2)\left(\log(4m+8)-1\right)$$

so condition (3.50) (i.e. (3.45)) holds whenenver

$$4\log\left(\frac{3}{2C}\right) + 2m\log\left(\frac{3}{2C}+1\right) + 2(m+2)\left(\log(4m+8)-1\right) \leq \log(C_4) \tag{3.52}$$

- Case *(ii)*

Second, consider the case (3.42). Let $T = k + R$ and observe that this choice makes the second integral I_2 disappear and we have

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4e^{2CR}D^2 + C\left(\frac{1}{R^2} + T^{-m}\epsilon\right)\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + C\frac{T^{-m}}{\epsilon}.$$

Setting

$$\epsilon = \frac{T^m}{R^2}$$

and using the choice of T implies that

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4e^{CR}D^2 + \frac{2C}{R^2}\|\tilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} + CR^2(k+R)^{-2m}.$$

Now choosing

$$R > 2\sqrt{C}, \tag{3.53}$$

so the coefficient in front of $\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}$ is less that 1/2 and

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4D^2 + C(k+R)^{-2m}.$$

The condition (3.42) means that

$$k + R \geq k + \frac{R}{2} = \frac{k}{2} + \frac{k+R}{2} \geq \frac{k}{2} + \frac{p}{2}\log\left(\frac{1}{D}\right) \geq \frac{\min\{1,p\}}{2}\left(k + \log\left(\frac{1}{D}\right)\right),$$

so we get (since $m > 0$)

$$\|\widetilde{q}\|^2_{H^{-s}(\mathbb{R}^n)} \leq Ck^4D^2 + C\left(\frac{\min\{1,p\}}{2}\left(k + \log\left(\frac{1}{D}\right)\right)\right)^{-2m},$$

so

$$\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} \leq Ck^2D + C\left(k + \log\left(\frac{1}{D}\right)\right)^{-m},$$

which is the desired result when we pick any $p > 0$.

To finish to proof we must choose appropriate R, p and C_4. First we pick $R > C_1 M$ large enough to satisfy (3.43) and (3.53). Then we choose p sufficiently small to satisfy (3.51). Finally, we choose C_4 large enough to satisfy (3.49) and (3.52).

□

Estimate (3.37) in Theorem 3.7 consists of two parts; a Lipschitz part and a logarithmic part. When k is small the logarithmic part dominates, but as k increases this part is dampen and the Lipschitz part dominates the stability. This means that the stability gets better when k increases. To see this denote $D = \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$. The transition from logarithmic dominance to Lipschitz dominance happens at k^* satisfying

$$(k^*)^2 D = \left(k^* + \log\left(\frac{1}{D}\right)\right)^{-m}.$$

Figure 3.1 shows an example of how k^* behaves as a function of D when $m = 1$ (corresponding to $n = 3$ and $s = 2$). k^* is seen to be a decreasing function in D.

Now for $k < k^*$ by (3.37) it holds that

$$k^2 D < (k^*)^2 D = \left(k^* + \log\left(\frac{1}{D}\right)\right)^{-m} < \left(k + \log\left(\frac{1}{D}\right)\right)^{-m},$$

so here the logarithmic part dominates. Now when k increases and $k > k^*$ it holds that

$$\left(k + \log\left(\frac{1}{D}\right)\right)^{-m} < \left(k^* + \log\left(\frac{1}{D}\right)\right)^{-m} = (k^*)^2 D < k^2 D,$$

so the Lipschitz part dominates the stability. This means that the area above the curve in Figure 3.1 corresponds to the Lipschitz part dominating and the area under to the logarithmic type.

As D goes to zero k^* grows, meaning we need a very large k to make sure we are the Lipschitz part dominates. But it also means that if there exists information

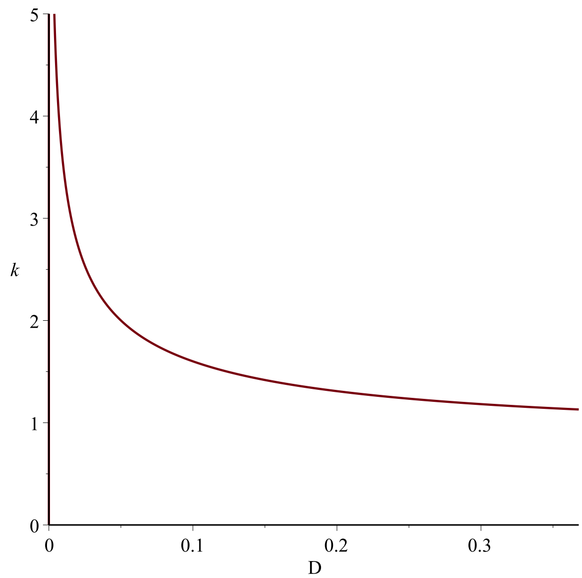

Figure 3.1: Behaviour of k^* over D for $m = 1$. When k is above the curve the stability is dominated by the Lipschitz part and when k is below by the logarithmic part.

about the error such as $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \sim \epsilon$ it can be determined at which value of k the transition happens. On the other hand, if the physical problem dictates that the frequency k is fixed or lies in a fixed interval, the logarithmic part dominates as $\epsilon \to 0$.

In general this shows that the stability increases with frequency. It means we can expect better stability for inverse problems dealing with high-frequency waves and this fact could be exploited when building reconstruction algorithms.

Note that the Lipschitz constant grows polynomial as k^2. k^2 comes from the fact that we used CGO solutions in the proof of Theorem 3.7. It is only an estimate and it is both important and interesting to see how the right-hand side of (3.37) behaves as a function of k. As mentioned it is shown in the zero frequency case that the logarithmic type stability is optimal, meaning that we cannot in general expect a better type of stability. To investigate if the constant k^2 is in some sense optimal, the next chapter examines how the right-hand side of (3.37) behaves for a specific choice of domain and potential. If the estimate is optimal, we expect the right-hand side of (3.37) to converge towards a constant value when k increases, since C and $\|\widetilde{q}\|_{H^{-s}(\Omega)}$ are independent of k. This is not a trivial exercise, since the operator norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ or an estimate hereof must be computed. Assuming that the estimate is optimal we would expect the operator norm to behave like k^{-2} for large k.

CHAPTER 4

Numerical investigation of optimality

Chapter 3 showed that the stability of the problem (1.14) increases with the frequency k in the sense that the logarithmic type stability is dampen when k increases. As mentioned before it is shown in for example [18] that for zero frequency the stability is of the logarithmic type and in [12] that this logarithmic stability is optimal in some sense for general potentials q. For large k the estimate (3.37) shows that the stability is of Lipschitz type with constant Ck^2. Though we know that (3.37) is optimal for $k = 0$, it is only an estimate, that is an upper bound, and it might be a conservative bound for $k \geq 1$. In this chapter we wish to investigate if the factor k^2 is optimal in the sense that the right hand side of (3.37) does not grow in k, but instead converges to $\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}$ when k increases. To evaluate (3.37) the operator norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ must be computed for a range of k, which is not a trivial task. Also $\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}$ can be computed, but since this is simply a value independent of k the exact value of $\|\tilde{q}\|_{H^{-s}(\mathbb{R}^n)}$ has no influence on the task at hand.

To numerically compute the operator norm or an estimate hereof it is shown in section 4.1 that the difference of two Dirichlét-to-Neumann maps is a compact operator on. This fact can be used to describe the operator norm in terms of the eigenvalues of the operator. Expressions for the eigenvalues are found in section 4.3 for the special case of q_1 being a spherically symmetric and piecewise constant potential and $q_2 = 0$. Recall that the estimate (3.37) is used to conclude how close we can expect two solutions of the inverse problem to be, when the corresponding data sets are close. This means that choosing two potentials in this way, we wish to see what happens when q_1 is close to zero.

In section 4.3 an investigation on the assumption that zero is not an eigenvalue of the Helmholtz equation with potential is carried out. The results found here will help to numerically estimate the operator norm in section 4.4. Finally the stability result is compared to the one derived in our main reference [9], where Cauchy data is used.

Some results in this chapter are numerical, but all based on theoretical investigations of the computed quantities and expressions.

4.1 Compactness

The first task at hand is to find an estimate of the operator norm $\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$. It turns out, that the Dirichlét-to-Neumann operator is completely determined by its spectral data, so we can approximate the operator norm by computing its eigenvalues ([14]). This is very useful in cases of particular choices of potential q as we will consider in his chapter.

To see why the operator norm of the difference Dirichlét-to-Neumann map can be described by its eiegnvalues, we will show that it is a compact operator. In general if an operator T is self-adjoint and compact on a Hilbert space it holds that

$$\max\left\{|(Tf,f)|\,\middle|\,f \in H, \quad \|f\| = 1\right\}$$

exists and equals the operator norm $\|T\|$. Moreover this maximum is attained for a normalized eigenvector with eigenvalue $\|T\|$ or $-\|T\|$ ([17] Corollary 6.7). The Spectral Theorem tells us that the absolute value of the eigenvalues can be arranged in a decreasing sequence (except from zeros) converging to zero, meaning that the first eigenvalue in this sequence is the operator norm. Let us now show that $\Lambda_q - \Lambda_0$ is in fact a compact operator.

The first step is to show boundedness of $\Lambda_q - \Lambda_0 : H^{1/2}(\partial\Omega) \to H^{m+1/2}(\partial\Omega)$ for any $m \in \mathbb{N}$. This is where the real work lies and the proof is rather technical. The crucial points in the proof is the use of q being compactly supported, so the Helmholtz equation with a potential becomes a simple Helmholtz equation close to the boundary, and also use of ellipticity of the partial differential equations is essential. The proof is based on part of the article [4]. In [4] compactness of the

Dircichlét-to-Neumann map for the conductivity equation is shown and we will here transfer the principles used to the Helmholtz equation with a potential.

Theorem 4.1 *Assume that $q \in L^\infty(\Omega)$ and that q has compact support, that is, $q = 0$ near the smooth boundary $\partial\Omega$. Then for any $m \in \mathbb{N}$*

$$\Lambda_q - \Lambda_0 : H^{1/2}(\partial\Omega) \to H^{m+1/2}(\partial\Omega)$$

is bounded and the operator norm is bounded as

$$\|\Lambda_q - \Lambda_0\|_{H^{1/2}(\partial\Omega)\to H^{m+1/2}(\partial\Omega)} \leq C\,\|q\|_{L^\infty(\Omega)}\,\|f\|_{H^{1/2}(\partial\Omega)}\,,$$

where C depends on k, Ω and m.

Proof.

Let $\Omega' \subset \Omega$ be an open domain such that $q = 0$ on $\Omega_0 = \Omega\backslash\overline{\Omega'}$. Let $\{\phi_n\}_{n=1}^m$ be a sequence of smooth functions, that has support near $\partial\Omega$ such that the following holds

- $0 \leq \phi_n \leq 1, n = 0, \ldots, m$
- $\phi_m = 1$ near $\partial\Omega$
- $\phi_0 = 0$ near $\partial\Omega'$
- $\phi_n = 1$ in $\Omega_{n+1} = \mathrm{supp}(\phi_{n+1})$ for $n = 1, \ldots, m-1$.

Now let u be the unique solution to (1.8) with $f \in H^{1/2}(\partial\Omega)$ on the boundary and let v solve $(\Delta + k^2)v = 0$ in Ω and $v = f$ on $H^{1/2}(\partial\Omega)$, so $u - v \in H_0^1(\Omega)$ solves the second order partial differential equation

$$(\Delta + k^2 + q)(u - v) = -qv. \tag{4.1}$$

ϕ_n is supported on Ω_n, so $\phi_n(u - v) \in H_0^1(\Omega_n)$ solves

$$\begin{aligned}(\Delta + k^2)(\phi_n(u - v)) &= \Delta\phi_n(u - v) + 2\nabla\phi_n \cdot \nabla(u - v) + \phi_n(\Delta + k^2)(u - v)\\ &= \Delta\phi_n(u - v) + 2\nabla\phi_n \cdot \nabla(u - v)\end{aligned}$$

in Ω_n, since $q = 0$ near $\partial\Omega$, so $(\Delta + k^2)(u - v) = 0$ in Ω_n. Since $\Delta + k^2$ is a second order elliptic differential operator, we have the following estimate (see e.g. [7])

$$\|\phi_n(u - v)\|_{H^{m+2}(\Omega)} \leq C\,\|\Delta\phi_n(u - v) + 2\nabla\phi_n \cdot \nabla(u - v)\|_{H^m(\Omega_n)}\,.$$

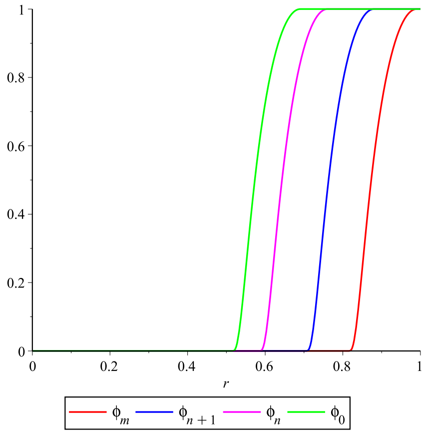

Figure 4.1: Example of behaviour of cut-off functions, where Ω is the unit ball and ϕ are assumed to be spherically symmetric. Here the radial part is plotted and $r = 0.5$ is $\partial\Omega'$.

Using that the Sobolev space $H^m(\Omega)$ is an algebra (see Theorem A.3) and that the square root is concave it holds that

$$\begin{aligned}
\|\phi_n(u-v)\|_{H^{m+2}(\Omega)} &\le C\,\|\Delta\phi_n\|_{H^m(\Omega_n)}\,\|(u-v)\|_{H^m(\Omega_n)} \\
&\quad + \|2\nabla\phi_n\|_{H^m(\Omega_n)}\,\|\nabla(u-v)\|_{H^m(\Omega_n)} \\
&\le C\left(\|(u-v)\|_{H^m(\Omega_n)} + \|\nabla(u-v)\|_{H^m(\Omega_n)}\right) \\
&\le C\Big(\sum_{|\alpha|\le m}\|D^\alpha(u-v)\|^2_{L^2(\Omega_n)} \\
&\quad + \sum_{|\alpha|\le m+1}\|D^\alpha(u-v)\|^2_{L^2(\Omega_n)}\Big)^{1/2} \\
&\le C\Big(\sum_{|\alpha|\le m}\|D^\alpha(u-v)\|^2_{L^2(\Omega_n)} + \left\|D^{m+1}(u-v)\right\|^2_{L^2(\Omega_n)}\Big)^{1/2} \\
&= C\,\|u-v\|_{H^{m+1}(\Omega_n)} \\
&\le C\,\|\phi_{n-1}(u-v)\|_{H^{m+1}(\Omega)}\,,
\end{aligned}$$

where C depends on k, Ω and ϕ_n. This holds for $n = 1, \ldots, m-1$, so by induction we get that

$$\begin{aligned}
\|\phi_m(u-v)\|_{H^{m+2}(\Omega)} &\leq C\,\|\phi_{m-1}(u-v)\|_{H^{m+1}(\Omega)} \\
&\leq C\,\|\phi_{m-2}(u-v)\|_{H^{m}(\Omega)} \\
&\ \ \vdots \\
&\leq C\,\|\phi_0(u-v)\|_{H^{2}(\Omega)}\,. \qquad (4.2)
\end{aligned}$$

Note also that

$$\begin{aligned}
\|\phi_0(u-v)\|_{H^{2}(\Omega)} &\leq C\,\|\Delta\phi_0(u-v)+2\nabla\phi_0\cdot\nabla(u-v)\|_{H^{0}(\Omega_n)} \\
&\leq C\,\|u-v\|_{H^{1}(\Omega_0)} \\
&\leq C\,\|u-v\|_{H^{1}(\Omega)}\,. \qquad (4.3)
\end{aligned}$$

Now since $u-v \in H^1_0(\Omega)$ satisfies (4.1), which is a second order elliptic operator it holds that

$$\begin{aligned}
\|u-v\|_{H^{1}(\Omega)} &\leq C\,\|-qv\|_{H^{-1}(\Omega)} \\
&\leq C\,\|q\|_{L^\infty(\Omega)}\,\|v\|_{H^{1}(\Omega)} \\
&\leq C\,\|q\|_{L^\infty(\Omega)}\,\|f\|_{H^{1/2}(\partial\Omega)}\,.
\end{aligned}$$

Combining the above results and using that the composition of the normal derivative and the trace operator is bounded, we can bound the Dirichlét-to-Neumann difference operator

$$\begin{aligned}
\|(\Lambda_q-\Lambda_0)f\|_{H^{m+1/2}(\partial\Omega)} &= \left\|\frac{\partial}{\partial\eta}(u-v)\right\|_{H^{m+1/2}(\partial\Omega)} \\
&= \left\|\frac{\partial}{\partial\eta}(\phi_m(u-v))\right\|_{H^{m+1/2}(\partial\Omega)}, \quad \text{since} \quad \phi_m = 1 \quad \text{near} \quad \partial\Omega \\
&\leq C\,\|\phi_m(u-v)\|_{H^{m+2}(\Omega)} \\
&\leq C\,\|\phi_0(u-v)\|_{H^{2}(\Omega)} \\
&\leq C\,\|u-v\|_{H^{1}(\Omega)} \\
&\leq C\,\|q\|_{L^\infty(\Omega)}\,\|f\|_{H^{1/2}(\partial\Omega)}\,.
\end{aligned}$$

□

Using that the embedding $H^{m+1/2}(\partial\Omega) \to H^{m+1/2-\epsilon}(\partial\Omega)$ is compact for any $\epsilon > 0$ ([1]) compactness is an easy consequence of Theorem 4.1.

Theorem 4.2 *The difference operator*

$$\Lambda_q - \Lambda_0$$

is compact on $H^{1/2}(\partial\Omega)$.

Proof. Theorem 4.1 tells us that

$$\Lambda_q - \Lambda_0 : H^{1/2}(\partial\Omega) \to H^{m+1/2}(\partial\Omega)$$

is bounded for any $m \in \mathbb{N}$. Now since the inclusion $H^{m+1/2}(\partial\Omega) \subset H^{m+1/2-\epsilon}(\partial\Omega)$ is compact for $m \in \mathbb{N}$ and for any $\epsilon > 0$, meaning that the identity operator

$$i : H^s(\partial\Omega) \hookrightarrow H^{m+1/2-\epsilon}(\partial\Omega)$$

is a compact embedding, the operator

$$i \circ (\Lambda_q - \Lambda_0) : H^{1/2}(\partial\Omega) \to H^s(\partial\Omega) \hookrightarrow H^{m+1/2-\epsilon}(\partial\Omega)$$

is compact as a composition of a bounded and a compact operator. Since this holds for any $m \in \mathbb{N}$ and $\epsilon > 0$ the result follows.

□

As mentioned, this result means that we can describe the map $\Lambda_q - \Lambda_0$ by its eigenvalues, where the largest corresponds to the operator norm. The next section deals with the problem of finding the eigenvalues of $\Lambda_q - \Lambda_0$ in a particular case of Ω and q. The rest of this chapter and the numerical results here will be based on this special case.

4.2 Eigenvalues

The problem of finding the largest eigenvalue for the Dirichlét-to-Neumann difference operator $\Lambda_{q_1} - \Lambda_{q_2}$ is now addressed. This will be done for specific choices of Ω, q_1 and q_2 as mentioned. The proofs of this section are build on the article [13], where an expression for the eigenvalues are derived for the Dirichlét-to-Neumann map of the conductivity problem. We will here use the same concepts to derive similar results for the Helmholtz equation with a potential. It turns out that we are dealing with Bessel's differential equations compared to an Euler type equation in [13]. This makes the computations more complicated, but the principles are the same.

Consider the special case where Ω is the unit sphere in $\mathbb{R}^3$ and where $q(x) = q(r)$ is spherically symmetric. We will choose $q_2 = 0$ and $q_1 = q(r)$ to be a piecewise constant function in r. The approach is to find general expressions for the eigenvalues of Λ_q and Λ_0, called λ_l^q and λ_l^0 respectively, and use these to express the eigenvalues of the difference operator by

$$\lambda_l = \lambda_q^q - \lambda_l^0.$$

We will consider the special case, where the spherically symmetric and piecewise constant q is given by

$$q(r) = \begin{cases} c & \text{when} \quad r \leq \rho \\ 0 & \text{when} \quad r > \rho \end{cases}, \tag{4.4}$$

where $0 < \rho < 1$. Note that this is not a continuous potential as assumed in Theorem 3.7, but we expect the estimate to still hold though it is important to note that the assumption of continuity is not satisfied.

The techniques used in this section relies mainly on separation of variables. Some long, but simple, computations are performed using the mathematical software Maple and left out here.

First Theorem 4.3 below will provide an expression for the eigenvalues of Λ_q for a general spherical symmetric potential $q(x) = q(r)$.

Theorem 4.3 *Let Ω be the unit sphere in $\mathbb{R}^3$ and assume that $q(x) = q(r)$ is spherically symmetric. Then the eigenfunctions of Λ_q are the spherical harmonic functions Y_l^m ([19] p. 275) and the eigenvalues are given as*

$$\lambda_l^q = \left.\frac{\partial R_l(r)}{\partial r}\right|_{r=1}, \tag{4.5}$$

where R_l solves the Bessel's differential equation (4.6).

Proof. Consider the Helmholtz equation with a potential where the boundary function is independent of r and given by a spherical harmonic function

$$\begin{aligned} (\Delta + k^2 + q(r))u_{lm} &= 0 \quad \text{in} \quad \Omega, \\ \left.u_{lm}\right|_{r=1} &= Y_l^m \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Split up u_{lm} into a radial and a spherical part $u(x, y, z) = u(r, \theta, \phi) = R(r)Y(\theta, \phi)$. Separation of variables then yields

$$R''Y + \frac{2}{r}R'Y + \frac{R}{r^2}\left(\frac{1}{\sin^2\theta}Y_{\phi\phi} + \frac{1}{\sin(\theta)}(\sin(\theta)Y_\theta)_\theta\right) + (q(r) + k^2)RY = 0.$$

Let γ be the separation constant, so

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+r^2(q(r)+k^2)=-\frac{1}{\sin^2\theta}\frac{Y_{\phi\phi}}{Y}-\frac{1}{\sin(\theta)Y}(\sin(\theta)Y_\theta)_\theta=\gamma.$$

This gives the two ordinary differential equations

$$R''+\frac{2}{r}R'+\left(q(r)+k^2-\frac{\gamma}{r^2}\right)R=0 \tag{4.6}$$

and

$$\frac{1}{\sin^2\theta}Y_{\phi\phi}+\frac{1}{\sin(\theta)}(\sin(\theta)Y_\theta)_\theta+\gamma Y=0 \tag{4.7}$$

for the radial and spherical part respectively. Now separating (4.7) again and using the boundary conditions $Y(\theta,\phi)$ of period 2π in ϕ and $Y(\theta,\phi)$ finite at $\theta=0,\pi$ it can be shown ([19] p. 272-275) that $\gamma=l(l+1)$, where l is a non-negative integer and $-l\le m\le l$, that the solutions of the spherical ordinary differential equation (4.7) are the spherical harmonic functions

$$Y_l^m(\theta,\phi)=P_l^{|m|}(\cos(\theta))e^{im\phi}. \tag{4.8}$$

Here $l=0,1,2,\dots$ and $-l\le m\le l$.

Now the radial part satisfies (4.6), which is a Bessel's differential equation, where we now have $\gamma=l(l+1)$. Denote the solution $R_l(r)$, since it does not depend on m. (4.6) is of second order, so the solution R_l will involve two unknown coefficients, that can be determined by the boundary conditions $R_l(1)=1$ and $R_l(0)$ is finite.

Combining these results gives the separated solution

$$u_{lm}=R_l(r)Y_l^m(\theta,\phi).$$

We can now see that Y_l^m are the eigenfunctions of Λ_q, since

$$\Lambda_qY_l^m(\theta,\phi)=\Lambda_q\left(u_{lm}\Big|_{r=1}\right)=\frac{\partial u_{lm}}{\partial\eta}\bigg|_{\partial\Omega}=\frac{\partial R_l(r)}{\partial r}\bigg|_{r=1}Y_l^m(\theta,\phi)=\lambda_lY_l^m(\theta,\phi).$$

Hence Y_l^m are the eigenfunctions with corresponding eigenvalues

$$\lambda_l=\frac{\partial R_l(r)}{\partial r}\bigg|_{r=1}.$$

□

Note that the eigenfunctions Y_l^m form a complete orthonormal set on the surface of the sphere in $L^2(\partial\Omega)$ ([19]).

Let us now find explicit expressions for the eigenvalues in the case of $q_1 = 0$ and $q_2 = q(r)$ given by (4.4). First, consider the simpler case where $q_1 = 0$.

Theorem 4.4 *The eigenvalues of Λ_0 are given by*

$$\lambda_l = l - k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}$$

for $l = 0, 1, 2, \ldots$.

Proof. Setting $q = 0$ turns (4.6) into

$$R^{0\prime\prime} + \frac{2}{r}R^{0\prime} + \left(k^2 - \frac{\gamma}{r^2}\right) R^0 = 0.$$

This is Bessel's differential equation and has the solution

$$R_l^0 = A_0\frac{J_{l+1/2}(kr)}{\sqrt{r}} + B_0\frac{N_{l+1/2}(kr)}{\sqrt{r}},$$

where J is the Bessel function of the first kind and N of second kind, also called the Neumann function. The Neumann function N blows up at $r = 0$, so we set $B_0 = 0$ to satisfy the boundary condition $R_l^0(0)$ being finite. This gives the solution

$$R_l^0 = A_0\frac{J_{l+1/2}(kr)}{\sqrt{r}}.$$

Now the boundary condition $R_l^0(1) = 1$ means that

$$1 = A_0 J_{l+1/2}(k), \tag{4.9}$$

so we get the solution

$$R_l^0 = \frac{J_{l+1/2}(kr)}{J_{l+1/2}(k)\sqrt{r}}.$$

The eigenvalues are now easy to compute by differentiating and we get

$$\lambda_l^0 = \frac{\partial R_l^0(r)}{\partial r}\bigg|_{r=1} = lA_0J_{l+1/2}(k) - A_0kJ_{l+3/2}(k) = l - k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}.$$

□

It will later be shown that the assumption of zero not being an eigenvalue of (1.8) with $q = 0$ means that $J_{l+1/2}(k) \neq 0$ for all $l \in \mathbb{N}_0$. Note that for a fixed k, $\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)} \to 0$ as $l \to \infty$, so $\lambda_l^0 \sim l$ for large l. As a function of k, that is for fixed l, λ_l^0 behaves like a π-periodic function for large k.

Let us now turn to the slightly more complicated problem of finding the eigenvalues λ_l^q, where q is given by (4.4).

Theorem 4.5 *Assume q is given by (4.4). The eigenvalues of Λ_q are then given as*

$$\lambda_l^q = l - B_2^l k N_{l+3/2}(k) + k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}\left(B_2^l N_{l+1/2}(k) - 1\right), \tag{4.10}$$

where

$$\begin{aligned} B_2^l = &\Big(kJ_{l+1/2}(\sqrt{k^2+c}\rho)J_{l+3/2}(k\rho) - \sqrt{k^2+c}J_{l+1/2}(k\rho)J_{l+3/2}(\sqrt{k^2+c}\rho)\Big) \\ &\Big/\Big(kN_{l+1/2}(k)J_{l+3/2}(k\rho)J_{l+1/2}(\sqrt{k^2+c}\rho) \\ &- kN_{l+3/2}(k\rho)J_{l+1/2}(\sqrt{k^2+c}\rho)J_{l+1/2}(k) \\ &- \sqrt{k^2+c}J_{l+1/2}(k\rho)J_{l+3/2}(\sqrt{k^2+c}\rho)N_{l+1/2}(k) \\ &+ \sqrt{k^2+c}N_{l+1/2}(k\rho)J_{l+3/2}(\sqrt{k^2+c}\rho)J_{l+1/2}(k)\Big). \end{aligned} \tag{4.11}$$

Proof. Since q is piecewise constant we solve the differential equation (4.6) in the two domains separately and then match transmission conditions at the jump.

First for $0 \leq r \leq \rho$ (4.6) turns into

$$r^2 R^{q_1\prime\prime} + 2rR^{q_1\prime} + r^2\left(c + k^2 - \frac{l(l+1)}{r^2}\right)R^{q_1} = 0.$$

Solving this gives

$$R_l^{q_1}(r) = A_1^l \frac{J_{l+1/2}(\sqrt{k^2+c}r)}{\sqrt{r}} + B_1^l \frac{N_{l+1/2}(\sqrt{k^2+c}r)}{\sqrt{r}}.$$

Since $R_l^{q_1}(r)$ must be finite at $r = 0$ we set B_1^l to zero and get

$$R_l^{q_1}(r) = A_1^l \frac{J_{l+1/2}(\sqrt{k^2+c}r)}{\sqrt{r}}.$$

For $\rho < r \leq 1$ we get the ordinary differential equation

$$r^2 R^{q_2}{}'' + 2rR^{q_2}{}' + r^2\left(k^2 - \frac{l(l+1)}{r^2}\right)R^{q_2} = 0$$

with solution

$$R_l^{q_2}(r) = A_2^l \frac{J_{l+1/2}(kr)}{\sqrt{r}} + B_2^l \frac{N_{l+1/2}(kr)}{\sqrt{r}}.$$

The eigenvalues are given as the derivative at $r = 1$, so they are computed as

$$\begin{aligned}
\lambda_l^q &= \left.\frac{\partial R_l^{q_2}(r)}{\partial r}\right|_{r=1} \\
&= \frac{\partial}{\partial r}\left(A_2^l \frac{J_{l+1/2}(kr)}{\sqrt{r}} + B_2^l \frac{N_{l+1/2}(kr)}{\sqrt{r}}\right)\Bigg|_{r=1} \\
&= l\left(B_2 N_{l+1/2}(k) + A_2^l J_{l+1/2}(k)\right) - k\left(B_2^l N_{l+3/2}(k) - A_2^l J_{l+3/2}(k)\right) \\
&= l - k\left(B_2^l N_{l+3/2}(k) + A_2^l J_{l+3/2}(k)\right).
\end{aligned}$$

The last equation is due to the fact that the boundary condition at $r = 1$ gives

$$1 = R_l^{q_2}(1) = A_2^l J_{l+1/2}(k) + B_2^l N_{l+1/2}(k).$$

Using this boundary condition we can also express A_2^l in terms of B_2^l

$$A_2^l = \frac{1 - B_2^l N_{l+1/2}(k)}{J_{l+1/2}(k)}.$$

This means that

$$R_l^{q_2}(r) = \frac{1 - B_2^l N_{l+1/2}(k)}{J_{l+1/2}(k)} \frac{J_{l+1/2}(kr)}{\sqrt{r}} + B_2^l \frac{N_{l+1/2}(kr)}{\sqrt{r}}$$

for $\rho < r \leq 1$.

Plugging this into the expression for the eigenvalues gives

$$\begin{aligned}
\lambda_l^q &= l - k\left(B_2^l N_{l+3/2}(k) + A_2^l J_{l+3/2}(k)\right) \\
&= l - k\left(B_2^l N_{l+3/2}(k) + \frac{\left(1 - B_2^l N_{l+1/2}(k)\right) J_{l+3/2}(k)}{J_{l+1/2}(k)}\right) \\
&= l - kB_2^l\left(N_{l+3/2}(k) - N_{l+1/2}(k)\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}\right) - k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}.
\end{aligned}$$

To determine B_2^l we need to derive transmission conditions at $r = \rho$. Remember that we are trying to find the operator norm to be able to investigate the optimimatility of the stability estimate in Theorem 3.7. This means that we must obey the assumptions taken here. First, $u \in H^s(\Omega)$ and $s > n/2 = 3/2$, so as a minimum $s = 2$. By the Sobolev embedding theorem ([5]) this means that u is continuous, so the first transmission condition is

$$u_{lm}(\rho^-, \theta, \phi) = u_{lm}(\rho^+, \theta, \phi),$$

which means that a Dirichlét condition must hold at the jump $r = \rho$

$$R_l^{q_1}(\rho) = R_l^{q_2}(\rho). \tag{4.12}$$

[1] Now to derive the second transmission condition, we will use the fact that u is the solution of (1.8). Let $\phi \in C_c^\infty(\Omega)$ be a test function. Dividing the integral into two, using the regions Ω_1, where $0 \le r \le \rho$ and Ω_2, where $\rho < r \le 1$, and integration by parts gives

$$\begin{aligned}
0 &= \int_\Omega (\Delta + k^2 + q)u\phi\, dx \\
&= -\int_\Omega \nabla u \cdot \nabla\phi + (k^2 + q)u\phi\, dx \\
&= -\int_{\Omega_1} \nabla u_1 \cdot \nabla\phi\, dx - \int_{\Omega_2} \nabla u_2 \cdot \nabla\phi\, dx + \int_\Omega (k^2+q)u\phi\, dx.
\end{aligned} \tag{4.13}$$

Performing integration by parts again and letting η be the unit outward normal derivative for the inner circle Ω_1 means that

$$-\int_{\Omega_1} \nabla u_1 \cdot \nabla\phi\, dx = \int_{\Omega_1} \Delta u_1\phi\, dx - \int_{r=\rho} \frac{\partial u_1}{\partial\eta}\phi\, ds$$

and

$$-\int_{\Omega_2} \nabla u_2 \cdot \nabla\phi\, dx = \int_{\Omega_2} \Delta u_2\phi\, dx + \int_{r=\rho} \frac{\partial u_2}{\partial\eta}\phi\, ds,$$

since the outward normal derivative for Ω_2 on the mutual boundary is in the opposite direction than before, so $-\eta$, and $\phi = 0$ on $r = 1$. Combining these two results means that

$$\begin{aligned}
0 &= \int_{\Omega_1} \Delta u_1\phi\, dx - \int_{r=\rho} \frac{\partial u_1}{\partial\eta}\phi\, ds + \int_{\Omega_2} \Delta u_2\phi\, dx + \int_{r=\rho} \frac{\partial u_2}{\partial\eta}\phi\, ds \\
&\quad + \int_\Omega (k^2+q)u\phi\, dx \\
&= \int_\Omega (\Delta + k^2 + q)u\phi\, dx - \int_{r=\rho} \frac{\partial u_1}{\partial\eta}\phi\, ds + \int_{r=\rho} \frac{\partial u_2}{\partial\eta}\phi\, ds.
\end{aligned}$$

[1]Even if we did only assume that $u \in H^1(\Omega)$, where u is not necessarily continuous, this would still hold under the assumption that u is continuous on the two domains $0 \le r \le \rho$ and $\rho < r \le 1$ (denote them Ω_i, $i = 1, 2$). This is due to the fact that $u \in H^1(\Omega)$ can be approximated by smooth functions, so it holds that the traces $T(u_1) = T(u_2)$ on the mutual boundary of Ω_1 and Ω_2. Using continuity of the trace operator, T, it then holds that $u_1 = u_2$ on $r = \rho$.

Hence for u to be a weak solution of (1.8), it must hold for all $\phi \in C_c^\infty(\Omega)$ that

$$\int_{r=\rho} \frac{\partial u_1}{\partial \eta}\phi \, ds = \int_{r=\rho} \frac{\partial u_2}{\partial \eta}\phi \, ds,$$

meaning that we require that

$$\left.\frac{\partial u_1}{\partial \eta}\right|_{r=\rho} = \left.\frac{\partial u_2}{\partial \eta}\right|_{r=\rho}.$$

Using that $u = R_l(r)Y_l^m(\theta, \phi)$ gives that

$$\left.\frac{\partial R_1}{\partial \eta}\right|_{r=\rho} = \left.\frac{\partial R_2}{\partial \eta}\right|_{r=\rho} \tag{4.14}$$

as the second transmission condition.

Matching the Dirichlét (4.12) and the Neumann (4.14) conditions at $r = \rho$ gives the two equations

$$A_1^l J_{l+1/2}(\sqrt{k^2+c}\rho) = \frac{1 - B_2^l N_{l+1/2}(k)}{J_{l+1/2}(k)} J_{l+1/2}(k\rho) + B_2^l N_{l+1/2}(k\rho) \tag{4.15}$$

and

$$\begin{aligned} &A_1^l \left(\frac{l}{\rho^{1/2}} J_{l+1/2}(\sqrt{k^2+c}\rho) - \sqrt{k^2+c} J_{l+3/2}(\sqrt{k^2+c}\rho) \right) = \\ &B_2^l \left(\frac{k N_{l+1/2}(k) J_{l+3/2}(k\rho)}{J_{l+1/2}(k)} - \frac{l}{\rho^{1/2}} \frac{N_{l+1/2}(k) J_{l+1/2}(k\rho)}{J_{l+1/2}(k)} \right. \\ &\qquad \left. - k N_{l+3/2}(k\rho) + \frac{l}{\rho^{1/2}} N_{l+1/2}(k\rho) \right) \\ &\qquad - k \frac{J_{l+3/2}(k\rho)}{J_{l+1/2}(k)} + \frac{l}{\rho^{1/2}} \frac{J_{l+1/2}(k\rho)}{J_{l+1/2}(k)}. \end{aligned} \tag{4.16}$$

Solving (4.15) and (4.16) (this is done in Maple) gives (4.11).

□

Note that the same procedure can be used if q is any piecewise constant function. In this case we would have more equations and more unknown coefficients, which could be determined by matching transmission conditions at the points, where q jumps.

Coming back to the eigenvalues of the difference operator $\Lambda_q - \Lambda_0$ we now know that they are given as

$$\begin{aligned}\lambda_l = \lambda_l^q - \lambda_l^0 &= l - kB_2^l\left(N_{l+3/2}(k) - N_{l+1/2}(k)\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}\right) - k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)} \\ &\quad - l + k\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)} \\ &= B_2^l k\left(\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)}N_{l+1/2}(k) - N_{l+3/2}(k)\right), \end{aligned} \tag{4.17}$$

where B_2^l is given by (4.11).

Given the frequency k we must find the l, where the largest eigenvalue occurs, but first we note that given k, λ_l has singularities close to some $l \in \mathbb{N}$. It turns out that these singularities corresponds to being near cases where zero is an eigenvalue of (1.8). This is the basis for the investigation in the next section.

4.3 Assumption of uniqueness

Let us now investigate what assumption we are actually making when assuming that zero is not a Dirichlét eigenvalue of the partial differential equation (1.8). To be able t compare the results with the investigation in the preceding an the next section, we are interested in special cases when q spherically symmetric and either constant 0 or given by (4.4). In this case consider the Dirichlét eigenvalue problem

$$\begin{aligned}(\Delta + k^2 + q(x))u(x) = \lambda u \quad &\text{in} \quad \Omega \subset \mathbb{R}^n \\ u = 0 \quad &\text{on} \quad \partial\Omega,\end{aligned}$$

where $q \in L^\infty(\Omega)$. Performing separation of variables is very similar to the separation of variables done above in Theorem 4.5 (think of $k^2 = k^2 - \lambda$), so λ will only figure in the ordinary differential equation derived for the radial part. The solution to this ODE is divided into two parts, since q is either zero ($c = 0$) or piecewise constant. The solution is given as

$$R_l(r) = \begin{cases} A_l\dfrac{J_{l+1/2}(\sqrt{k^2+c-\lambda_l}r)}{\sqrt{r}}, & 0 < r \le \rho \\ B_l\dfrac{J_{l+1/2}(\sqrt{k^2-\lambda_l}r)}{\sqrt{r}} + C_l\dfrac{N_{l+1/2}(\sqrt{k^2-\lambda_l}r)}{\sqrt{r}}, & \rho < r \le 1 \end{cases}, \tag{4.18}$$

where A_l, B_l and C_l are arbitrary constants and $l = 0, 1, 2, \ldots$. Inserting the Dirichlét boundary condition $u = 0$ at $r = 1$ means that

$$B_l J_{l+1/2}(\sqrt{k^2 - \lambda_l}) + C_l N_{l+1/2}(\sqrt{k^2 - \lambda_l}) = 0$$

and since we are interested in the case when $\lambda_l = 0$ is an eigenvalue we get

$$B_l J_{l+1/2}(k) + C_l N_{l+1/2}(k) = 0.$$

Using transmission conditions at $r = \rho$ we can for example express B_l as a function of C_l. Doing this will mean that the above equation can be written as the constant C_l multiplied with a fraction. The numerator of this expression is zero when

$$\begin{aligned} 0 = &\sqrt{k^2 + c} J_{l+1/2}(k\rho) N_{l+1/2}(k) J_{l-1/2}(\sqrt{k^2 + c}\rho) \\ &- k J_{l+1/2}(\sqrt{k^2 + c}\rho) N_{l+1/2}(k) J_{l-1/2}(k\rho) \\ &+ k J_{l+1/2}(\sqrt{k^2 + c}\rho) J_{l+1/2}(k) N_{l-1/2}(k\rho) \\ &- \sqrt{k^2 + c} N_{l+1/2}(k\rho) J_{l+1/2}(k) J_{l-1/2}(\sqrt{k^2 + c}\rho). \end{aligned} \tag{4.19}$$

Hence when assuming that zero is not an eigenvalue, we are actually assuming that k and c are given such that (4.19) does not hold.

Let us see what happens to (4.19) when k is *much larger* than c and we can assume that $\sqrt{k^2 + c} \sim k$ or when $c = 0$.

$$\begin{aligned} 0 &\simeq k J_{l+1/2}(k\rho) N_{l+1/2}(k) J_{l-1/2}(k\rho) - k J_{l+1/2}(k\rho) N_{l+1/2}(k) J_{l-1/2}(k\rho) \\ &\quad + k J_{l+1/2}(k\rho) J_{l+1/2}(k) N_{l-1/2}(k\rho) - k N_{l+1/2}(k\rho) J_{l+1/2}(k) J_{l-1/2}(k\rho) \\ &\simeq k J_{l+1/2}(k) \left(J_{l+1/2}(k\rho) N_{l-1/2}(k\rho) - N_{l+1/2}(k\rho) J_{l-1/2}(k\rho) \right), \end{aligned}$$

so assuming $k > 0$ we get $J_{l+1/2}(k) = 0$. This means that when k is *large* in the case $c \neq 0$ or for all k in the case $c = 0$ we assume that

$$J_{l+1/2}(k) \neq 0, \tag{4.20}$$

since otherwise zero would be an eigenvalue and the solution to (1.8) not unique. Considering the case where $q = 0$, we exactly need to assume that (4.20) holds to be sure that we have a unique solution. Figure 4.2 shows a plot of the right hand side of (4.19) in blue and $J_{l+1/2}(k)$ in red for $c = 10$, $\rho = 0.7$, $l = 0$ as a function of k. Here we can see that the difference between the zeros of the two functions very quickly go to zero when k grows.

Let us consider an asymptotic expansion of the Bessel functions. As $z \to \infty$ the Bessel function goes like (see for example [19])

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(k^{-3/2}).$$

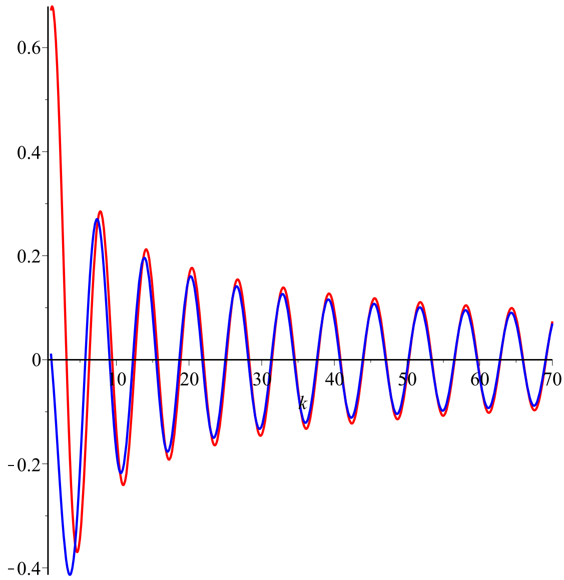

Figure 4.2: Right hand side of (4.19) plotted in blue and $J_{l+1/2}(k)$ in red for $c = 10$, $\rho = 0.7$, $l = 0$ as a function of k. It is seen that the difference of the intersections with zero goes to zero when k increases.

Remembering that l is an integer we have for $k \to \infty$

$$\begin{aligned} J_{l+1/2}(k) &= \sqrt{\frac{2}{\pi k}} \cos\left(k - \frac{\left(l+\frac{1}{2}\right)\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(k^{-3/2}) \\ &= \sqrt{\frac{2}{\pi k}} \cos\left(k - l\frac{\pi}{2} - (l+1)\frac{\pi}{4}\right) + \mathcal{O}(k^{-3/2}). \end{aligned}$$

This expression can be expanded in linear combinations of cos and sin. We will here choose a particular sequence of $k's$ to simplify things and remember to disregard the $l \in \mathbb{N}_0$ that give rise to zeros. We choose

$$k = \left(\frac{n}{2} + \frac{1}{4}\right)\pi, \tag{4.21}$$

to be furthest away from zeros of cos and sin. Note that we especially have to be careful when k is small, since then the asymptotic expansion does not hold.

4.4 Operator norm

Before actually searching for the operator norm as the largest eigenvalue, let us consider how we expect the operator norm to behave as k increases.

The operator norm is given as

$$\begin{aligned} \|\Lambda_q - \Lambda_0\| &= \sup_{\substack{f,g\in H^{1/2}(\partial\Omega) \\ \|f\|=\|g\|=1}} |\langle(\Lambda_q - \Lambda_0) f, g\rangle| \\ &= \sup_{\substack{f,g\in H^{1/2}(\partial\Omega) \\ \|f\|=\|g\|=1}} \left|\int_\Omega quv\, dx\right| \\ &\leq \sup_{\substack{f,g\in H^{1/2}(\partial\Omega) \\ \|f\|=\|g\|=1}} \|q\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \end{aligned} \tag{4.22}$$

where u solves (1.8) and v solves $(\Delta+q)v = 0$ in Ω and $v = g$ on $\partial\Omega$. This means that we wish to investigate how the L^2-norm of u and v behaves as a function of k. Recall by the separation of variables in section 4.3 that the k-dependency only occurs in the radial part of u and v.

Let us start with the simpler task, $\|v\|_{L^2(\Omega)}$. We know that the solution of $(\Delta + q)v = 0$ can be separated as $v_{lm} = R_l(r)Y_l^m(\theta,\phi)$, so

$$\begin{aligned} \|v_{lm}\|_{L^2(\Omega)} &= \left(\int_\Omega |R_l(r)Y_l^m(\theta,\phi)|^2\, dx\right)^{1/2} \\ &= \left(\int_0^1 |R_l(r)|^2 r^2\, dr\right)^{1/2} \left(\int_{S^2} |Y_l^m(\theta,\phi)|^2 \sin(\theta)\, ds\right)^{1/2}. \end{aligned}$$

We know that $R_l(r) = \frac{J_{l+1/2}(kr)}{J_{l+1/2}(k)\sqrt{r}}$, so

$$\left(\int_0^1 |R_l(r)|^2 r^2\, dr\right)^{1/2} = \left(\int_0^1 \left|\frac{J_{l+1/2}(kr)}{J_{l+1/2}(k)\sqrt{r}}\right|^2 r^2\, dr\right)^{1/2}.$$

Choosing k as (4.21) and using the asymptotic expansion we get for $k \to \infty$

$$\left|\frac{J_{l+1/2}(kr)}{J_{l+1/2}(k)\sqrt{r}}\right|^2 r^2 \simeq \left|\frac{\sqrt{\frac{2}{kr}}}{\sqrt{\frac{2}{k}}}\right|^2 r = 1,$$

so we expect $\|v\|_{L^2(\Omega)} \to 1$ as $k \to \infty$.

This is also seen if we we fix l and set $l = 0$.

$$\left(\int_0^1 |R_0(r)|^2 r^2 \, dr\right)^{1/2} = \left(\int_0^1 \left|\frac{\sin(kr)}{\sqrt{r}\sin(k)}\right|^2 r \, dr\right)^{1/2}$$
$$= \frac{1}{|\sin(k)|}\left(\int_0^1 \sin^2(kr) \, dr\right)^{1/2}$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sin^2(k)} - \frac{1}{\tan(k)}\frac{1}{k}\right)^{1/2}.$$

Both $1/\tan(k)$ and $\frac{1}{\sin^2(k)}$ are 2π-periodic functions, so when k is large $1/k$ goes to zero and the above expression goes to a 2π-periodic function. If we choose k as (4.21) and expand the square root we see that it exactly converges towards 1

$$\left(\int_0^1 |R_0(r)|^2 r^2 \, dr\right)^{1/2} = \frac{1}{\sqrt{2}}\left(2 - \frac{1}{k}\right)^{1/2} = 1 - \frac{1}{4k} - \frac{1}{k^2} + \ldots \to 1,$$

as $k \to \infty$. Note that the convergence rate is $1/k$.

For larger l similar behaviour of the L^2-norm is seen, only the k^{-1} convergence occurs for larger k.

Now coming to the L^2-norm of u, we again know that u can be separated into solutions $u_{lm} = R_l(r)Y_l^m(\theta, \phi)$, where the k- (and now also q-) dependency occurs in the radial solution. In this case we compute $\|R_0\|_{L^2(\Omega)}$ numerically, since the expression for R_0 is rather complicated. The integral is evaluated in `MATLAB` (see B.1) and a plot on a logarithmic scale of the difference between the computed L^2-norm and 1 is seen in Figure 4.3. It is observed that $\|u\|_{L^2(\Omega)}$ behaves as $\|v\|_{L^2(\Omega)}$; the norm converges towards constant 1 as k increases with a rate of k^{-1}. Note that this is of course computed with fixed values of c and ρ (here $c = 0.01$ and $\rho = 0.7$).

Summing up, when k is chosen as in (4.21) we expect the operator norm of $\Lambda_q - \Lambda_0$ to converge towards a constant value depending on q as k increases. Note that this result assumes that $l = 0$ is a good estimate of the position of the dominating eigenfunction.

Let us now try to find the largest eigenvalue of $|\lambda_l|$. Recall

$$\lambda_l = B_2^l k \left(\frac{J_{l+3/2}(k)}{J_{l+1/2}(k)} N_{l+1/2}(k) - N_{l+3/2}(k)\right),$$

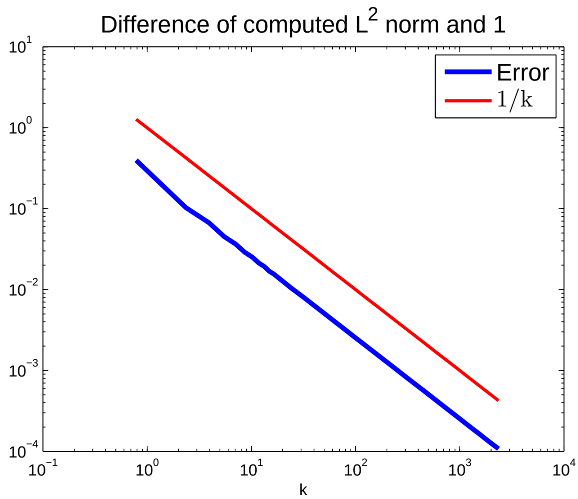

Figure 4.3: L^2-norm of the radial part of u for $l = 0$, $\rho = 0.7$, $c = 0.01$ (left) and convergence rate k^{-1} towards 1 (right).

where B_2^l is given by (4.11), so

$$\|\Lambda_q - \Lambda_0\| \simeq k \max_{l \in \mathbb{N}} \left\{ \left|B_2^l\right| \left| \frac{J_{l+3/2}(k)}{J_{l+1/2}(k)} N_{l+1/2}(k) - N_{l+3/2}(k) \right| \right\}. \tag{4.23}$$

A first naive attempt to find the above operator norm would be to compute the eigenvalues $|\lambda_l|$ for a range of $l \in \mathbb{N}$ and choose the largest. As mentioned even though we have chosen k as (4.21) there are still $l \in \mathbb{N}_0$ that correspond to being close to a zero of (4.19). To handle this problem we will only compute $|\lambda_l|$ for l chosen in such a way, that we are not close to these singularities. See Figure 4.4 for a visualization of (4.19) for a given $k = 10\frac{\pi}{2} + \frac{\pi}{4}$ (and $c = 0.01$, $\rho = 0.7$). To avoid being close to a zero of this function, we numerically find the roots of (4.19) and choose the integer l's that are closest to the midpoint between two roots, which correspond to being close to the extrema points as seen in the plot. The `MATLAB` function doing this is seen in Appendix B.2. Here the Zero Eigenvalue Function is (4.19), which as mentioned behaves a lot like $J_{l+1/2}(k)$ at least for *large* k. It can be shown ([21] p. 156) that $J_{l+1/2}(k)$ has no zeros when $k \in \left(0, l + \frac{1}{2}\right)$, that is when $l + \frac{1}{2} > k > 0$, so when $l > \lceil k - \frac{1}{2} \rceil$. After this point $J_{l+\frac{1}{2}}(k)$ (and also (4.19) in general) decays to zero as a function of l, which is also seen in Figure 4.4. This means that if we allow to large l (4.19) will be close to zero and the value of the corresponding eigenvalues blow up and become very large. Another way of choosing a range of l that does not come too close to zero, is to simply throw away the integer l that give rise a (4.19) being too close to zero. Here *too* means a given tolerance, that must be chosen.

Both methods give the same result the first being faster since it chooses less l.

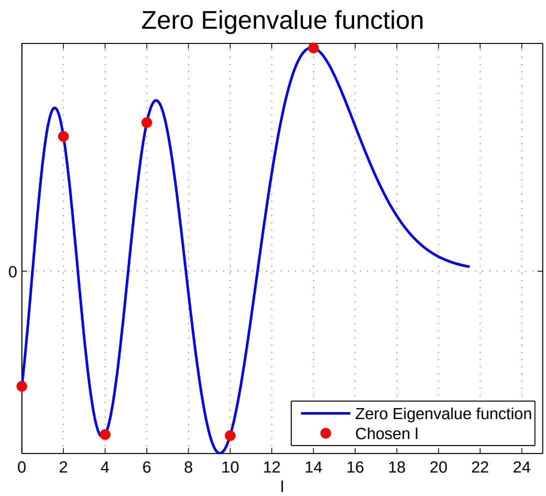

Figure 4.4: The right hand side of (4.19) is plotted in blue, where the integer l chosen by the MATLAB function seen in B.2 are plotted as red dots.

Now we wish to find the operator norm, or at least an estimate hereof, for a range of k to see what happens when k increases. To do this we use the above method for every k to find a list of acceptable integer l's and then compute the operator norm for these. We then choose the largest as our estimate of the operator norm. The MATLAB function doing this can be seen in Appendix B.3. It turns out that the largest eigenvalue always correspond to the lowest value of l and as k increases the value chosen is constant $l = 0$. As mentioned when $k \to \infty$ the Bessel functions of order $l + \frac{1}{2}$ are almost independent of k, so we would also expect this behaviour. For smaller k, the chosen values (4.21) are not as good as for larger k, so for some small k, $l = 0$ is too close to a zero of (4.19) and $l = 1$ is chosen instead.

We cannot be sure that the chosen eigenvalues correspond to the exact operator norm for the chosen k, but we know that it is at least is a lower bound. When we earlier examined the expectation of the operator norm we too did this for a fixed value $l = 0$, so we expect these chosen eigenvalues to converge towards a constant value when k increases. Let us call the eigenvalues for the computed operator norm or estimate hereof. It turns out that they converge towards $c\rho$ at rate k^{-1}, which seems very reasonable remembering (4.22). Seeing that the operator norm is zero when $c = 0$ is also a good indication of the correctness of our numerical results. Figure 4.5 shows a double logarithmic plot of the difference between $c\rho$ and the computed operator norm. In this particular case, the coefficient B_2^l and the expression concerning Bessel's functions in the parentheses in (4.23) both go to zero at rate $k^{-0.5}$. The factor k is then the reason for the convergence to a constant value when k increases.

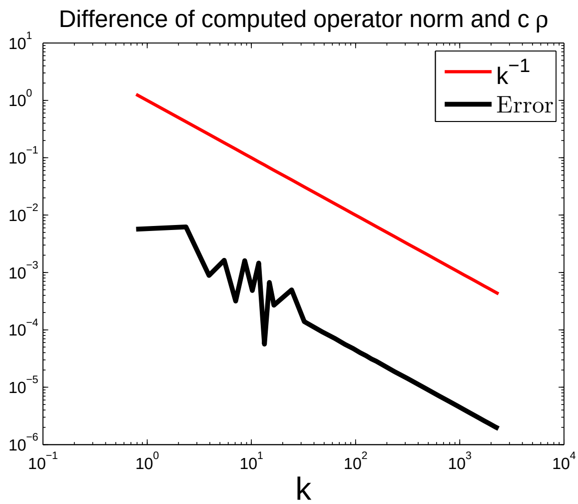

Figure 4.5: Difference between $c\rho$ and the computed operator norm is seen to converge to zero at rate k^{-1}.

We are now finally ready to compare the numerical results with the stability estimate found in chapter 3. Figure 4.6 shows a double logarithmic plot of the right hand-side of the stability estimate (3.37). The logarithmic part of the stability is plotted in red, the Lipschitz part in blue and the combined estimate in black. The constant green value is the L^2-norm of q, which is a lower bound of $\|\widetilde{q}\|_{H^s(\mathbb{R}^n)}$. As expected the estimate follows the logarithmic part for small k and when k gets larger it follows the Lipschitz part. It is seen that the Lipschitz part grows like k^2 which is expected, since we have just shown that the computed operator norm approaches a constant value and not zero. This means that for large k (3.37) is a very conservative estimate and in this sense not optimal. The result suggests that in this special case of the unit sphere Ω, spherically symmetric and piecewise constant q and k chosen as (4.21), a stability estimate with k^0 instead of k^2 is a better estimate. Even though we have only computed an estimate of the operator norm, the values computed are a lower bound for the true operator norm, so it cannot go to zero at rate k^{-2}, which would have made the estimate optimal.

This does not contradict the result in chapter 3, on the contrary it means that we can expect better Lipschitz stability for large k in this special case.

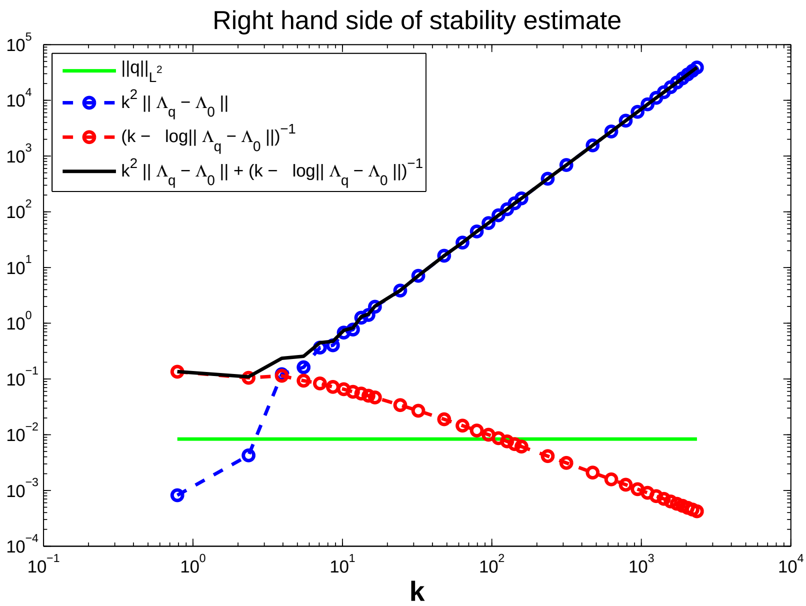

Figure 4.6: Estimate visualized as a function of k for $\rho = 0.7$ and $c = 0.01$.

4.5 Cauchy data

At last, let us compare the stability result obtained in chapter 3 with the result derived in the article [9]. The discussion of the relationship between the Cauchy data and the Dirichlét-to-Neumann map is inspired by [6] and results not shown can be found here. As mentioned, the main article, [9], used in chapter 3 uses Cauchy data instead of the Dirichlét-to-Neumann map. This is done to avoid the assumption that zero cannot be an eigenvalue of (1.8), which is the reason for the difficulties in the preceding numerical investigations. The problem when zero is an eigenvalue is that the Dirichlét-to-Neumann map, is not well-defined, since there can be several solutions to the Helmholtz equation with a potential.

Now if we assume that $u \in H^1(\Omega)$ is a solution to (1.8) we can define the Dirichlét-to-Neumann map weakly, without assuming that zero is not an eigenvalue. This is done by defining

$$\int_{\partial\Omega} \Lambda_q f \, g \, ds = \int_{\Omega} \nabla u \cdot \nabla e_g \, dx - \int_{\Omega} (k^2 + q) u e_g \, dx,$$

where $g \in H^{1/2}(\partial\Omega)$ and $e_g \in H^1(\Omega)$ is an extension of g.

We can then define the Cauchy data associated with (1.8) as the set

$$C_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega} \right), \quad \text{where } u \text{ is a solution to} \quad (1.8) \right\}.$$

Now if zero is not an eigenvalue of (1.8) the Cauchy data corresponds to the graph of the Dirichlét-to-Neumann map ([6]), that is

$$C_q = \left\{ (f, \Lambda_q f) \,\middle|\, f \in H^{1/2}(\partial\Omega) \right\},$$

since then u is the unique solution to (1.8).

When using Cauchy data as in [9] the inverse problem is formulated as the problem of finding q, when knowing C_q. Now to measure the distance between two Cauchy data sets, the authors use the following

$$\operatorname{dist}(C_{q_1}, C_{q_2}) = \max \left\{ \sup_{(f,g)\in C_{q_1}} \inf_{(\tilde{f},\tilde{g})\in C_{q_2}} \frac{\left\|(f,g) - (\tilde{f},\tilde{g})\right\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}}, \right.$$
$$\left. \sup_{(f,g)\in C_{q_2}} \inf_{(\tilde{f},\tilde{g})\in C_{q_1}} \frac{\left\|(f,g) - (\tilde{f},\tilde{g})\right\|_{H^{1/2}\oplus H^{-1/2}}}{\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}}} \right\},$$

where the norm on $H^{1/2} \oplus H^{-1/2}$ is

$$\|(f,g)\|_{H^{1/2}\oplus H^{-1/2}} = \left(\|f\|^2_{H^{1/2}(\partial\Omega)} + \|g\|^2_{H^{-1/2}(\partial\Omega)} \right)^{1/2}.$$

Again if we are in the case where zero is not an eigenvalue, that is when the Cauchy data set is simply the graph of the Diriclét-to-Neumann map, the following bounds hold (see [6])

$$\frac{\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}}{\sqrt{1 + \|\Lambda_{q_1}\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}}\sqrt{1 + \|\Lambda_{q_2}\|^2_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}}}$$
$$\leq \operatorname{dist}(C_{q_1}, C_{q_2}) \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}. \tag{4.24}$$

The estimate obtained in [9] is very similar to the one we found in chapter 3, with the only difference being the Lipschitz constant. Where we have k^2 [9] has k^4. Using the above first bound, if $\|\Lambda_{q_l}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ goes as k, we can bound

$$k^2 \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq Ck^4 \operatorname{dist}(C_{q_1}, C_{q_2})$$

and the two estimates correspond.

Let us see how $\|\Lambda_q\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ behaves in the case of spherically symmetric piecewise constant potential q considered in this chapter. In the simple case where $q = 0$, k chosen as (4.21) and for $l = 0$ we get

$$|\lambda_0^0| = \left| k \frac{J_{3/2}(k)}{J_{1/2}(k)} \right| = \left| \frac{\sin(k) - k\cos(k)}{\sin(k)} \right| = |1 \pm k|,$$

where the sign depends on k. So in this case

$$\|\Lambda_q\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq Ck,$$

and we get

$$\begin{aligned}\|\widetilde{q}\|_{H^{-s}(\mathbb{R}^n)} &\leq Ck^2 \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ &\quad + C\left(k + \log \frac{1}{\left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}}\right)^{-(2s-n)} \\ &\leq Ck^4 \left\|\Lambda_{q_1} - \Lambda_{q_2}\right\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + C\left(k + \log \frac{1}{\operatorname{dist}(C_{q_1}, C_{q_2})}\right)^{-(2s-n)},\end{aligned} \tag{4.25}$$

which is exactly the result of [9].

Now in general we have that

$$\|\Lambda_q f\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \leq C_k k^2 \|u\|_{L^2(\Omega)},$$

where C_k is the constant depending on the elliptic Helmholtz equation with potential, hence it depends of k (among other quantities). It would be interesting to see if the dependence of k in C_k could be determined and a general relationship between the Dirichlét-to-Neumann map and the Cauchy data found. At least in the case considered in this chapter, the two results correspond to each other.

Conclusion

In this thesis the results from [9] have been investigated in detail and a similar result using the Dirichlét-to-Neumann map has been derived. It is concluded that the stability of the inverse problem of determining the potential q inside a domain from knowledge of the Dirichlét-to-Neumann map on the boundary is poor, but increases with frequency. For a low frequency parameter a logarithmic type estimate holds, which corresponds to known results for the conductivity equation and the problem of electrical impedance tomography. For high frequency problems a Lipschitz type stability dominates, which corresponds to the stability of the linearised problem. For a fixed $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ it is possible to solve an implicit equation and find the value of the frequency, where the problem goes from being of logarithmic type stability to a Lipschitz type. In general for a fixed k, when $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ goes to zero we will end up in the logarithmic part. The fact that the stability increases with frequency indicates that the linearisation is a better approximation of the non-linear problem for high frequencies. This corresponds to our intuition since the linearisation is performed around zero potential and the zeroth order term, $q + k^2$, in the partial differential can be thought of as being dominated by k^2 for frequencies k much larger than the size of the potential q.

The Lipschitz constant grows polynomially with the frequency as k^2. This factor is due to the use of complex geometric optics solutions in the derivation of the stability estimate. Here k appears naturally as a consequence of the underlying partial differential equation, the Helmholtz equation with a potential. From the investigations in chapter 4 it is deduced that the particular problem in the unit sphere in $\mathbb{R}^3$, where the potential is assumed to be spherical symmetric and piecewise constant, obeys the derived stability estimate for a specific sequence of frequencies. It is seen that a Lipschitz constant of k^0 seems more optimal in this special case in the sense that the estimate then decays towards a constant value when frequency increases. This is due to a derived estimate of the operator norm of the difference between two Dirichlét-to-Neumann maps. In this particular setting the operator norm does not converge to zero with rate k^{-2}, but converges to a constant value that depends on the potential q. Hence the Lipschitz part $Ck^2 \|\Lambda_{q_1} - \Lambda_{q_2}\|$ of the estimate grows like k^2 and is therefore not optimal in the sense that the estimate is very conservative for large k of the specific sequence.

Comparing the stability estimate found in this thesis using the Dirichlét-to-Neumann operator to the result in [9] where Cauchy-data is used shows a difference of factor k^2 in the Lipschitz part. The Dirichlét-to-Neumann operator can be bounded by a constant depending on k and the Cauchy-data, and in the particular setting of chapter 4 the constant is of order k^2. This indicates that the relationship between the two estimates holds. However, we cannot conclude whether the k^2 factor when using the Dirichlét-to-Neumann is in general optimal or not. The numerical results found here are specific to one case, which is chosen to make it possible to solve the ordinary differential equations given by the separation of variables and find the eigenvalues of the Dirichlét-to-Neumann difference operator. In this case it is not optimal, but has even better stability for high frequencies.

Outlook

This thesis is an investigation of the relationship between the stability and the frequency of the Helmholtz equation with a potential. Parameter dependent stability is a phenomenon observed numerically in several cases. It would be very interesting to derive similar results and test the optimality for other inverse problems relating to this problem. For example, considering the acoustic equation $(\Delta+k^2q)u = 0$, a similar estimate is derived in [15]. One could also imagine changing the sign of the Laplacian in the Helmholtz equation with potential and see how the results would change. The difficulties arising in chapter 4 are due to the assumption that zero is not an eigenvalue and they could be avoided by replacing Δ with $-\Delta$. This would give another problem and new CGO solutions must be found when trying to derive a stability estimate. Intuition dictates that the stability of these two problems are the same so it would be interesting to investigate optimality of this problem and compare it to the results obtained here.

The reason for considering Cauchy data in [9] is to get around this problem of assuming that zero is not an eigenvalue and get an estimate for all $k \geq 1$. The optimality of this problem is also interesting and non-trivial.

Since the discussion of optimality in this work only deals with one very specific choice of potential it would be of great interest to examine how a more general case behaves. The reason for the simple choice in this work is due to the analytic solvability of Bessel's differetial equation. The case of a more general potential is no trivial case. It would however be desirable to work with a continuous potential, as assumed in the stability estimate. In this case one could imagine solving the ordinary differential equations obtained by separation of variables numerically and in this way estimate the operator norm.

Finally, it would also be of interest to investigate the error made when performing the linearisation in chapter 2 and see if it decays with frequency, as the intuition from this thesis dictates. Another way to examine the linearisation could be to use a linear reconstruction method and see if it works better in the high frequency case than in the low frequency case.

Appendix A

Sobolev Spaces

Some definitions and results regarding Sobolev spaces are listed here. The theory is based mostly on the works [1], [5] and [16]. Only results that are not generally a part of a first introduction to Sobolev Spaces are included.

Recall the definition of Sobolev spaces for non-negative integer k and domain Ω as given in for example [5].

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) : \; D^\alpha f \in L^2(\Omega) \textit{ for all } |\alpha| \leq k \right\}, \tag{A.1}$$

where D^α is the weak derivative of order α. The norm is given as

$$\|f\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{1/2}.$$

A.1 Sobolev spaces of real order

We start out by considering the domain $\Omega = \mathbb{R}^n$. Recall for $f \in L^2(\mathbb{R}^n)$ the Fourier Transform is defined as

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx.$$

Using the Fourier Transform it is possible to give an equivalent definition of the Sobolev space $H^k(\mathbb{R}^n)$ for non-negative integer k. It can be shown (see for example [5]) that $f \in L^2(\mathbb{R}^n)$ lies in $H^k(\mathbb{R}^n)$ if and only if

$$(1 + |\xi|^k)\widehat{f} \in L^2(\mathbb{R}^n).$$

As a motivation we use the fact that the Fourier transform is isometric on $L^2(\mathbb{R}^n)$ and that derivatives of the Fourier Transform comes out as multiplication.

$$\|D^\alpha f\|_{L^2(\mathbb{R}^n)} = \left\|\widehat{D^\alpha f}\right\|_{L^2(\mathbb{R}^n)} = \left\|(i\xi)^\alpha \widehat{f}\right\|_{L^2(\mathbb{R}^n)}.$$

Using this means that

$$\begin{aligned}
\|f\|^2_{H^k(\mathbb{R}^n)} &= \sum_{|\alpha|\le k} \|D^\alpha f\|^2_{L^2(\mathbb{R}^n)} \\
&= \sum_{|\alpha|\le k} \left\| (i\xi)^\alpha \widehat{f}\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\widehat{f}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} |\xi^\alpha|^2 \right) |\widehat{f}(\xi)|^2 \, d\xi.
\end{aligned}$$

It can also be shown that there exist constants A, B depending only on n and k such that

$$A(1 + |\xi|^2)^k \le \sum_{|\alpha|\le k} |\xi^\alpha|^2 \le B(1 + |\xi|^2)^k,$$

which means that the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} = \left\|(1 + |\xi|^2)^{s/2} \hat{f}\right\|_{L^2(\mathbb{R}^n)} \qquad \text{(A.2)}$$

is equivalent to the norm of $H^k(\mathbb{R}^n)$ for all non-negative integers $s = k$. The definition (A.2) suggests a definition for all $s \ge 0$ and not only integers.

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \ (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\Omega) \right\}.$$

In practice when working with partial differential equations in the weak formulation, we require that the solution variable lies in a Sobolev space of integer order. It is included here, since it may sometimes be easier to work with this new definition.

Note that if $s > r$, then

$$H^s(\mathbb{R}^n) \subset H^r(\mathbb{R}^n), \tag{A.3}$$

since for $f \in H^s(\mathbb{R}^n)$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^n} (1+|\xi|^2)^r |\hat{f}(\xi)|^2 \, d\xi &\leq \int_{\mathbb{R}^n} (1+|\xi|^2)^{r+(s-r)} |\hat{f}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty.\end{aligned}$$

The following theorem is given as an exercise in [5] and some sub-results are based on [16].

Theorem A.1 *The Sobolev space $H^s(\mathbb{R}^n)$ is an algebra for $s > \frac{n}{2}$, meaning that for $u, v \in H^s(\mathbb{R}^n)$ it holds that*

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)},$$

where C is a constant depending on n and s.

Proof.

Let $u, v \in H^s(\mathbb{R}^n)$. To conclude that $H^s(\mathbb{R}^n)$ is an algebra we have to show that their product $uv \in H^s(\mathbb{R}^n)$, that is, we wish to bound

$$\int_{\mathbb{R}^n} |1+|\xi|^2|^s \widehat{uv}(\xi)|^2 \, d\xi. \tag{A.4}$$

It takes several sub-results to do this. The first is a result concerning the Fourier transform of a product. For simplicity denote

$$\langle \xi \rangle = |1+|\xi|^2|^{1/2}.$$

By the inverse Fourier transform and Fubini's theorem we get

$$\begin{aligned}\widehat{fg}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int f(x)g(x)e^{-\pi i x\cdot\xi} \, dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \widehat{g}(\eta) e^{\pi i x\cdot\eta} \, d\eta \, e^{-\pi i x\cdot\xi} \, dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{g}(\eta) \int_{\mathbb{R}^n} f(x) e^{-\pi i x\cdot(\xi-\eta)} \, dx \, d\eta \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta) \, d\eta \\ &= \frac{1}{(2\pi)^{n/2}} (\widehat{f} * \widehat{g})(\xi).\end{aligned}$$

Now to be able to bound the entire integrand in (A.4) we wish to bound $\langle\xi\rangle^{2s}$. Considering the above estimate it would be desirable to find a bound consisting of $\langle\eta\rangle^s$ and $\langle\xi-\eta\rangle^s$.

Let $\xi,\eta\in\mathbb{R}^n$ and note that

$$\begin{aligned}\langle\xi\rangle^s &= |1+|\xi|^2|^{s/2}\\ &\le |1+|\xi|^2+|\xi-2\eta|^2|^{s/2}, \quad \text{since } |\xi-2\eta|^2\ge 0\\ &= |1+|\xi|^2+|\xi|^2+4|\eta|^2-4|\xi||\eta||^{s/2}\\ &= |1+2|\eta|^2+2(|\xi|^2+|\eta|^2-2|\xi||\eta|)|^{s/2}\\ &= \left|1+2|\eta|^2+2|\xi-\eta|^2\right|^{s/2}\\ &\le 2^{s/2}\left|1+|\eta|^2+|\xi-\eta|^2\right|^{s/2}\\ &\le 2^{s/2}\left|1+|\eta|^2+1+|\xi-\eta|^2\right|^{s/2}.\end{aligned}$$

We wish to split this up in two, so we get $2^{s/2}\left(\langle\eta\rangle^s+\langle\xi-\eta\rangle^s\right)$. To do this use that $f(x)=x^p$ is convex for $x\ge 0$ and $p\ge 1$, so from the triangle inequality and Jensen's inequality it holds for $p\ge 1$

$$|a+b|^p\le(|a|+|b|)^p\le\frac{2^p}{2}\left(|a|^p+|b|^p\right)$$

and for $0\le p\le 1$, $f(x)=x^p$ is concave and $f(0)=0$, so for a $0<t<1$

$$f(tx)=f(tx+(1-t)\cdot 0)\ge tf(x)+(1-t)\cdot f(0)=tf(x),$$

which leads to

$$f(x)=f\left((x+y)\frac{x}{x+y}\right)\ge\frac{x}{x+y}f(x+y),$$

and

$$f(x)+f(y)\ge\frac{x}{x+y}f(x+y)+\frac{y}{x+y}f(x+y)=f(x+y).$$

so

$$|a+b|^p\le|a|^p+|b|^2,$$

for $0\le p\le 1$. In general we have that

$$|a+b|^p\le\max\left\{1,\frac{2^p}{2}\right\}\left(|a|^p+|b|^p\right)$$

and using this for $p = s/2$, $a = 1 + |\eta|^2$ and $b = 1 + |\xi - \eta|^2$ means that we have the bound

$$\langle \xi \rangle^s \leq C \left(|1 + |\eta|^2|^{s/2} + |1 + |\xi - \eta|^2|^{s/2} \right),$$

where C depends on s. Using these two results we can bound the integrand of (A.4) and we get

$$\begin{aligned}
\langle \xi \rangle^s |\widehat{uv}(\xi)| &= C \langle \xi \rangle^s | \left(\widehat{u} * \widehat{v} \right)(\xi)| \\
&= C \langle \xi \rangle^s \left| \int_{\mathbb{R}^n} \widehat{u}(\xi - \eta) \widehat{v}(\eta) \, d\eta \right| \\
&\leq C \int_{\mathbb{R}^n} \langle \xi \rangle^s \left| \widehat{u}(\xi - \eta) \widehat{v}(\eta) \right| \, d\eta \\
&\leq C \int_{\mathbb{R}^n} \left(\left|1 + |\eta|^2\right|^{s/2} + \left|1 + |\xi - \eta|^2\right|^{s/2} \right) \left| \widehat{u}(\xi - \eta) \widehat{v}(\eta) \right| \, d\eta \\
&= C \int_{\mathbb{R}^n} \langle \eta \rangle^s \left| \widehat{u}(\xi - \eta) \widehat{v}(\eta) \right| \, d\eta + C \int_{\mathbb{R}^n} \left|1 + |\xi - \eta|^2\right|^s \left| \widehat{u}(\xi - \eta) \widehat{v}(\eta) \right| \, d\eta \\
&= C \left(|\widehat{u}| * |\langle \cdot \rangle^s \widehat{v}| \right)(\xi) + C \left(|\langle \cdot \rangle^s \widehat{u}| * |\widehat{v}| \right)(\xi),
\end{aligned}$$

where C is a constant depending on n and s. Let us now plug this into (A.4) and use Young's inequality for convolutions (see [3]), that states that

$$\|f * g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)},$$

so we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |1 + |\xi|^2|^s \widehat{uv}(\xi)|^2 \, d\xi &\leq C \int_{\mathbb{R}^n} |\left(|\widehat{u}| * |\langle \cdot \rangle^s \widehat{v}| \right)(\xi)|^2 + |\left(|\langle \cdot \rangle^s \widehat{u}| * |\widehat{v}| \right)(\xi)|^2 \, d\xi \\
&= C \left\| |\widehat{u}| * |\langle \cdot \rangle^s \widehat{v}| \right\|_{L^2(\mathbb{R}^n)}^2 + C \left\| |\langle \cdot \rangle^s \widehat{u}| * |\widehat{v}| \right\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq C \left\| \langle \cdot \rangle^s \widehat{v} \right\|_{L^2(\mathbb{R}^n)}^2 \left\| \widehat{u} \right\|_{L^1(\mathbb{R}^n)}^2 + C \left\| \langle \cdot \rangle^s \widehat{u} \right\|_{L^2(\mathbb{R}^n)}^2 \left\| \widehat{v} \right\|_{L^1(\mathbb{R}^n)}^2 \\
&\leq C \|v\|_{H^s(\mathbb{R}^n)}^2 \left\| \widehat{u} \right\|_{L^1(\mathbb{R}^n)}^2 + C \|u\|_{H^s(\mathbb{R}^n)}^2 \left\| \widehat{v} \right\|_{L^1(\mathbb{R}^n)}^2 .
\end{aligned}$$

The last result needed is a bound of the L^1-norm of a Fourier transform in terms of the Sobolev norm of the functions itself.

To get this we use that $s > \frac{n}{2}$ implies that

$$\int_{\mathbb{R}^n} \left|1 + |\xi|^2\right|^{-s} \, d\xi < \infty. \tag{A.5}$$

This can be seen by performing the following change of variables. Let $\xi = r\eta$, where $r \geq 0$, $\eta \in \mathbb{R}^n$ and $\|\eta\| = 1$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|1+|\xi|^2\right|^{-s} \, d\xi &= \int_0^\infty \int_{\|\eta\|=1} \left|1+r^2\right|^{-s} \, d\eta \, r^{n-1} \, dr \\
&= \int_{\|\eta\|=1} d\eta \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} \, dr \\
&\leq \alpha(n) \int_0^\infty \frac{(1+r)^{n-1}}{(1+r^2)^s} \, dr, \quad \text{when} \quad n \geq 1 \\
&\leq \alpha(n) \int_0^\infty \frac{(1+r)^{n-1}}{(1+r)^{2s}} \, dr, \quad \text{since} \quad (1+r)^2 \geq 1+r^2 \\
&= \alpha(n) \int_0^\infty (1+r)^{n-1-2s} \, dr \\
&= \alpha(n) \frac{1}{n-2s} \, \left[(1+r)^{n-2s}\right]_{r=0}^\infty \\
&= \frac{\alpha(n)}{2s-n} \quad \text{when} \quad n-2s < 0. \qquad \text{(A.6)}
\end{aligned}$$

Here $\alpha(n)$ denotes the volume of the unit ball in R^n: $\{\eta \in \mathbb{R}^n \big| \, \|\eta\| = 1\}$ and $\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$. Using this gives the desired inequality

$$\begin{aligned}
\|\widehat{u}\|^2_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)| \, d\xi \\
&= \int_{\mathbb{R}^n} \langle\xi\rangle^s \langle\xi\rangle^{-s} \, |\widehat{u}(\xi)| \, d\xi \\
&\leq \left(\int_{\mathbb{R}^n} \left|1+|\xi|^2\right|^{-s} \, d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} \left|1+|\xi|^2\right|^s |\widehat{u}(\xi)|^2 \, d\xi\right)^{1/2} \\
&\leq C \, \|u\|_{H^s(\mathbb{R}^n)}, \quad \text{when} \quad s > \frac{n}{2}
\end{aligned}$$

and we can conclude that

$$\|uv\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |1+|\xi|^2|^s |\widehat{uv}(\xi)|^2 \, d\xi\right)^{1/2} \leq C \, \|v\|_{H^s(\mathbb{R}^n)} \, \|u\|_{H^s(\mathbb{R}^n)} \, .$$

□

We would like to use similar results in the case of open bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. We have the definition (A.1) and we wish to relate it to the Sobolev spaces defined by use of the Fourier transform in $\mathbb{R}^n$. We define

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) : \exists g \in H^s(\mathbb{R}^n), \quad g\Big|_{\Omega} = f, \right\} \tag{A.7}$$

with norm

$$\|f\|_{H^k(\Omega)} = \inf_{\substack{g \in H^k(\mathbb{R}^n) \\ f = g|_\Omega}} \|g\|_{H^k(\mathbb{R}^n)} \tag{A.8}$$

The results (A.3) and Theorem A.3 extend to this case (see for example [1]). The main tool for relating these two spaces is the extension operator.

Theorem A.2 *([1])*
Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that $\partial\Omega$ is smooth. Then for any non-negative integer s there exists an extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^n)$, meaning that E is a linear and bounded operator, mapping functions defined a.e. in Ω into functions defined a.e. in $\mathbb{R}^n$ and for every k, $0 \le k \le s$, it holds for $E : H^k(\Omega) \to H^k(\mathbb{R}^n)$ that

$$Eu(x) = u(x) \quad \text{a.e. in } \Omega,$$
$$\|Eu\|_{H^k(\mathbb{R}^n)} \le K \|u\|_{H^k(\Omega)} .$$

Now using Theorem A.2 we can apply this result on a bounded domain Ω.

Theorem A.3 *Assume that $\Omega \subset \mathbb{R}^n$ is bounded and has smooth boundary $\partial\Omega$. Let s be a non-negative integer satisfying $s > \frac{n}{2}$. Then $H^s(\Omega)$ is a Banach algebra.*

Proof. Let $u, v \in H^s(\Omega)$. Then

$$\begin{aligned} \|uv\|_{H^s(\Omega)} &= \inf_{\tilde{uv} \in H^s(\mathbb{R}^n), uv = \tilde{uv}|_\Omega} \|\tilde{uv}\|_{H^s(\mathbb{R}^n)} \\ &\le \|EuEv\|_{H^s(\mathbb{R}^n)} \\ &\le C \|Eu\|_{H^s(\mathbb{R}^n)} \|Ev\|_{H^s(\mathbb{R}^n)} \quad \text{by} \quad \textit{Theorem A.1} \\ &\le C \|u\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} \quad \text{by} \quad \textit{Theorem A.2}. \end{aligned}$$

□

We also wish to define the Sobolev Space $H^s(\Omega)$ for $s < 0$ and it turns out that $H^{-s}(\Omega)$ is the dual space of $H_0^s(\Omega)$, that is,

$$H^{-s}(\Omega) = \{\textit{all bounded linear functionals on } H_0^s(\Omega)\} \tag{A.9}$$

The norm is gives by

$$\|u\|_{(H^s(\mathbb{R}^n))'} = \sup_{\substack{v \in H^s(\mathbb{R}^n) \\ \|v\| \neq 0}} \frac{\left| \int_{\mathbb{R}^n} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \right|}{\left(\int_{\mathbb{R}^n} |1 + |\xi|^2|^s \left|\widehat{v}(\xi)\right|^2 \ d\xi \right)^{1/2}}$$

which can be shown is equivalent to

$$\|u\|_{H^{-s}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |1 + |\xi|^2|^{-s} \left|\widehat{u}(\xi)\right|^2 \ d\xi \right)^{1/2}.$$

Sobolev spaces of negative order contain elements that are no longer L^2-functions, but are distributions.

We also wish to consider the spaces $H^s(\partial\Omega)$. In boundary value problems it is necessary to satisfy some given boundary conditions, meaning that we need to be able to restrict functions in a Sobolev space to the boundary. It is not obvious how we make sense of the value of for example a function $u \in H^1(\Omega)$ on the boundary $\partial\Omega$. The question is *how smooth* does the data on the boundary need to be for the function in the Sobolev space to take such values. For example if we have an infinitely smooth function, the boundary function is simply its restriction to the boundary, which is well-defined in this case. Let us introduce a trace operator, that restricts a function defined in Ω to its boundary $\partial\Omega$. Define T_0 and T_1 by

$$\begin{aligned} T_0 :& H^1(\Omega) \to H^{1/2}(\partial\Omega), \\ T_0 u =& u\Big|_{\partial\Omega} \end{aligned}$$

and

$$\begin{aligned} T_1 :& H^1(\Omega) \to H^{-1/2}(\partial\Omega), \\ T_1 u =& \frac{\partial u}{\partial \eta}\Big|_{\partial\Omega}, \end{aligned}$$

where we define $H^{1/2}(\Omega)$ as

$$H^{1/2}(\Omega) = \left\{ f \in L^2(\partial\Omega) : u \in H^1(\Omega), T_0 u = f \right\} \tag{A.10}$$

with norm

$$\|f\|_{H^{1/2}(\partial\Omega)} = \inf_{\substack{u \in H^1(\Omega) \\ u|_{\partial\Omega} = f}} \|u\|_{H^1(\Omega)}$$

and the dual space

$$H^{-1/2}(\partial\Omega) = \left(H^{1/2}(\partial\Omega) \right)' \tag{A.11}$$

with norm

$$\|g\|_{H^{-1/2}(\partial\Omega)} = \inf_{\substack{u \in H^{-1}(\Omega) \\ u|_{\partial\Omega} = g}} \|u\|_{H^{-1}(\Omega)} .$$

See for example [16] for a deeper investigation of these spaces.

A.2 Useful inequality

The following theorem is given as an exercise in [5] in the case $\Omega = \mathbb{R}^n$.

Theorem A.4 *If $u \in H^s(\Omega)$ for $s > n/2$, then $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{H^s(\Omega)} , \tag{A.12}$$

where C depends only on s, n.

Proof.
Let $u \in H^s(\Omega)$, $x \in \Omega$ and let $\tilde{u} \in H^s(\mathbb{R}^n)$ be an arbitrary extension of u to $\mathbb{R}^n$. Then by use of the Fourier Transform, Hölder's inequality and Theorem A.5

$$\begin{aligned}
|u(x)| &= \left| \left(\mathcal{F}^{-1} \mathcal{F} \tilde{u} \right)(x) \right| \\
&= \frac{1}{(2\pi)^{n/2}} \left| \int_{\mathbb{R}^n} \widehat{\tilde{u}}(\xi) e^{i\xi \cdot x} \, d\xi \right| \\
&\leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left| \widehat{\tilde{u}}(\xi) \right| \, d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left| \widehat{\tilde{u}}(\xi) \right| \frac{|1 + |\xi|^2|^{s/2}}{|1 + |\xi|^2|^{s/2}} \, d\xi \\
&\leq \frac{1}{(2\pi)^{n/2}} \left(\int_{\mathbb{R}^n} \left| \widehat{\tilde{u}}(\xi) \right|^2 |1 + |\xi|^2|^s \, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} |1 + |\xi|^2|^{-s} \, d\xi \right)^{1/2} \\
&\leq C \|\tilde{u}\|_{H^s(\mathbb{R}^n)} .
\end{aligned}$$

Since this holds for all extensions $\tilde{u}$ of u, it must also hold for the smallest, so we have for all $x \in \Omega$

$$\begin{aligned}
|u(x)| &\leq C \inf_{\substack{\tilde{u} \in H^s(\mathbb{R}^n), \\ \tilde{u}|_\Omega = u}} \|\tilde{u}\|_{H^s(\mathbb{R}^n)} \\
&= C \|u\|_{H^s(\Omega)} ,
\end{aligned}$$

meaning that

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{H^s(\Omega)}$$

as desired. □

Appendix B

Matlab code

This appendix includes the main MATLAB functions written and used in chapter 4.

B.1 Computation of L^2-norm of radial part of u

```
function i = l2u(k,L,c,rho)
% Input:    k: frequency parameter
%           L: number eigenvalue
%           c: potential function
%           rho: position of jump
% Output:   i: L2 norm of radial part of solution to the Helmholtz ...
    with potential (delta + k^2 + q)u = 0


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% Numerator of B2
T =   k.*(besselj(L+1/2, (k.^2+c).^(1/2)*rho).*besselj(L+3/2, k*rho)...
     )...
    - (k.^2+c).^(1/2).*(besselj(L+3/2, (k.^2+c).^(1/2)*rho).*besselj...
        (L+1/2, k*rho) );

%Denominator of B2
```

```
N1 = besselj(L+1/2, sqrt(k.^2+c)*rho).*  besselj(L+3/2, k*rho).* ...
                 bessely(L+1/2, k);
N2 = besselj(L+1/2, sqrt(k.^2+c)*rho).*  besselj(L+1/2, k).* ...
                     bessely(L+3/2, k*rho);
N3 = bessely(L+1/2, k).*                besselj(L+1/2, k*rho).* ...
                 besselj(L+3/2, sqrt(k.^2+c)*rho);
N4 = besselj(L+1/2, k).*                bessely(L+1/2, k*rho).* ...
                 besselj(L+3/2, sqrt(k.^2+c)*rho);

N = k.*(N1 - N2) -sqrt(k.^2+c).*(N3 - N4);

% Coefficient B2
B2 = T./N;

% Numerator of A1
Ta = B2.*besselj(L+1/2, k*rho).*bessely(L+1/2, k)-B2.*bessely(L+1/2,...
     k*rho).*besselj(L+1/2, k)-besselj(L+1/2, k*rho);
%Denominator of A1
Na = besselj(L+1/2, sqrt(k.^2+c)*rho).*besselj(L+1/2, k);

% Coefficient A1
A1 = -Ta./Na;

% |R_l|^2 r^2 in 0 < r < rho
f1 = @(r)  ( A1.*(besselj(L+1/2, sqrt(k.^2+c).*r )./sqrt(r)) ).^2 .*...
    r.^2;

% |R_l|^2 r^2 in  rho < r < 1
f2 = @(r)   ((1-B2.*bessely(L+1/2,k))./(  besselj(L+1/2,k)).*(...
    besselj(L+1/2,k.*r))./(sqrt(r))...
    + B2.*(bessely(L+1/2,k.*r))./(sqrt(r))).^2.*r.^2;

% integrate over both intervals
i1 = integral(f1,0,rho,'ArrayValued',true);
i2 = integral(f2,rho,1,'ArrayValued',true);

% combined integral
i = (i1 + i2).^(1/2);
```

B.2 Chosen l

```
function L = findls(k,c,rho)
% Input:     k: frequency parameter
%            c: potential function
%            rho: position of jump
% Output:    L: integer values, that are as far away from the zeros ...
    of the function J0 defined below

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L = [];

% Zero Eigenvalue function
J0 = chebfun( @(l) (besselj(l+1/2, k*rho).*sqrt(k.^2+c).*bessely(l...
    +1/2, k).*besselj(l-1/2, sqrt(k.^2+c)*rho)...
    -bessely(l+1/2, k*rho).*sqrt(k.^2+c).*besselj(l+1/2, k).*besselj...
        (l-1/2, sqrt(k.^2+c)*rho)...
    +besselj(l+1/2, sqrt(k.^2+c)*rho).*k.*bessely(l-1/2, k*rho).*...
        besselj(l+1/2, k)....
    -besselj(l+1/2, sqrt(k.^2+c)*rho).*k.*bessely(l+1/2, k).*besselj...
        (l-1/2, k*rho)),[0 k]);

% Roots
r = roots(J0);

% First two values of L
if length(r)<2
mi =  fminbnd(J0,0,k);
ma =  fminbnd(-J0,0,k);

L = [L round(mi)];
L = [L round(ma)];

else

mi =  fminbnd(J0,0,r(2));
ma =  fminbnd(-J0,0,r(2));

L = [L round(mi)];
L = [L round(ma)];
end

% Choose L as midpoins of the roots
l = ((r(2:end)-r(1:end-1))/2 + r(1:end-1));
L = [L l'];

% Last value of L
if length(r)>2

    mi =  fminbnd(J0,r(end-1),k);
    ma =  fminbnd(-J0,r(end-1),k);
```

```
    L = [L (mi)];
    L = [L (ma)];
end

% L are the integers closest to the above found values.
L = round(L);
```

B.3 Operator norm

```
function [Eig,Eig0,Eigq,L] = operatornorm(k,c,rho)
% Input:     k: frequency parameter
%            c: potential function
%            rho: position of jump
% Output:    Eig: Computed operator norm of the Dircihlet—to—Neumann ...
    differenceoperator Lambda_q — Lambda_0
%            Eig0: Eigenvalue when q = 0 at chosen L
%            Eigq: Eigenvalue when is piecewise constant c,0 at ...
    chosen L
%            L: Number of eigenvalues chosen

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% Initialize
Eig = zeros(1,length(k));
Eig0 = zeros(1,length(k));
Eigq = zeros(1,length(k));
L = zeros(1,length(k));

% Loop over all frequencies
for i = 1:length(k)

    % find l that need to be investigated
    l = findls(k(i),c,rho);

    % Denominator
    T =   k(i)*(besselj(l+1/2, (k(i)^2+c)^(1/2)*rho).*besselj(l+3/2,...
          k(i)*rho) )...
       — (k(i)^2+c)^(1/2)*(besselj(l+3/2, (k(i)^2+c)^(1/2)*rho).*...
            besselj(l+1/2, k(i)*rho) );

    % Numerator
    N1 = besselj(l+1/2, sqrt(k(i)^2+c)*rho).*  besselj(l+3/2, k(i)*...
        rho).*              bessely(l+1/2, k(i));
    N2 = besselj(l+1/2, sqrt(k(i)^2+c)*rho).*  besselj(l+1/2, k(i))....
        *                   bessely(l+3/2, k(i)*rho);
    N3 = bessely(l+1/2, k(i)).*                besselj(l+1/2, k(i)*...
        rho).*              besselj(l+3/2, sqrt(k(i)^2+c)*rho);
    N4 = besselj(l+1/2, k(i)).*                bessely(l+1/2, k(i)*...
        rho).*              besselj(l+3/2, sqrt(k(i)^2+c)*rho);

    N = k(i)*(N1 — N2) —sqrt(k(i)^2+c)*(N3 — N4);

    % Coefficient
    B2 = T./N;

    % Parantheses
    e = k(i)*(  besselj(l+3/2*ones(1,length(l)), k(i)) ./ (besselj(l...
        +1/2*ones(1,length(l)), k(i)))...
```

```
        .*bessely(l+1/2*ones(1,length(l)), k(i))  - bessely(l+3/2*...
            ones(1,length(l)), k(i)));

    % Eigenvalues for all l
    eigl =  abs(B2).*abs(e);

    % Largest eigenvalue
    maxeig = max(eigl);

    % Position of largest eigenvalue
    id = (eigl==maxeig);

    % Find chosen l
    l_valg = find(id,1,'first');
    L(i) = l(l_valg);

    % Coefficient at L
    B2 = B2(l_valg);

    % Eigenvalue at L when q = 0
    eig0 =  L(i) - k(i)*besselj( L(i)+3/2, k(i)) ./(besselj( L(i)...
        +1/2, k(i)));
    % Eigenvalue at L when q = c,0
    eigq = L(i) +k(i)*(B2.*bessely(L(i)+1/2, k(i))-1).*besselj(L(i)...
        +3/2, k(i))./besselj(L(i)+1/2, k(i))...
        -B2*k(i).*bessely(L(i)+3/2, k(i));

    % Load values
    Eig0(i) = abs(eig0);
    Eigq(i) = abs(eigq);

    Eig (i) = maxeig;
end
```

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