SECTION FOR DIGITAL SIGNAL PROCESSING
DEPARTMENT OF MATHEMATICAL MODELLING
TECHNICAL UNIVERSITY OF DENMARK

Course 04362 Digital Signal Processing: Solutions to Problems in Proakis and Manolakis, Digital Signal Processing, 3rd Edition

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3rd Edition

Preface

This note is a supplement to the textbook used in the DTU Course 04362 Digital Signal:

> J.G. Proakis & D.G. Manolakis: *Digital Signal Processing: Principles, Algorithms and Applications*, 3rd edition, Upper Saddle River, New Jersey: Prentice-Hall, Inc., 1996.

Unless anything else is mentioned, all page and equation references are with respect to the course textbook.

The note is partially based on material by Simon Boel Pedersen used in the former DTU course 4232 Digital Signal Processing.

Jan Larsen
Lyngby, September 1998

The manuscript was typeset in 12 points Times Roman and Pandora using LaTeX 2_ε

SOLUTION TO PROBLEM 9.1

The center frequency is $F_c = 50$ and the bandwidth is $B = 20$. According to Ch. 9.1. the minimum sampling frequency is given by $F_{s,min} = 1/T' = 2Br'/r$ where $r' = F_c/B + 1/2 = 50/20 + 1/2 = 3$ and $r = \lfloor r' \rfloor = 3$. That is, $F_{s,min} = 2B = 40$. In general, we require that the p'th and the $(p+1)$'th replication of the spectrum in the sampled signal does not interfere with the original signal spectrum as shown in Fig 9.1.

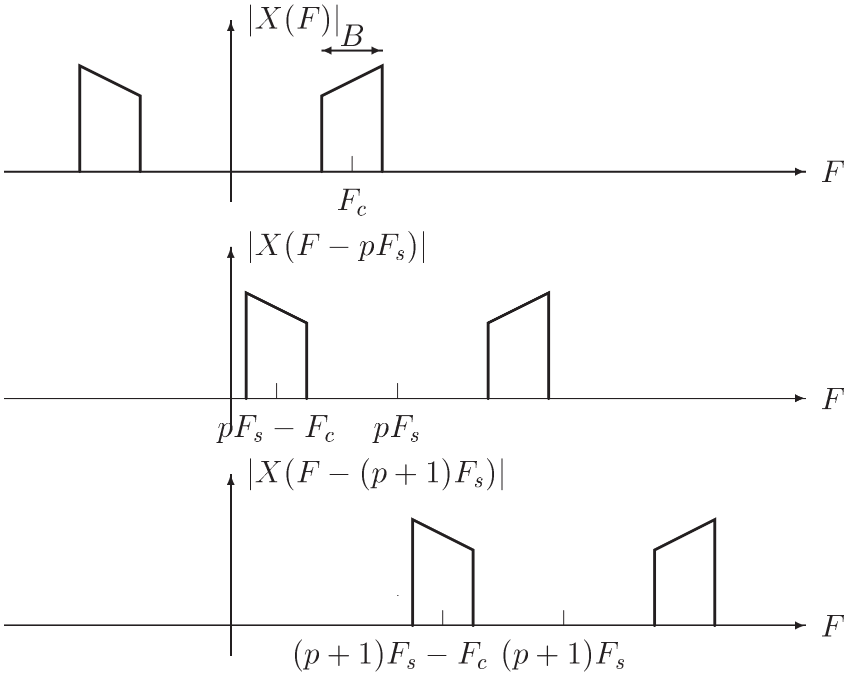

Fig. 9.1

This gives the inequalities:

$$\begin{aligned} F_c - B/2 &> pF_s - F_c + B/2 \\ F_c + B/2 &< (p+1)F_s - F_c - B/2 \end{aligned}$$

Here we assume $F_c > B/2$ and p is an integer. Rewriting yields:

$$\begin{aligned} p &< \frac{2F_c - B}{F_s} \\ p+1 &> \frac{2F_c + B}{F_s} \end{aligned}$$

Thus

$$\frac{B}{F_s} - 1 < p - \frac{2F_C}{F_s} < -\frac{B}{F_s}$$

which implies the natural condition:

$$F_s > 2B$$

The minimum sampling frequency is achieved when p is maximized. Using the natural condition, the first inequality gives

$$p < \frac{B}{F_s}\left(\frac{2F_c}{B} - 1\right) < \frac{F_c}{B} - \frac{1}{2},$$

and the second gives

$$p + 1 > \frac{F_c}{B} + \frac{1}{2} > \frac{B}{F_s}\left(\frac{2F_c}{B} + 1\right)$$

$\Updownarrow$

$$p > \frac{F_c}{B} - \frac{1}{2}.$$

Choosing p as the integer, $p_{\max} = \lfloor \frac{F_c}{B} - \frac{1}{2} \rfloor$, ensures that the replicated parts of the sampled signal are distributed equally along the frequency axis. Applying $p_{\max}$ in the inequalities results in:

$$2B\frac{r'}{r} < F_s < 2B\frac{s'}{s}$$

where $r' = F_c/B + 1/2$, $r = \lfloor r' \rfloor$, $s' = F_c/B - 1/2$, and $s = \lfloor s' \rfloor$.

SOLUTION TO PROBLEM 9.4

Solution to Question (a): Using the sampling frequency $F_s = 2B$, $X(f)$ will be a periodic replication of the original spectrum $X_a(F)$ with period F_s cf. Eq. (4.2.85) $X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$. The spectrum of $y(n)$, say $Y(f)$ is obtained by using the fact that multiplication in the time-domain corresponds to convolution in the frequency domain. Finally, $y_1(t)$ is found by low-pass filtering $y(n)$ using a passband $F \in [0; B]$.

$X_a(F)$ has ideal low-pass filter characteristic, say $X_a(f) = 1$ $|F| \leq B$, and zero otherwise. The spectrum $X(f) = F_s$, $\forall\, |f| \leq 1/2$, i.e., constant. Thus $x(n) = F_s\delta(n)$. $y(n) = x^2(n) = F_s^2\delta(n)$ with spectrum $Y(f) = F_s^2$, $\forall\, |f| \leq 1/2$. Consequently, by Eq. (4.2.87) $Y_1(F) = F_s$ for $|F| \leq B$, and zero otherwise. That is, $y_1(t) = F_s \cdot x_1(t)$.

The spectrum of $s_a(t) = x_a^2(t)$ is found using convolution in the frequency domain, i.e.,

$$S_a(F) = \int_{-\infty}^{\infty} X_a(S)X_a(F-S)dS = \begin{cases} 2B - |F| & , -2B \leq F \leq 2B \\ 0 & , \text{otherwise} \end{cases}$$

The sketch of the spectra is shown in Fig. 9.4.1.

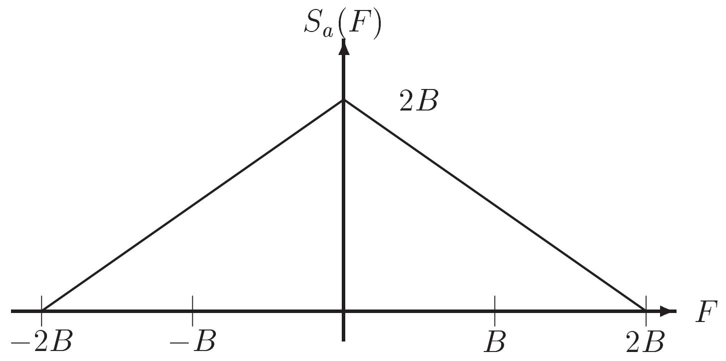

Fig. 9.4.1(a)

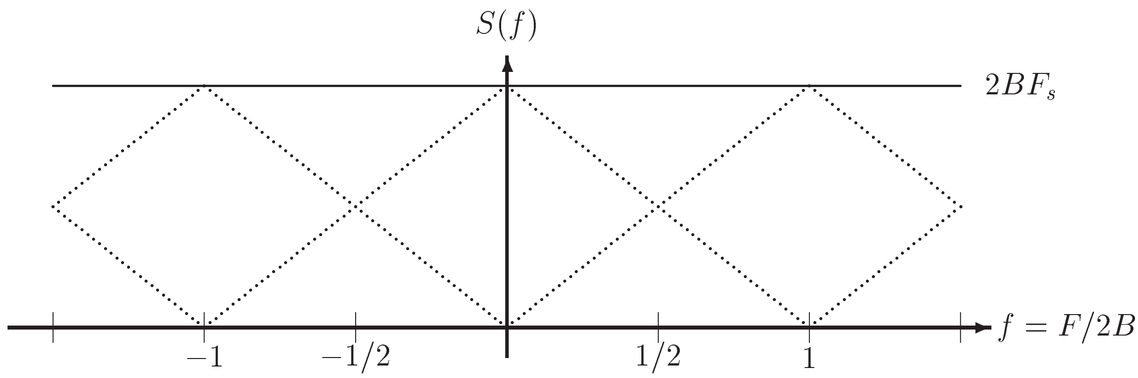

Fig. 9.4.1(b)

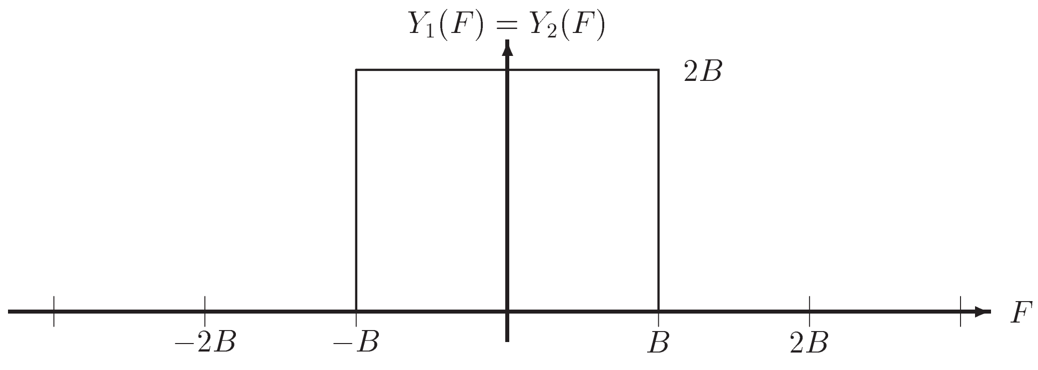

Fig. 9.4.1(c)

Thus $y_2(t) = 2Bx(t) = 2By_1(t)/F_s = y_1(t)$.

Solution to Question (b): $x_a(t) = \cos(2\pi F_0 t)$ with spectrum $X_a(F) = 0.5 \cdot \delta(F \pm F_0)$, where $F_0 = 20\,\text{Hz}$.

Using $F_s = 50\,\text{Hz} > 2F_0$ the sampling theorem is fulfilled. $X(F/F_s)$ has components of amplitude $F_s/2$ for $F = \pm(20 + p50)\,\text{Hz}$, $p = 0, 1, 2, \cdots$. By using the rule of convolution in the frequency domain $Y(F/F_s)$ has components for $F = \pm 10\,\text{Hz}$ with amplitude $F_s/4$ and a DC component ($F = 0$) with amplitude $F_s/2$. By ideal D/A we have $y_1(t) = 1/2 + 1/2 \cdot \cos(2\pi(10\,\text{Hz})t)$. $s_a(t) = x_a^2(t) = \cos^2(2\pi F_0 t) = 1/2 + 1/2\cos(2\pi 2F_0 t)$, i.e., $X_a(F)$ has a DC component with amplitude $1/2$ and two components at $F = \pm 40\,\text{Hz}$ with amplitude $1/4$. Sampling with $F_s = 50\,\text{Hz}$ will now cause aliasing. $S(F/F_s)$ will have components at $F = 0$ with amplitude $F_s/2$ and two components at $F = \pm 10\,\text{Hz}$ with amplitude $F_s/4$. Ideal D/A gives $y_2(t) = y_1(t)$.

When sampling with $F_s = 30\,\text{Hz}$ the sampling theorem is not fulfilled. $x(n)$ will have components at $F = \pm 10\,\text{Hz}$ with amplitudes $F_s/2$. $y(n)$ consequently have components at $F \pm 10\,\text{Hz}$ with amplitudes $F_s/4$ at a DC component with amplitude $1/2$. Reconstruction gives $y_1(t) = 1/2 + 1/2 \cdot \cos(2\pi(10\,\text{Hz})t)$. Sampling $s_a(t)$ with $F_s = 30\,\text{Hz}$ results in $s(n)$ with a DC component with amplitude $F_s/2$ and two components $F = \pm 10\,\text{Hz}$ with amplitudes $F_s/4$. Reconstruction gives $y_2(t) = y_1(t)$.

SOLUTION TO PROBLEM 4.83

Solution to Question (a): From Fig. P4.83 we notice that frequency-inversion scrambling is done by shifting the spectrum by $f = 1/2$ ($\omega = \pi$), thus $Y(f) = X(f - 1/2)$. The corresponding time domain operation is found by using the frequency shift property – or modulation theorem Ch. 4.3.2: $y(n) = (-1)^n x(n)$.

Solution to Question (b): The unscrambler is a similar frequency shift of $f = 1/2$, thus $x(n) = (-1)^n y(n) = (-1)^{2n} x(n) = x(n)$.

SOLUTION TO PROBLEM 9.5

A analog TV signal $x_a(t)$ is corrupted by an echo signal. The received signal is:

$$s_a(t) = x_a(t) + \alpha \cdot x_a(t - \tau), \; |\alpha| < 1, \; \tau > 0$$

By using the time shifting property, the spectrum of $s_a(t)$ is given by:

$$\begin{aligned} S_a(F) &= X_a(F) + \alpha \cdot X_a(F) e^{-j2\pi F\tau} \\ &= X_a(F) \cdot \left(1 + \alpha e^{-j2\pi F\tau}\right) \end{aligned}$$

Since the bandwidth of $x_a(t)$ is B, clearly the bandwidth of $s_a(t)$ is also B. Choosing the sampling frequency in accordance with the sampling theorem $F_s > 2B$ and such that $\tau = n_0/F_s = n_0 T$, where n_0 is a positive integer, and T is the sampling interval, the digital equivalent becomes:

$$s(n) = x(n) + \alpha x(n - n_0)$$

with spectrum

$$S(f) = X(f) \cdot \left(1 + \alpha e^{-j2\pi f n_0}\right)$$

In order to get $y(n) = x(n)$ and consequently $y_a(t) = x_a(t)$ when using an ideal reconstruction, i.e., a BL interpolator with cut-off frequency B, we need a filter $H(f)$ which is the inverse of $1 + \alpha e^{-j2\pi f n_0}$, thus:

$$H(f) = \frac{1}{1 + \alpha e^{-j2\pi f n_0}}$$

The corresponding transfer function is $H(z) = 1/(1 + \alpha z^{-n_0})$, i.e., the filter is an all-pole filter with poles found by solving the binomial equation of degree n_0:

$$z^{n_0} = -\alpha$$

The n_0 poles is therefore given by:

$$z_p = \begin{cases} \sqrt[n_0]{|\alpha|} \cdot e^{j(2\pi p/n_0)} & , -1 < \alpha < 0, \; p = 0, 1, \cdots, n_0 - 1 \\ \sqrt[n_0]{|\alpha|} \cdot e^{j(\pi/n_0 + 2\pi p/n_0)} & , 0 \le \alpha < 1, \; p = 0, 1, \cdots, n_0 - 1 \end{cases}$$

The poles are inside the unit circle, i.e., $|z_p| < 1$ for $|\alpha| < 1$, thus the filter is (BIBO) stable and causal.

If $n_0 = 1$ the filter is a low-pass filter for $\alpha < 0$ cf. Ch. 4.5.2. For $\alpha > 0$ the filter is the corresponding high-pass filter found by using the modulation theorem (frequency shifting) Ch. 4.3.2. This is easily seen as $h(n)(-1)^n$ corresponds to $H(f - 1/2)$. Thus shifting $H(f)$ gives:

$$H(f - 1/2) = \frac{1}{1 + \alpha e^{-j2\pi(f-1/2)}} = \frac{1}{1 + \alpha e^{j\pi} e^{-j2\pi f}} = \frac{1}{1 - \alpha e^{-j2\pi f}}$$

Denote by $H_{n_0}(f)$ the filter using n_0. The filters of n_0 is easily found from $H_1(f)$. Since $H_1(z) = 1/(1 - \alpha z^{-1})$, then $H_{n_0}(z) = H_1(z^{n_0})$, see Ch. 4.5.5 and 10.3. That is, the spectrum $H_{n_0}(f) = H_{n_0}(z)|_{|z|=e^{j2\pi f}}$ is found by

$$H_{n_0}(f) = H_1(n_0 f)$$

That is, $H_{n_0}(f)$ is a comb filter cf. Ch. 4.4.5. Fig. 9.5.1 and 9.5.2 show typical spectra.

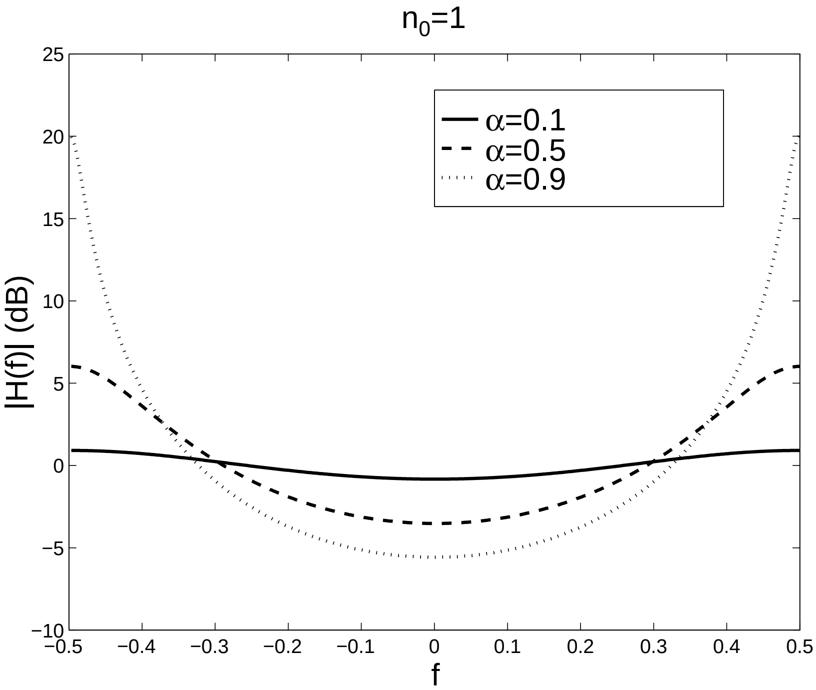

Fig. 9.5.1

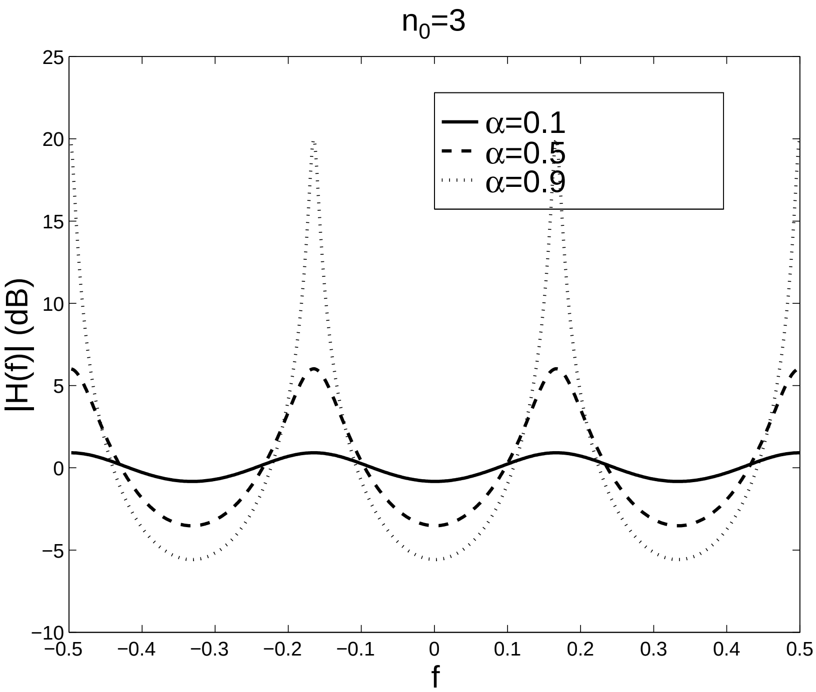

Fig. 9.5.2

SOLUTION TO PROBLEM 9.7

$x(n)$ is a stationary random signal with mean $E[x(n)] = 0$, variance $V[x(n)] = \sigma_x^2$ and autocorrelation function, $\gamma_{xx}(m)$. We consider using a delta modulator.

Solution to Question (a): Since $d(n) = x(n) - ax(n-1)$ and $E[x(n)] = 0$ also $E[d(n)] = 0$, i.e., the variance σ_d^2 of $d(n)$ is found by evaluating

$$\begin{aligned} \sigma_d^2 &= E[d^2(n)] \\ &= E\left[(x(n) - ax(n-1))^2\right] \\ &= E[x^2(n)] - 2aE[x(n)x(n-1)] + a^2E[x^2(n-1)] \\ &= \sigma_x^2 - 2a\gamma_{xx}(1) + a^2\sigma_x^2 \\ &= \sigma_x^2\left(1 + a^2 - 2a\rho_{xx}(1)\right) \end{aligned}$$

where we used that the normalized autocorrelation function (Ch. 2.6.2) $\rho_{xx}(m) = \gamma_{xx}/\gamma_{xx}(0) = \gamma_{xx}/\sigma_x^2$.

Solution to Question (b): An extremum of σ_d^2 is found by solving

$$\frac{\partial \sigma_d^2}{\partial a} = 0$$

That is, the optimal value a^* is given by

$$\frac{\partial \sigma_d^2}{\partial a} = 2\sigma_x^2\left(a^* - \rho_{xx}(1)\right) = 0 \;\Leftrightarrow\; a^* = \rho_{xx}(1)$$

Since $\partial^2\sigma_x^2/\partial a^2 = 2\sigma_x^2 > 0$ the extremum is a minimum. By substituting a^* we get:

$$\sigma_d^2 = \sigma_x^2\left(1 + \rho_{xx}^2(1) - 2\rho_{xx}(1)\rho_{xx}(1)\right) = \sigma_x^2\left(1 - \rho_{xx}^2(1)\right)$$

Solution to Question (c): $\sigma_d^2 < \sigma_x^2$ is obtained when $1 - \rho_{xx}^2(1) < 1$, i.e., $\rho_{xx}^2(1) > 0$ which always is the case except when $x(n)$ is a white noise signal for which $\gamma_{xx}(m) = \sigma_x^2\delta(m)$.

Solution to Question (d): In order to study the second order prediction error $d(n) = x(n) - a_1x(n-1) - a_2x(n-2)$ we define two column vectors:

$$\boldsymbol{a} = (a_1, a_2)^\top, \qquad \boldsymbol{x} = (x(n-1), x(n-2))^\top$$

Then we express $d(n) = x(n) - \boldsymbol{x}^\top\boldsymbol{a}$. The variance of $d(n)$ is found by first evaluating $d^2(n)$:

$$\begin{aligned} d^2(n) &= \left(x(n) - \boldsymbol{x}^\top\boldsymbol{a}\right)^2 \\ &= x^2(n) - 2x(n)\boldsymbol{x}^\top\boldsymbol{a} + \boldsymbol{a}^\top\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{a} \end{aligned}$$

Performing the expectation $E[\cdot]$

$$E[d^2(n)] = E[x^2(n)] - 2E[x(n)\boldsymbol{x}^\top]\boldsymbol{a} + \boldsymbol{a}^\top E[\boldsymbol{x}\boldsymbol{x}^\top]\boldsymbol{a}$$

where

$$\begin{aligned} E[x(n)\boldsymbol{x}^\top] &= (E[x(n)x(n-1)], E[x(n)x(n-2)]) = (\gamma_{xx}(1), \gamma_{xx}(2)) \\ E[\boldsymbol{x}\boldsymbol{x}^\top] &= \begin{bmatrix} E[x(n-1)x(n-1)] & E[x(n-1)x(n-2)] \\ E[x(n-1)x(n-2)] & E[x(n-2)x(n-2)] \end{bmatrix} = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(1) & \gamma_{xx}(0) \end{bmatrix} \end{aligned}$$

Since $\gamma_{xx}(0) = \sigma_x^2$ we define

$$\begin{aligned} \boldsymbol{P}^\top &= \frac{E[x(n)\boldsymbol{x}^\top]}{\sigma_x^2} = (\rho_{xx}(1), \rho_{xx}(2)) \\ \boldsymbol{R} &= \frac{E[\boldsymbol{x}\boldsymbol{x}^\top]}{\sigma_x^2} = \begin{bmatrix} 1 & \rho_{xx}(1) \\ \rho_{xx}(1) & 1 \end{bmatrix} \end{aligned}$$

Notice that $\boldsymbol{R}$ is positive definite, i.e., $\boldsymbol{q}^\top\boldsymbol{R}\boldsymbol{q} > 0$ for all $\boldsymbol{q} \neq \boldsymbol{0}$ since

$$\sigma_x^2\left(\boldsymbol{q}^\top\boldsymbol{R}\boldsymbol{q}\right) = E[\boldsymbol{q}^\top\boldsymbol{x}\boldsymbol{x}^\top\boldsymbol{q}] = E[(\boldsymbol{q}^\top\boldsymbol{x})^2] > 0, \ \boldsymbol{q} \neq \boldsymbol{0}$$

Now we have the expression

$$\sigma_d^2 = \sigma_x^2\left(1 - 2\boldsymbol{P}^\top\boldsymbol{a} + \boldsymbol{a}^\top\boldsymbol{R}\boldsymbol{a}\right)$$

Extremum is found by solving $\partial\sigma_d^2/\partial\boldsymbol{a} = \boldsymbol{0}$, i.e.,

$$\frac{\partial\sigma_d^2}{\partial\boldsymbol{a}} = \sigma_x^2(-2\boldsymbol{P} + 2\boldsymbol{R}\boldsymbol{a}) = \boldsymbol{0}$$

That is the optimal $\boldsymbol{a}$, say $\boldsymbol{a}^*$, is

$$\boldsymbol{a}^* = \boldsymbol{R}^{-1}\boldsymbol{P}$$

Note that $\boldsymbol{R}^{-1}$ exists since it is positive definite. Moreover, the optimum is a minimum since the second order derivative matrix $\partial^2\sigma_d^2/\partial\boldsymbol{a}\partial\boldsymbol{a}^\top = \boldsymbol{R}$ is positive definite.

By substituting the optimal value $\boldsymbol{a}^*$ into the expression for σ_d^2 we get

$$\begin{aligned} \sigma_d^2 &= \sigma_x^2\left(1 - 2\boldsymbol{P}^\top\boldsymbol{R}^{-1}\boldsymbol{P} + \boldsymbol{P}^\top\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{P}\right) \\ &= \sigma_x^2\left(1 - \boldsymbol{P}^\top\boldsymbol{R}^{-1}\boldsymbol{P}\right) \end{aligned}$$

$\boldsymbol{P}^\top\boldsymbol{R}^{-1}\boldsymbol{P} \geq 0$ since $\boldsymbol{R}$ is positive definite and equals zero if $\boldsymbol{P} = \boldsymbol{0}$, i.e., when $\rho_{xx}(m) = 0$ for $m > 1$ (white noise). Thus, except for the white noise case, $\sigma_d^2 < \sigma_x^2$ is always fulfilled. Further note it is possible to show $\boldsymbol{P}^\top\boldsymbol{R}^{-1}\boldsymbol{P} \leq 1$ since $\sigma_d^2 \geq 0$.

SOLUTION TO PROBLEM 9.11

Solution to Question (a): Consider the rewritten second order SDM model shown in Fig. 9.11.1

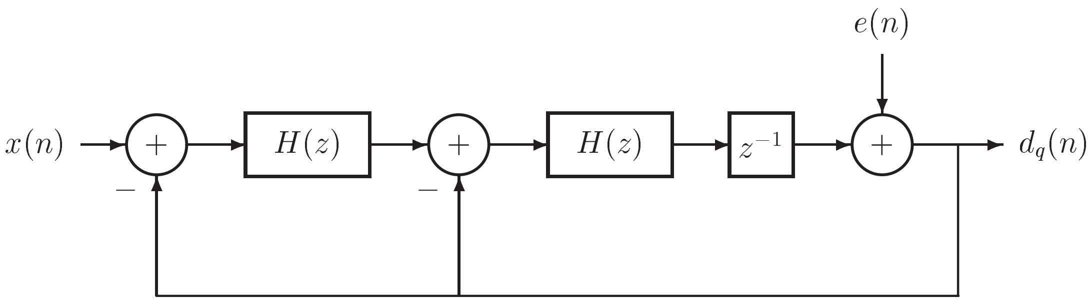

Fig. 9.11.1

where $H(z)$ cf. Fig. P9.11 is defined by $H(z) = 1/(1-z^{-1})$. Using the superposition principle we can write:

$$D_q(z) = -D_q(z)H(z)z^{-1}(1 + H(z)) + H^2(z)z^{-1}X(z) + E(z)$$

That is,

$$D_q(z) = \frac{H^2(z)z^{-1}}{1 + H(z)z^{-1}(1 + H(z))}X(z) + \frac{1}{1 + H(z)z^{-1}(1 + H(z))}E(Z)$$

By using the expression for $H(z)$, the noise transfer function is

$$H_n(z) = \frac{1}{1 + H(z)z^{-1}(1 + H(z))} = (1 - z^{-1})^2$$

and the signal transfer function is

$$H_s(z) = \frac{H^2(z)z^{-1}}{1 + H(z)z^{-1}(1 + H(z))} = z^{-1}$$

Solution to Question (b): The magnitude frequency response is found by evaluating at $|H_n(z)|$ at $z = e^{j\omega}$, i.e.,

$$\begin{aligned}
|H_n(\omega)| &= \left|\left(1 - e^{-j\omega}\right)^2\right| = \left|\left(1 - 2e^{-j\omega} + e^{-j2\omega}\right)\right| \\
&= \left|e^{-j\omega}\left(e^{j\omega} - 2 + e^{-j\omega}\right)\right| \\
&= |-2 + 2\cos(\omega)|
\end{aligned}$$

The noise magnitude responses of a first and a second order SDM is shown in Fig. 9.11.2

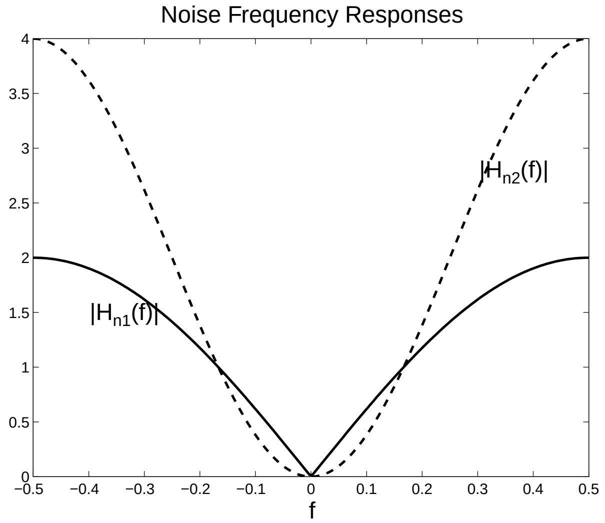

Fig. 9.11.2

The 6 dB difference is due to an extra zero for the 2nd order system.

Solution to Question (c): Using the fact that the quantization noise is white, it has the power spectral density $S_e(F) = \sigma_e^2/F_s$ where F_s is the sampling frequency and σ_e^2 is the noise power. Using Eq. (9.2.19) where B is the bandwidth of the signal $x_a(t)$, we get

$$\begin{aligned}
\sigma_n^2 &= \int_{-B}^{B} |H_n(F/F_s)|^2 S_e(F)\, dF \\
&= 4\frac{\sigma_e^2}{F_s}\int_{-B}^{B} (-1+\cos(2\pi F/F_s))^2\, dF \\
&= \frac{2\sigma_e^2}{\pi}\int_{-2\pi B/F_s}^{2\pi B/F_s} (-1+\cos(\omega))^2\, d\omega \\
&= \frac{2}{\pi}\sigma_e^2\left[\frac{3\omega}{2}+\frac{\sin(2\omega)}{4}-2\sin(\omega)\right]_{-2\pi B/F_s}^{2\pi B/F_s} \\
&= \frac{2}{\pi}\sigma_e^2\left[\frac{6\pi B}{F_s}+\frac{\sin(4\pi B/F_s)}{2}-4\sin(2\pi B/F_s)\right]
\end{aligned}$$

Taylor series expansion of $\sin(x) = x - x^3/3! + x^5/5!$ for $x \ll 1$ corresponding to $F_s \gg B$ gives:

$$\sigma_n^2 \approx \frac{\pi^4\sigma_e^2}{5}\left(\frac{2B}{F_s}\right)^5$$

A doubling of F_s gives a reduction of σ_n^2 by a factor of 32 which corresponds to approx. 15 dB.

SOLUTION TO PROBLEM 9.12

Given $x(n) = \cos(2\pi n/N)$, $n = 0, 1, \cdots, N-1$. The signal is reconstructed through an ideal D/A converter with sampling interval T corresponding to a sampling frequency $F_s = 1/T$.

Solution to Question (a): In general, cf. Ch. 4.2, a periodic digital signal has normalized frequency components at $f = k/N$, $k = 0, 1, \cdots, N-1$, and consequently frequency components at $F = kF_s/N$. A cosine has spectral components for $k = 1$ and $k = N-1$ only. Thus, $F_0 = F_s/N$.

Solution to Question (b): If F_s is fixed and we only use the given lookup table, the only possibility is to sample $x(n)$. Downsampling a discrete-time signal with a factor of D is equivalent to

$$x_{\text{sam}}(n) = x(n) \sum_{q=-\infty}^{\infty} \delta(n - qD)$$

The spectrum is found by (see e.g., Ch. 10 or the textbooks in signal analysis)

$$X_{\text{sam}}(f) = \frac{1}{D} \sum_{q=0}^{D-1} X(f - q/D)$$

The spectrum of the signal $x(n)$ considered as a aperiodic signal of length N can be interpreted as the corresponding period signal multiplied with a square wave signal of length N. Thus the spectrum is the convolution of the spectrum of the square wave signal and the cosine signal:

$$X(f) = \frac{\sin \pi n f N}{\sin \pi f} e^{-j\pi f(N-1)} * (\delta(f - 1/N)/2 + \delta(f + 1/N)/2)$$

The spectrum is $X(1/N) = X(-1/N) = 1/2$ and has zeros at $f = k/N$ $k \neq \pm 1$. The sampled signals spectrum is a replication of $X(f)$ with q/D. Thus if D is a divisor in N, say $N = KD$, $X_{\text{sam}}(f) = 1/D$ for $f = 1/N + q/D = (1 + qK)/N$, $q = 0, 1, \cdots, D-1$ and zero for all other multiples of $1/N$. The infinitely replicated signal is forming a periodic signal with frequencies at $k = 1, 1+K, 1+2K, \cdots, 1+(D-1)K$ and $k = -1 + K, -1 + 2K, \cdots, -1 + DK$. Changing to the new sampling frequency $1/D$ gives components at D/N corresponding to $F_0 = DF_s/N$. The number of possible frequencies is thus the number of all possible products of prime factors in N. If e.g., $N = 12 = 2 \cdot 2 \cdot 3$ the possible $D \in \{2, 3, 4, 6\}$.

If D is not a divisor in N then $X_{\text{sam}}(f)$ will have non-zero values for most multiples of $1/N$ thus the replicated signal is not a cosine.

This is shown in the following Figures 9.12.1–4.

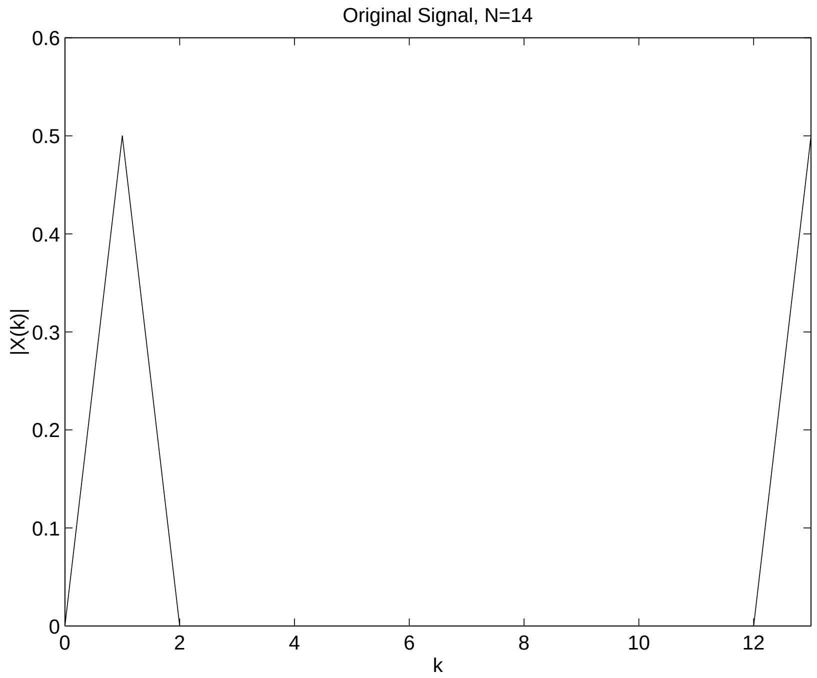

Fig. 9.12.1

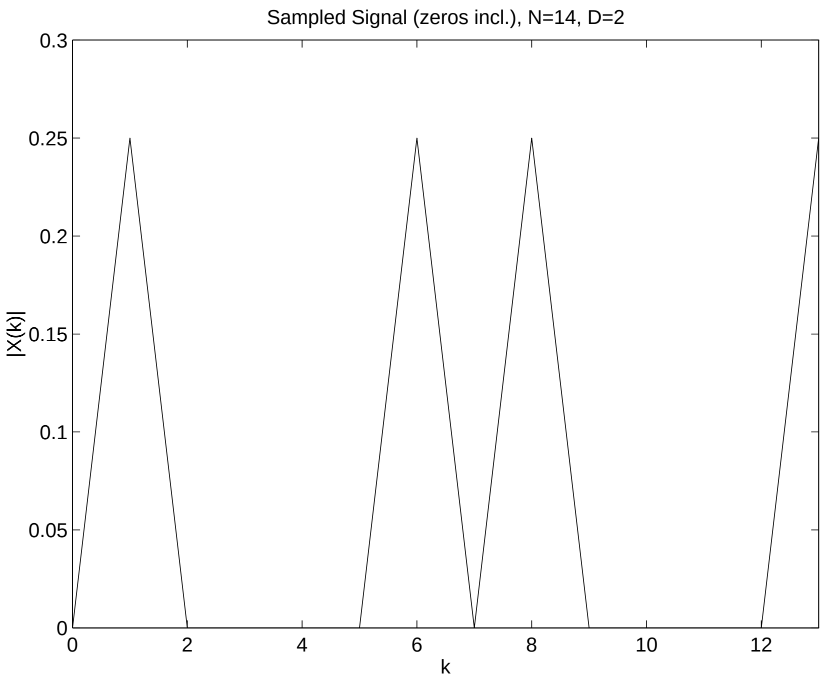

Fig. 9.12.2

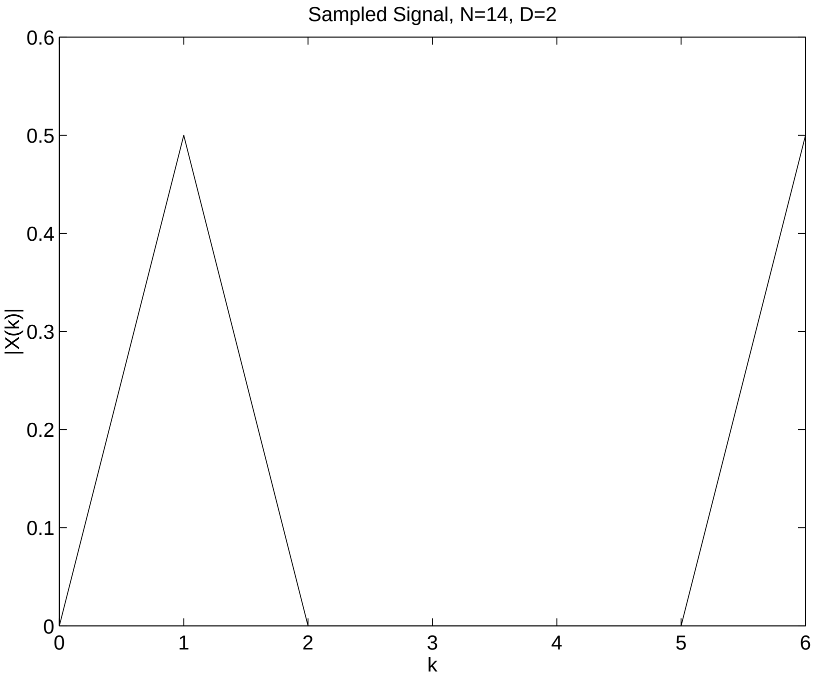

Fig. 9.12.3

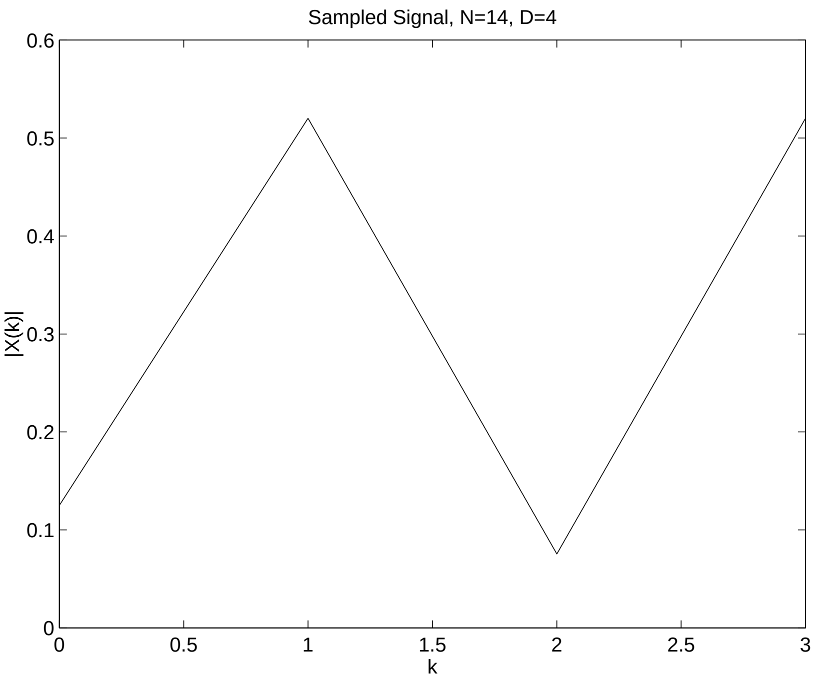

Fig. 9.12.4

SOLUTION TO PROBLEM 5.20

Solution to Question (a): $X(k) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} x(n)$ where N is an even number and $x(n)$ fulfill the symmetry property:

$$x(n+N/2) = -x(n),\ n = 0, 1, \cdots, N/2-1$$

Recall that $W_N = e^{-j2\pi/N}$. By evaluating $X(k)$ we get:

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)W_N^{kn} \\
&= \sum_{n=0}^{N/2-1} x(n)W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)W_N^{kn} \\
&= \sum_{n=0}^{N/2-1} x(n)W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{k(n+N/2)} \\
&= \sum_{n=0}^{N/2-1} x(n)W_N^{kn} + \sum_{n=0}^{N/2-1} -x(n)W_N^{kN/2}W_N^{kn} \\
&= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{k+1}x(n)\right] W_N^{kn}
\end{aligned}$$

Here we used the fact that $W_N^{kN/2} = e^{-j\pi k} = (-1)^k$. Now for $k = 2k'$, $k' = 0, 1, \cdots, N/2-1$ we have: $x(n) + (-1)^{k+1}x(n) \equiv 0$, thus $X(2k') = 0$.

Solution to Question (b): Evaluating the odd harmonics we get:

$$\begin{aligned}
X(2k'+1) &= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2k'+2}x(n)\right] W_N^{(2k'+1)n} \\
&= \sum_{n=0}^{N/2-1} \left[2x(n)W_N^n\right] W_N^{2k'} \\
&= \sum_{n=0}^{N/2-1} \left[2x(n)W_N^n\right] W_{N/2}^{k'}
\end{aligned}$$

That is, $X(2k'+1) \overset{\text{DFT}}{\underset{N/2}{\longleftrightarrow}} y(n)$ where $y(n) = 2x(n)W_N^n$, $n = 0, 1, \cdots, N/2-1$.