The Frame Set of Gabor Systems with B-spline Generators

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Summary

This thesis is concerned with computational and theoretical aspects of Gabor systems in time-frequency analysis and, in particular, the representation of signals in terms of time-frequency shifts of B-splines constituting a frame. Frames are systems of "simple" functions or building blocks which deliver ways of analysing and representing signals in a stable manner, even in the presence of noise. Because of these desirable properties, frames play an important role in both harmonic analysis and signal processing.

One of the fundamental problems in Gabor analysis is to determine for which sampling and modulation rates, controlled by two parameters $a > 0$ and $b > 0$, respectively, the corresponding time-frequency shifts of a given generator constitutes a frame. The so-called *frame set* of a generator is the parameter values $(a, b) \in \mathbb{R}_+^2$ for which the associated Gabor system generated by the generator function is a frame.

Except for the first B-spline very little is known about the frame set for B-splines. This thesis adds a considerable amount of new information on the frame set for B-splines. We first review some of the known characteristics of the frame set for B-splines. We then prove a new domain of parameter values (a, b) for which the Gabor system generated by B-splines is indeed a frame. Furthermore, we examine some of the unknown areas numerically, both in Matlab and Maple. From these simulations, we discover new parameter values (a, b) which do not belong the frame set of the B-splines of order two. This, in turn, disproves a recent conjecture by Karlheinz Gröchenig. Finally, we formulate two new conjectures on the frame set of the B-splines of order two based on our numerical and theoretical findings.

Resumé

Dette speciale omhandler numeriske og teoretiske aspekter af Gabor systemer i tids-frekvens analyse og specielt repræsentation af signaler ved brug af tids-frekvens forskydninger af B-splines der udgør en frame. Frames er systemer af "simple" funktioner som giver en metode til stabil analyse og repræsentation af signaler, selv hvis de indeholder støj. Grundet disse fordelagtige egenskaber spiller frames en vigtig rolle i både harmonisk analyse og signalbehandling.

Et af de grundlæggende problemer i Gabor analyse er at bestemme for hvilke sampling og modulations rater, kontrolleret af parametrene $a > 0$ og $b > 0$, de tilhørende tids-frekvensforskydninger af en given generator udgør en frame. Den såkaldte *frame-mængde* af en generator g er de parameterværdier $(a, b) \in \mathbb{R}^2_+$, for hvilke Gabor systemet genereret af g er en frame.

Bortset fra for den første B-spline er det ikke meget der er kendt omkring frame-mængden for B-splines. Dette speciale tilføjer betydelig ny viden om frame-mængden for B-splines. Først undersøger vi nogle af de kendte beskrivelser af frame-mængden for B-splines. Derefter beviser vi en ny mængde af parameterværdier (a, b), for hvilke Gabor systemet genereret af B-splines er en frame. Derudover undersøger vi en del af de ukendte områder numerisk både i Matlab og i Maple. Fra disse simuleringer opdager vi nye parameterværdier (a, b), der ikke tilhører frame-mængden for den B-spline, der har orden to. Derved modbeviser vi en formodning af Karlheinz Gröchenig. Til sidst formulerer vi to nye formodninger om frame-mængden for B-splines baseret på vores numeriske undersøgelser.

Preface

This thesis is submitted as partial fulfilment of the requirements for the M.Sc. degree in Mathematical Modelling and Computing. The work has been carried out in the period between February and June 2015 at DTU Compute at the Technological University of Denmark.

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I would like to thank my supervisor Jakob Lemvig for many great meetings and a never failing enthusiasm.

A guide to the reader

The thesis consists of 4 sections and an Appendix. Section 2 to 4 provide a theoretical background on frames in Hilbert spaces, B-splines and Gabor systems. Section 5 contains both known and new results for the frame set of B-splines. The appendix provides Matlab code for the numerical parts of section 5 and a Maple sheet with calculations used in section 5.

Notation is self-explanatory or it is introduced in the main text.

Contents

1 Introduction

Frames are a useful tool to analyze and represent complex signals in terms of smaller and simpler building blocks. In the analysis process a frame converts the signal to a sequence of coefficients in $\ell^2(\mathbb{N})$. In the synthesis process a so-called dual frame is used to bring back the original signal from the sequence of coefficients in a stable way, even if the analysis coefficients are corrupted by noise.

Gabor system are used for time-frequency analysis of signals, where the building blocks are time-frequency shifts of a given generator function along the translation lattice $a\mathbb{Z}$ and modulation lattice $b\mathbb{Z}$ for some fixed $a, b > 0$. The Gabor system generated by a generator function $g \in L^2(\mathbb{R})$ is given by

$$\{e^{2\pi i bmx} g(x - na)\}_{n,m\in\mathbb{Z}}.$$

In the analysis transform using Gabor frames one obtains information about the time-frequency content of a signal.

One of the fundamental problems in Gabor analysis is to determine for which sampling and modulation rates, controlled by the two parameters $a > 0$ and $b > 0$, respectively, the Gabor system generated by a given generator function $g \in L^2(\mathbb{R})$ is a frame. The set of parameter values $(a, b) \in \mathbb{R}^2_+$ for which the associated Gabor system generated by generator function g is a frame is called the *frame set of g*.

This thesis considers Gabor systems, where the generator g is a B-spline. We study B-splines since these generators are widely used in applications as they have several useful properties, e.g., they are piecewise polynomial and have a good time-frequency localisation. However, even though B-splines are among the standard choices of Gabor generators, very little is known about their frame set. Hence, our main goal of this work is to give a detailed study of the frame set of B-splines. We study the known results on the frame set of the B-splines and based on these we develop methods to extend the frame set. We also use numerical methods to investigate the frame set further and thus find new results about the frame set.

Section 2 gives an introduction to frames in general Hilbert spaces. We give the basic definitions of frames and present results about frames and dual frames. Section 3 introduces both the cardinal and symmetric B-splines. We present and prove fundamental properties of the B-splines.

Section 4 introduces the Gabor systems on rectangular lattices $a\mathbb{Z} \times b\mathbb{Z}$. We introduce the Zak transform of functions in $L^2(\mathbb{R})$. This is an important tool in numerical analysis of the Gabor system. We also prove that results about frame properties also hold for translates of the generator g.

Section 5 is the largest and most important section of the thesis. It presents known and new results about Gabor systems with B-spline generators. Subsection 5.1 to 5.3 consider B-splines of any order as generators while we focus on the B-splines of order $N \geq 2$ from Subsection 5.4 and onward. We use numerical methods to get some intuition in Subsection 5.2 and Subsection 5.6. In Subsection 5.3 we use numerical methods to estimate the frame bounds for some Gabor systems. In Subsection 5.7 we prove a new (a, b)-value for which the Gabor system generated by the second B-spline is not a frame, thus disproving a conjecture by Karlheinz Gröchenig.

The thesis ends with a summary of the known and new results about the frame set of B-splines with graphical representation for the second and third B-spline. Subsection 5.9 gives suggestions for further studies based on the work in this thesis.

2 Frames in Hilbert spaces

The purpose of this section is to give an introduction to frames in separable Hilbert spaces. First the basic definitions are given. Then some properties of frames are given. Throughout this section $\mathcal{H} \neq 0$ will denote a separable Hilbert space. Unless otherwise stated the proofs in this section will follow the proofs given in [1, Sections 5.1-5.2].

First we need the definition of a frame.

Definition 2.1 (Frame)**.** *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if constants $A, B > 0$ exist, such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

If the upper frame condition holds, then $\{f_k\}_{k=1}^{\infty}$ is said to be a Bessel sequence with bound B. The constants A, B are called the frame bounds for $\{f_k\}_{k=1}^{\infty}$. The highest possible A and lowest possible B such that (2.1) still holds are called the optimal frame bounds. It can also be the case that the optimal frame bounds coincide. In that case we have a special type of frame known as a tight frame.

Definition 2.2. *A sequence $\{f_k\}_{k=1}^{\infty}$ in $\langle$ is said to be a tight frame for $\mathcal{H}$ if a constant $A > 0$ exists, such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = A\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.2}$$

From the definition of a frame it can easily be seen that a frame satisfies

$$\overline{span}\{f_k\}_{k=1}^{\infty} = \mathcal{H} \tag{2.3}$$

since the frame conditions make sure that if an element $f \in \mathcal{H}$ is orthogonal to every element f_k of a frame, then necessarily $f = 0$. Since $\mathcal{H}$ is complete, this is sufficient to imply (2.3) by [5, Theorem 3.6-2].

The synthesis and analysis operators are two important operators for a frame. The operators work on sequences in $\ell^2(\mathbb{N})$ and elements in $\mathcal{H}$, respectively. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. Then the synthesis operator is defined as

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k, \tag{2.4}$$

and the analysis operator is defined as

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}. \tag{2.5}$$

For any Bessel sequence $\{f_k\}_{k=1}^{\infty}$ with bound B, the synthesis operator T given by (2.4) is well-defined and bounded with $\|T\| \leq \sqrt{B}$ by [1, Theorem 3.1.3]. The upper frame condition directly shows that T^* is well-defined and bounded with $\|T^*\| \leq \sqrt{B}$, since

$$\|T^*f\|^2 = \|\{\langle f, f_k\rangle\}_{k=1}^{\infty}\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \Leftrightarrow \|T^*f\| \leq \sqrt{B}\|f\|.$$

It can be shown [1, Lemma 3.1.1] that T^* is indeed the adjoint operator of T, which justifies the notation.

By composing T and T^* the frame operator is obtained

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = TT^*f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k. \tag{2.6}$$

Some of the important properties of the frame operator are stated in Lemma 2.3. In the proof of Lemma 2.3 we will use a partial ordering on the set of self-adjoint operators on $\mathcal{H}$. The partial ordering is given by

$$U_1 \leq U_2 \Leftrightarrow \langle U_1x, x\rangle \leq \langle U_2x, x\rangle, \quad \forall x \in \mathcal{H}. \tag{2.7}$$

To show that this is in fact a partial ordering we need to show that it is reflexive, antisymmetric and transitive. Let U_1, U_2 and U_3 be self-adjoint operators on $\mathcal{H}$. Then

$$\langle U_1x, x\rangle = \langle U_1x, x\rangle, \quad \forall x \in \mathcal{H},$$

and thus we see that $U_1 \leq U_1$ showing that (2.7) is reflexive. Now assume that $U_1 \leq U_2$ and $U_2 \leq U_1$, then

$$\langle U_1x, x\rangle \leq \langle U_2x, x\rangle, \ \forall x \in \mathcal{H} \text{ and } \langle U_2x, x\rangle \leq \langle U_1x, x\rangle, \ \forall x \in \mathcal{H}.$$

Hence

$$\langle U_1x, x\rangle = \langle U_2x, x\rangle, \quad \forall x \in \mathcal{H}.$$

If $\mathcal{H}$ is assumed to be complex, this gives us

$$\langle U_1x, x\rangle - \langle U_2x, x\rangle = \langle (U_1 - U_2)x, x\rangle = 0, \ \forall x \in \mathcal{H},$$

and thus $U_1 - U_2 = 0$ by [5, Theorem 3.9-3(b)], that is, $U_1 = U_2$. This shows that (2.7) is antisymmetric. Finally, assume that $U_1 \leq U_2$ and $U_2 \leq U_3$, then

$$\langle U_1x, x\rangle \leq \langle U_2x, x\rangle, \ \forall x \in \mathcal{H} \text{ and } \langle U_2x, x\rangle \leq \langle U_3x, x\rangle, \ \forall x \in \mathcal{H}.$$

This shows that

$$\langle U_1x, x\rangle \leq \langle U_3x, x\rangle, \quad \forall x \in \mathcal{H}.$$

Thus $U_1 \leq U_3$ and we have shown that (2.7) is transitive. Since (2.7) satisfies the three properties it is a partial ordering.

Lemma 2.3. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame bounds A, B and frame operator S. Then the following holds*

(i) S is bounded, invertible, self-adjoint and positive.

(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame operator S^{-1} and frame bounds B^{-1}, A^{-1}.

(iii) If A, B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$ then B^{-1}, A^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$.

Proof. (i): S is bounded since it is composed of two bounded operators. Furthermore,

$$\|S\| = \|TT^*\| = \|T\| \; \|T^*\| = \|T\|^2 \leq B.$$

It is easy to show that S is self-adjoint since

$$S^* = (TT^*)^* = T^{**}T^* = TT^* = S.$$

Note that

$$\langle Sf, f\rangle = \langle TT^*f, f\rangle = \langle T^*f, T^*f\rangle = \|T^*f\|^2 = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2. \tag{2.8}$$

It can be seen from (2.8) that the frame condition (2.1) can be written as

$$A\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.9}$$

This way of writing the frame condition immediately shows that S is positive due to the lower bound. Using the partial ordering given in (2.7) we can rewrite(2.9) as

$$AI \leq S \leq BI \tag{2.10}$$

We can rearrange (2.10) to get $-BI \leq -S \leq -AI$, that is, $0 \leq I - B^{-1}S \leq \frac{B-A}{B}I$. From this it follows that

$$\begin{aligned}\|I - B^{-1}S\| &= \sup_{\|f\|=1} \|(I - B^{-1}S)f\| \\ &= \sup_{\|f\|=1} |\langle (I - B^{-1}S)f, f\rangle| \leq \frac{B-A}{B} < 1.\end{aligned}$$

Hence, by using Neumann series [5, Theorem 7.3-1] it is seen that $B^{-1}S$ is invertible and thus S itself is invertible as B is simply a constant.

(ii): We start by noting that since S is self-adjoint, then S^{-1} is also self-adjoint. Using this we show that

$$\sum_{k=1}^{\infty} |\langle f, S^{-1}f_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle S^{-1}f, f_k\rangle|^2 \leq B\|S^{-1}f\|^2 \forall f \in \mathcal{H}.$$

Here the equality comes from S^{-1} being self-adjoint and the inequality comes from the fact that $\{f_k\}_{k=1}^{\infty}$ is a frame. Furthermore, since S^{-1} is bounded, we have $\|S^{-1}f\| \leq \|S^{-1}\| \, \|f\|$, so

$$\sum_{k=1}^{\infty} |\langle f, S^{-1}f_k\rangle|^2 \leq B\|S^{-1}\|^2 \, \|f\|^2.$$

Hence, $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a Bessel sequence. Therefore the frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is well-defined and by definition its action on an element $f \in \mathcal{H}$ is

$$\begin{aligned} f \mapsto \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle S^{-1}f_k = S^{-1}\left[\sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k\right] &= S^{-1}\left[S\left(S^{-1}f\right)\right] \\ &= S^{-1}f. \end{aligned} \tag{2.11}$$

The first equality in (2.11) follows from the fact that S^{-1} is bounded and thus continuous and the second equality uses the definition of the frame operator. Thus we have shown that S^{-1} is the frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The operator S^{-1} commutes with both S and I and they are all self-adjoint. This means that we can use [1, Theorem 2.4.2] and multiply the inequalities (2.10) with S^{-1} whereby we can obtain

$$B^{-1}I \leq S^{-1} \leq A^{-1}I,$$

which is the same as

$$B^{-1}\|f\|^2 \leq \langle S^{-1}f, f\rangle \leq A^{-1}\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Using (2.11) this means that

$$B^{-1}\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle S^{-1}f, f_k\rangle|^2 \leq A^{-1}\|f\|^2, \quad \forall f \in \mathcal{H},$$

which shows that $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds B^{-1} and A^{-1}.

(iii): We aim to prove this property by contradiction. Let B be the optimal upper bound for $\{f_k\}_{k=1}^{\infty}$ and assume that the optimal lower bound for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is $C > B^{-1}$. By using the result from *(ii)* we see that $\{S^{-1}\left(S^{-1}f_k\right)\}_{k=1}^{\infty} = \{f_k\}_{k=1}^{\infty}$ has upper bound $C^{-1} < B$, but this contradicts the initial assumption. Thus, B^{-1} is the optimal lower bound for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The result for the optimal upper bound A^{-1} can be proved similarly as shown in [1]. □

The new frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ described in Lemma 2.3 is called the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$. In general, two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames if $f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k$ holds for all $f \in \mathcal{H}$. It can be shown that dual frames are indeed frames. We will use this later in Section 5 to prove that certain functions generate frames by showing that they have a dual.

For the Gabor frames that will be introduced in Section 4 the canonical dual will also have the Gabor structure. However, this is not the case for any structure. As an example the canonical dual of a wavelet frame will not necessarily have the wavelet structure, though it can be guaranteed if the frame is tight.

The result given in Theorem 2.4 is very important for frames. It shows that any elements in a Hilbert space $\mathcal{H}$ can be represented using a frame $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$. The theorem shows that frames can be seen as a kind of generalisation of a basis. The frame differs from an orthonormal basis in the sense that the representation is not necessarily unique.

Theorem 2.4. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame for $\mathcal{H}$ with operator S. Then*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}, \tag{2.12}$$

and

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}. \tag{2.13}$$

Proof. Let f be in element in $\mathcal{H}$. Using the fact that $f = SS^{-1}f$ and the fact that S^{-1} is self-adjoint we get

$$f = S\left(S^{-1}f\right) = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H}.$$

Similarly, using the fact that $f = S^{-1}Sf$ and the fact that S is self-adjoint we get

$$f = S^{-1}\left(Sf\right) = \sum_{k=1}^{\infty} \langle Sf, S^{-1} f_k \rangle S^{-1} f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \quad \forall f \in \mathcal{H}.$$

□

Theorem 2.4 shows that an element in $\mathcal{H}$ can be represented entirely by the coefficients $\{\langle f, S^{-1} f_k \rangle\}_{k=1}^{\infty}$, which are known as the frame coefficients. However, to find these coefficients we would need to be able to calculate the effect of S^{-1}. In general this can be a difficult task, but in special cases simple results exist. One such example is tight frames.

Corollary 2.5. *If $\{f_k\}_{k=1}^{\infty}$ is a tight frame with frame bound $A > 0$, then the dual frame is $\{A^{-1} f_k\}_{k=1}^{\infty}$ and*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k = \frac{1}{A} \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}. \tag{2.14}$$

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a tight frame with frame bound A. Then

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = A\|f\|, \quad \forall f \in \mathcal{H}.$$

It has previously been shown that this is the same as

$$\langle Sf, f \rangle = A\|f\| = \langle Af, f \rangle \Leftrightarrow \langle (S - AI)\, f, f \rangle = 0, \quad \forall f \in \mathcal{H}.$$

By [1, Lemma 2.4.3] this implies $S - AI = 0$, that is, $S = AI$, which shows that $S^{-1} = A^{-1}I$. Finally the result follows from applying S^{-1} to (2.12). □

It has already been mentioned that a frame can be seen as generalisation of bases. Now we look at some relationships between Riesz bases and frames. A Riesz basis for a Hilbert space $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$ where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis and $U : \mathcal{H} \to \mathcal{H}$ us a bounded bijective operator.

Theorem 2.6. *Let $\{f_k\}_{k=1}^{\infty}$ be a Riesz basis for $\mathcal{H}$, then it is also a frame for $\mathcal{H}$ and the Riesz basis bounds coincide with the frame bounds. The unique dual Riesz basis of $\{f_k\}_{k=1}^{\infty}$ equals the canonical dual frame $\{S^{-1}f_k\}_{k=1}^{\infty}$.*

The proof for Theorem 2.6 follows from the properties of a Riesz basis and can be found in [1, Theorem 5.2.1].

Theorem 2.6 shows that a Riesz basis will always be a frame, but one may wonder what conditions are needed in order for a frame to be a Riesz basis. Theorem 2.7 gives a sufficient condition for a frame to also be a basis.

Theorem 2.7. *If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then the following are equivalent.*

(i) $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for* $\mathcal{H}$.

(ii) *If* $\sum_{k=1}^{\infty} c_k f_k = 0$ *for some* $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, *then* $c_k = 0, \quad \forall k \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$: Assume that $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ and that there exists a sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ such that $\sum_{k=1}^{\infty} c_k f_k = 0$. We know that a Riesz basis can be related to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ by some bounded, bijective operator U on $\mathcal{H}$. This can be written as $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ and from this it follows that

$$0 = \sum_{k=1}^{\infty} c_k f_k = \sum_{k=1}^{\infty} c_k U e_k = U \sum_{k=1}^{\infty} c_k e_k.$$

Since U is bijective this implies that $\sum_{k=1}^{\infty} c_k e_k = 0$ and since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis this shows that $c_k = 0, \ \forall k \in \mathbb{N}$.
$(ii) \Rightarrow (i)$: Assume (ii) is true. Then the synthesis operator T associated with $\{f_k\}_{k=1}^{\infty}$ will be well-defined, bounded and injective. Furthermore, since $\{f_k\}_{k=1}^{\infty}$ is a frame T is also surjective. Now, let $\{\delta_k\}_{k=1}^{\infty}$ denote the canonical orthonormal basis for $\ell^2(\mathbb{N})$. Now we can relate $\{f_k\}_{k=1}^{\infty}$ to $\{\delta_k\}_{k=1}^{\infty}$ using T as $T\delta_k = f_k$. Then the result follows directly from the definition of a Riesz basis. □

A frame that is not a Riesz basis is called an overcomplete frame and the reason can be seen from Theorem 2.7. Indeed, if $\{f_k\}_{k=1}^{\infty}$ is a frame, but not a basis, it follows from Theorem 2.7 that there exists a sequence of coefficients $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \setminus \{0\}$ such that

$$f = \sum_{k=1}^{\infty} c_k f_k = 0.$$

Using (2.12) this shows that any element $f \in \mathcal{H}$ has several representations in terms of the frame $\{f_k\}_{k=1}^{\infty}$. Indeed, by using such coefficients $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}) \setminus \{0\}$ we get

$$f=\sum_{k=1}^{\infty}\langle f,S^{-1}f_k\rangle f_k=\sum_{k=1}^{\infty}\left(\langle f,S^{-1}f_k\rangle+c_k\right)f_k=0.$$

This result shows that there are other ways of representing an element $f\in\mathcal{H}$. It does not guarantee that the sequence of coefficients $\{\langle f,S^{-1}f_k\rangle+c_k\}_{k=1}^{\infty}$ come from a frame as in (2.12). However, it is possible to prove that every overcomplete frame has other dual frames than the canonical dual.

Theorem 2.8. *Assume that $\{f_k\}_{k=1}^{\infty}$ is an overcomplete frame. Then there exists frames $\{g_k\}_{k=1}^{\infty}\neq\{S^{-1}f_k\}_{k=1}^{\infty}$ for which*

$$f=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k,\quad \forall f\in\mathcal{H}. \tag{2.15}$$

Proof. The proof is split in two cases. First we assume that $f_\ell=0$ for some $\ell\in\mathbb{N}$ for this index we have $S^{-1}f_\ell=0$. Let $g_k=S^{-1}f_k$ for all $k\neq\ell$ and choose g_ℓ to be any non-zero element of $\mathcal{H}$. Then the frame decomposition (2.12) shows that (2.15) is satisfied, since the terms $\langle f,g_\ell\rangle f_\ell=\langle f,S^{-1}f_\ell\rangle f_\ell=0$ and $\langle f,g_k\rangle f_k=\langle f,S^{-1}f_k\rangle f_k,\ \forall k\neq\ell$. By construction $\{g_k\}_{k=1}^{\infty}\neq\{S^{-1}f_k\}_{k=1}^{\infty}$.

In the other case we assume $f_\ell\neq 0$ for all $k\in\mathbb{N}$. Since $\{f_k\}_{k=1}^{\infty}$ is an overcomplete frame it follows from Theorem 2.6 that there exists a sequence $\{c_k\}_{k=1}^{\infty}\in\ell^2(\mathbb{N})\setminus\{0\}$ such that

$$\sum_{k=1}^{\infty}c_kf_k=0.$$

For some $\ell\in\mathbb{N}$ we have $c_\ell\neq 0$, and we can take the corresponding term out of the sum to get

$$f_\ell=\frac{-1}{c_\ell}\sum_{k\neq\ell}c_kf_k.$$

We want to use this to show that $\{f_k\}_{k\neq\ell}$ is a frame for $\mathcal{H}$. It is enough to prove that $\{f_k\}_{k\neq\ell}$ has a lower frame bound since the upper frame bound for $\{f_k\}_{k=1}^{\infty}$ will still hold for $\{f_k\}_{k\neq\ell}$. Note that for any $f\in\mathcal{H}$ the Cauchy-Schwarz inequality shows that

$$\begin{aligned}|\langle f,f_\ell\rangle|^2=\left|\langle f,\frac{-1}{c_\ell}\sum_{k\neq\ell}c_kf_k\rangle\right|^2&=\left|\frac{-1}{c_\ell}\sum_{k\neq\ell}c_k\langle f,f_k\rangle\right|^2\\&\leq\frac{1}{|c_\ell|^2}\sum_{k\neq\ell}|c_k|^2\sum_{k\neq\ell}|\langle f,f_k\rangle|^2.\end{aligned}$$

Let $C=\frac{1}{|c_\ell|^2}\sum_{k\neq\ell}|c_k|^2$. Then letting A denote the lower frame bound for $\{f_k\}_{k=1}^{\infty}$, this implies that

$$A\|f\|^2\leq\sum_{k=1}^{\infty}|\langle f,f_k\rangle|^2=\sum_{k\neq\ell}|\langle f,f_k\rangle|^2+|\langle f,f_\ell\rangle|^2$$

$$\leq (1+C)\sum_{k\neq \ell} |\langle f, f_k\rangle|^2.$$

Hence, $\{f_k\}_{k\neq\ell}$ satisfies the lower frame condition with frame bound $\frac{A}{1+C}$.

Let $\{g_k\}_{k\neq\ell}$ denote the canonical dual frame of $\{f_k\}_{k\neq\ell}$ and let $g_\ell = 0$. Then (2.15) holds for $\{g_k\}_{k=1}^{\infty}$, but it is different from the canonical dual for $\{f_k\}_{k=1}^{\infty}$, since $S^{-1}f_\ell \neq 0 \Leftrightarrow S^{-1}f_\ell \neq g_\ell$. □

As the inverse of frame operator can be difficult to calculate it is not always easy to find the expansions in (2.12) and (2.13). Therefore it is interesting to know that other dual frames exist, but it is not certain when the different duals should be used. Different cases will be discussed specifically for Gabor frames in a later section.

3 B-Splines

Now we focus on a specific class of functions in $L^2(\mathbb{R})$ namely the cardinal B-splines and the symmetric B-splines. To define the cardinal B-splines $N_n(x)$ we start by defining the first function $N_1(x)$ as

$$N_1(x) := \chi_{[0,1]}(x). \tag{3.1}$$

Then the cardinal B-splines of higher order are defined recursively using convolution as

$$N_{n+1}(x) := N_n * N_1(x) = \int_{-\infty}^{\infty} N_n(x-t)N_1(t)dt = \int_0^1 N_n(x-t)dt. \tag{3.2}$$

We calculate the expression for the function $N_2(x)$ since it will be useful for examples:

$$N_2(x) = \int_0^1 N_1(x-t)dt = \int_0^1 \chi_{[0,1]}(x-t)dt = \begin{cases} x, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & otherwise \end{cases} \tag{3.3}$$

A plot of $N_2(x)$ is given in Figure 1.

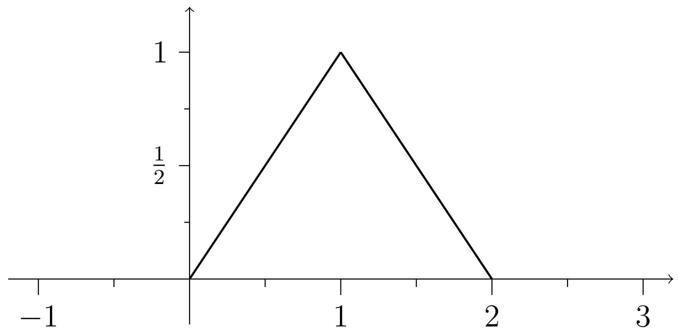


Figure 1: The second B-spline $N_2(x)$.

Some basic and important properties of the B-splines are given in Theorem 3.1.

Theorem 3.1. *Given $n \in \mathbb{N}$, the B-spline N_n has the following properties*

(i) *supp* $N_n = [0, n]$ *and* $N_n > 0$ *on* $]0, n[$,

(ii) $\int_{-\infty}^{\infty} N_n(x)dx = 1$,

(iii) *For* $n \ge 2$,

$$\sum_{k\in\mathbb{Z}} N_n(x-k) = 1, \quad \forall x \in \mathbb{R}. \tag{3.4}$$

For $n = 1$ *the formula (3.4) holds for all* $x \in \mathbb{R} \setminus \mathbb{Z}$.

(iv) *For any continuous function* $f : \mathbb{R} \to \mathbb{C}$,

$$\int_{-\infty}^{\infty} N_n(x)f(x)dx = \int_{[0,1]^n} f(x_1 + \cdots + x_n)dx_1 \cdots dx_n \tag{3.5}$$

Proof. The proofs for these properties all rely on induction and the formula (3.2).
(i): It is trivial to see that (i) holds for $N_1(x) = \chi_{[0,1]}$. Now assume that (i) holds for $N_n(x)$ for some $n \in \mathbb{N}$ and consider $N_{n+1}(x)$. For $t \in [0,1]$ the function $N_n(x-t)$ can only be non-zero for $x \in]0, n+1[$, since $N_n > 0$ on $]0, n[$. Thus, by (3.2) *supp* $N_{n+1} \subseteq [0, n+1]$. On the other hand if $x \in]0, n+1[$, then there exists a $t \in [0,1]$ such that $x - t \in [0, n]$ and thus by the induction hypothesis $N_n(x-t) > 0$. By using (3.2) this shows that $N_{n+1} > 0$ and it also shows that *supp* $N_{n+1} = [0, n+1]$.

(ii): For $n = 1$ we have

$$\int_{-\infty}^{\infty} N_1(x)dx = \int_{-\infty}^{\infty} \chi_{[0,1]}(x)dx = \int_0^1 1dx = 1.$$

Now assume that (ii) holds for $N_n(x)$ for some $n \in \mathbb{N}$ and consider $N_{n+1}(x)$.

$$\begin{aligned}\int_{-\infty}^{\infty} N_{n+1}(x)dx &= \int_{-\infty}^{\infty}\int_0^1 N_n(x-t)dtdx \\ &= \int_0^1\int_{-\infty}^{\infty} N_n(x-t)dxdt \\ &= \int_0^1\int_{-\infty}^{\infty} N_n(y)dydt = \int_0^1 1dt = 1.\end{aligned}$$

The first equality comes from (3.2). The second follows from Tonelli's Theorem for non-negative functions. The third comes from using the substitution $y = x - t$ on the inner integral. The fourth comes by using the induction hypothesis.

(iii): For $n = 2$ we can see that there will only be two non-zero terms in the sum (3.4), since *supp* $N_2 = [0, 2]$ and the translation parameter k is an integer. The two non-zero terms arise when $x - k = x - \lfloor x \rfloor \in [0, 1]$ and $x - k = x - \lfloor x \rfloor + 1 \in [1, 2[$. Using (3.3) this shows that

$$\sum_{k \in \mathbb{Z}} N_2(x-k) = (x - \lfloor x \rfloor) + 2 - (x - \lfloor x \rfloor + 1) = 1.$$

Hence (iii) holds for $n = 2$. Now assume that (iii) holds for $N_n(x)$ for some $n \in \mathbb{N}\backslash\{1\}$ and consider $N_{n+1}(x)$.

$$\sum_{k \in \mathbb{Z}} N_{n+1}(x-k) = \sum_{k \in \mathbb{Z}} \int_0^1 N_n(x-k-t)dt$$

Using the substitution $y = x - t$ we get

$$\begin{aligned}\sum_{k \in \mathbb{Z}} N_{n+1}(x-k) &= \sum_{k \in \mathbb{Z}} \left(-\int_x^{x-1} N_n(y-k)\right) dy \\ &= \sum_{k \in \mathbb{Z}} \left(\int_{x-1}^{x} N_n(y-k)\right) dy\end{aligned}$$

$$= \int_{x-1}^{x} \left(\sum_{k\in\mathbb{Z}} N_n(y-k) \right) dy.$$

Finally, using the induction hypothesis, we get

$$\sum_{k\in\mathbb{Z}} N_{n+1}(x-k) = \int_{x-1}^{x} \left(\sum_{k\in\mathbb{Z}} N_n(y-k) \right) dy = \int_{x-1}^{x} 1dy = 1.$$

For $n = 1$ it is clear that in cases where $x \in \mathbb{R} \setminus \mathbb{Z}$, the sum in (3.4) will only have one term which will be equal to one and thus (3.4) holds. However, in the case $x \in \mathbb{Z}$ the sum in (3.4) will have two terms which will both be equal to one and thus

$$\sum_{k\in\mathbb{Z}} N_1(x-k) = 2, \quad \forall x \in \mathbb{Z}.$$

(iv): Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function. First we check that (3.5) holds for $n = 1$. For $n = 1$ we have

$$\int_{-\infty}^{\infty} N_1(x) f(x) dx = \int_{-\infty}^{\infty} \chi_{[0,1]}(x) f(x) dx = \int_{[0,1]} f(x) dx.$$

Thus (3.5) holds for $n = 1$.
Now assume that (iv) holds for $N_n(x)$ for some $n \in \mathbb{N}$ and consider $N_{n+1}(x)$. Then

$$\int_{-\infty}^{\infty} N_{n+1}(x) f(x) dx = \int_{-\infty}^{\infty} \int_{0}^{1} N_n(x-t) dt f(x) dx.$$

Using a substitution by $y = x - t$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{0}^{1} N_n(x-t) dt f(x) dx &= \int_{-\infty}^{\infty} \int_{0}^{1} N_n(y) f(y+t) dt dy \\ &= \int_{-\infty}^{\infty} N_n(y) \int_{0}^{1} f(y+t) dt dy. \end{aligned}$$

The result of the inner integral can be written as some function of y, say $F(y) = \int_0^1 f(y+t)dt$. Then by using the induction hypothesis, we get

$$\begin{aligned} \int_{-\infty}^{\infty} N_n(y) \int_{0}^{1} f(y+t) dt dy &= \int_{-\infty}^{\infty} N_n(y) F(y) dy \\ &= \int_{[0,1]^n} F(x_1 + \cdots x_n) dx_1 \cdots dx_n. \end{aligned}$$

Finally, using the definition of $F(y)$, we get

$$\begin{aligned} \int_{[0,1]^n} F(x_1 + \cdots x_n) dx_1 \cdots dx_n &= \int_{[0,1]^n} \int_{[0,1]} f(x_1 + \cdots x_n + t) dt dx_1 \cdots dx_n \\ &= \int_{[0,1]^{n+1}} f(x_1 + \cdots x_{n+1}) dx_1 \cdots dx_{n+1}. \end{aligned}$$

$\square$

It will later be seen that the property (3.4) is important in relation to Gabor frames. The formula (3.5) can be used to prove the following result about the Fourier transform of the B-splines.

Corollary 3.2. *For $n \in \mathbb{N}$, the Fourier transform of the B-spline N_n is given by*

$$\widehat{N_n}(\gamma) = \left(\frac{1 - e^{-2\pi i \gamma}}{2\pi i \gamma} \right)^n \tag{3.6}$$

Proof. Using the definition of the Fourier transform we have

$$\begin{aligned}
\widehat{N_n}(\gamma) = \int_{-\infty}^{\infty} N_n(x) e^{-2\pi i x \gamma} dx &= \int_{[0,1]^n} e^{-2\pi i (x_1 + \cdots + x_n)\gamma} dx_1 \cdots dx_n \\
&= \int_{[0,1]^n} e^{-2\pi i x_1 \gamma} \cdots e^{-2\pi i x_n \gamma} dx_1 \cdots dx_n \\
&= \int_{[0,1]} e^{-2\pi i x_1 \gamma} dx_1 \cdots \int_{[0,1]} e^{-2\pi i x_n \gamma} dx_n \\
&= \left(\int_{[0,1]} e^{-2\pi i x \gamma} dx \right)^n = \left(\left[-\frac{e^{-2\pi i x \gamma}}{2\pi i \gamma} \right]_0^1 \right)^n \\
&= \left(\frac{1 - e^{-2\pi i \gamma}}{2\pi i \gamma} \right)^n
\end{aligned}$$

□

The B-splines discussed so far have support on the positive part of the x-axis. However, there is also a symmetric version of the B-splines. For $n \in \mathbb{N}$ define

$$B_n(x) := T_{-\frac{n}{2}} N_n(x) = N_n \left(x + \frac{n}{2} \right). \tag{3.7}$$

The functions B_n are called symmetric B-splines since they are supported on an interval that is symmetric around zero. In the same way that the cardinal B-splines N_n are defined by (3.1) and (3.2), the symmetric B-splines B_n can be defined recursively by

$$B_1(x) := \chi_{[-1/2,1/2]}(x), \quad B_{n+1}(x) := B_n * B_1(x), n \in \mathbb{N}. \tag{3.8}$$

Thus, we get the result

$$B_{n+1}(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} B_n(x - t).$$

Since the symmetric B-splines $B_n(x)$ are simply translations of the B-splines $N_n(x)$, several properties of $B_n(x)$ are direct consequences of the results for $N_n(x)$. Some properties for $B_n(x)$ are given in Corollary 3.9.

Corollary 3.3. *For $n \in \mathbb{N}$, the symmetric B-spline B_n has the following properties:*

(i) For $n \geq 2$,

$$\sum_{k \in \mathbb{Z}} B_n(x-k) = 1, \quad \forall x \in \mathbb{R}. \tag{3.9}$$

For $n = 1$ the formula holds for $x \in \mathbb{R}$, except those that can be written in the form $x = m + \frac{1}{2},\ m \in \mathbb{Z}$.

(ii)

$$\widehat{B_n}(\gamma) = \left(\frac{e^{\pi i \gamma} - e^{-\pi i \gamma}}{2\pi i \gamma}\right)^n = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^n \tag{3.10}$$

Proof. (i): It is trivial that (i) still holds for $B_n(x)$ and $n \geq 2$. The translation of $N_n(x)$ is equivalent to using (iii) from (3.1) with $x + \frac{n}{2}$ and thus the result is still correct since it holds for all $x \in \mathbb{R}$. Similarly, for $n = 1$ the result also still holds for a.e. $x \in \mathbb{R}$. However, the exceptions have also been translated so the equality does not hold for any $x = m + \frac{1}{2},\ m \in \mathbb{Z}$.

(ii): Here we use the fact that $\widehat{T_a f}(\gamma) = E_{-a}\hat{f}(\gamma)$. This gives

$$\begin{aligned}
\widehat{B_n}(\gamma) = \widehat{T_{-\frac{n}{2}} N_n(x)}(\gamma) &= e^{-2\pi i\left(-\frac{n}{2}\right)\gamma}\widehat{N_n}(\gamma) \\
&= e^{n\pi i\gamma}\left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^n \\
&= \left(e^{\pi i\gamma}\left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)\right)^n \\
&= \left(\frac{e^{\pi i\gamma} - e^{-\pi i\gamma}}{2\pi i\gamma}\right)^n
\end{aligned}$$

□

4 Gabor Frames

In Section 2 we considered general frames in abstract Hilbert spaces. We now go into the more specific case of Gabor systems in the space $L^2(\mathbb{R})$. Some results here are stated without proof, they are from [1, Chapter 9].

First, let us recall the translation and modulation operators on $L^2(\mathbb{R})$.

$$\begin{aligned}
&\text{Translation by } a \in \mathbb{R}, \quad T_a : \ L^2(\mathbb{R}) \to L^2(\mathbb{R}), \ (T_a f)(x) = f(x-a),\\
&\text{Modulation by } b \in \mathbb{R}, \quad E_b : \ L^2(\mathbb{R}) \to L^2(\mathbb{R}), \ (E_b f)(x) = e^{2\pi i b x} f(x).
\end{aligned}$$

Gabor systems are built using the translation and modulation operators. One can either look at an integral representation where all possible translations and modulations are considered or restrict the operations to a lattice in phase space. Here the focus will be on the latter case, in particular, we look at systems with translations and modulations on a rectangular lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$. We now give the definition of a Gabor system.

Definition 4.1. *Let E_b and T_a be the modulation and translation operators on $L^2(\mathbb{R})$ and let g be some function in $L^2(\mathbb{R})$. Then for $a, b > 0$ the collection of functions*

$$\mathcal{G}(g, a, b) := \{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}} \tag{4.1}$$

is called a Gabor system.

The function $g \in L^2(\mathbb{R})$ that is used to generate the Gabor system is called the generator function or the window function. Using Definition 4.1 a Gabor frame for $L^2(\mathbb{R})$ is a frame of the form (4.1) where $a, b > 0$ are given parameters and $g \in L^2(\mathbb{R})$ is a given function.

One of the fundamental problems within Gabor analysis is for which parameters $(a, b) \in \mathbb{R}^2_+$ the Gabor system $\mathcal{G}(g, a, b)$ is a frame. For a Gabor system $\mathcal{G}(g, a, b)$, the frame set $\mathcal{F}(g)$ is exactly the set of parameters (a, b) that make the system a frame. So to state it formally, the frame set for a function g is

$$\mathcal{F}(g) = \{(a, b) \in \mathbb{R}^2_+ \mid \mathcal{G}(g, a, b) \text{ is a frame}\}.$$

Frame sets can be different depending on the generator function g. In Section 5 we will look at the frame set for Gabor systems with B-spline generators. However, in this section we will focus on results that hold for Gabor systems with an arbitrary generator $g \in L^2(\mathbb{R})$.

Theorem 4.2 gives some results about the frame set that hold for any $g \in L^2(\mathbb{R})$. Other results in this section will help us determine whether a Gabor system is a frame for a given set of parameters.

Theorem 4.2. *Let g be a function in $L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following holds:*

(i) If $ab > 1$, then $\mathcal{G}(g, a, b)$ cannot be a frame for $L^2(\mathbb{R})$.

(ii) If $\mathcal{G}(g, a, b)$ is a frame then

$$ab = 1 \Leftrightarrow \mathcal{G}(g, a, b) \text{ is a Riesz basis.} \tag{4.2}$$

By Theorem 4.2 we can focus the investigation of the frame set $\mathcal{F}(g)$ of any generator $g \in L^2(\mathbb{R})$ on values of a and b where $ab \leq 1$. This area is represented in Figure 2. If we are on the hyperbola $ab = 1$, then the frame property implies that the Gabor system is also a Riesz basis.

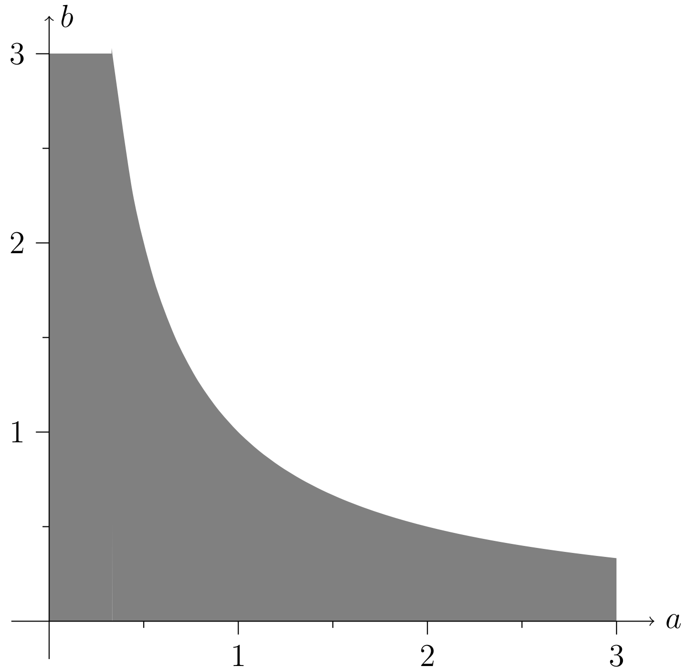


Figure 2: The gray area represents the possible frame set for a function $g \in L^2(\mathbb{R})$.

Now we will look at various methods to determine whether the Gabor system $\mathcal{G}(g, a, b)$ is a frame for a given generator function $g \in L^2(\mathbb{R})$ and given parameters $(a, b) \in \mathbb{R}^2_+$. The Zak transform which is given in Definition 4.3 is one tool we can use to determine this.

Definition 4.3. *Let f be a function in $L^2(\mathbb{R})$. Then the Zak transform $Z_\lambda f$ of f is a function of two real variables, defined as*

$$(Z_\lambda f)(t, \nu) = \sqrt{\lambda} \sum_{k \in \mathbb{Z}} f(\lambda(t-k))\, e^{2\pi i k \nu}, \quad t, \nu \in \mathbb{R}. \tag{4.3}$$

For $f \in W(\mathbb{R})$, the Zak transform is defined pointwise and is bounded on $\mathbb{R}^2$. Here $W(\mathbb{R})$ denotes the Wiener space defined by:

$$W(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) \mid \|f\|_W < \infty\},$$

where

$$\|f\|_W = \sum_{k \in \mathbb{N}} \operatorname*{ess\,sup}_{x \in [0,1]} |N_n(x+k)|.$$

If $f \in W(\mathbb{R}) \cup C^0(\mathbb{R})$, then, by [3, Lemma 8.2.1(c)], the Zak transform $Z_\lambda f$ is also continuous. For general functions in $L^2(\mathbb{R})$ we have to carefully consider how the definition of the Zak transform is interpreted. It can be shown [1, Lemma 9.7.1], that the series defining $Z_\lambda f$ converges in $L^2([0, 1[^2)$ for all $f \in L^2(\mathbb{R})$.

From the definition of the Zak transform we can easily see that is 1-periodic in ν since

$$\begin{aligned}(Z_\lambda f)(t,\nu+1) &= \sqrt{\lambda}\sum_{k\in\mathbb{Z}} f(\lambda(t-k))\, e^{2\pi ik(\nu+1)} \\ &= \sqrt{\lambda}\sum_{k\in\mathbb{Z}} f(\lambda(t-k))\, e^{i(2\pi k\nu+2\pi k)} \\ &= \sqrt{\lambda}\sum_{k\in\mathbb{Z}} f(\lambda(t-k))\, e^{2\pi ik\nu} \\ &= (Z_\lambda f)(t,\nu).\end{aligned}$$

The final equality comes from the 2π-periodicity of the complex exponential. Furthermore, the Zak tranform is quasi-periodic in t since

$$\begin{aligned}(Z_\lambda f)(t+1,\nu) &= \sqrt{\lambda}\sum_{k\in\mathbb{Z}} f(\lambda(t+1-k))\, e^{2\pi ik\nu} \\ &= \sqrt{\lambda}\sum_{\ell\in\mathbb{Z}} f(\lambda(t-\ell))\, e^{2\pi i(\ell+1)\nu} \\ &= \sqrt{\lambda}\sum_{\ell\in\mathbb{Z}} f(\lambda(t-\ell))\, e^{2\pi i\ell\nu} e^{2\pi i\nu} \\ &= e^{2\pi i\nu}(Z_\lambda f)(t,\nu).\end{aligned}$$

We recall that the absolute value of the complex exponential with a purely imaginary exponent is 1. Hence, if we are only interested in the absolute value of the Zak transform then it will be sufficient to consider the Zak transform for $(t,\nu)\in[0,1]\times[0,1]$. In any case it will be easy to compute the Zak transform for any point $(t,\nu)\in\mathbb{R}^2_+$ when you already know its values for $(t,\nu)\in[0,1]\times[0,1]$.

In Proposition 4.4 we look at some special properties when we look at parameters a,b such that $ab=1$. This type of sampling is called *critical sampling*. The critical sampling gives the opportunity of getting Riesz bases.

Proposition 4.4. *Let $g\in L^2(\mathbb{R})$ and $a,b>0$ with $ab=1$ be given. Then the following holds:*

(i) $\mathcal{G}(g,a,b)$ is complete in $L^2(\mathbb{R})$ if and only if $Z_a g\neq 0$, a.e.

(ii) $\mathcal{G}(g,a,b)$ is a Bessel sequence with bound B if and only if $|Z_a g|^2\leq B$, a.e.

(iii) $\mathcal{G}(g,a,b)$ is a frame for $L^2(\mathbb{R})$ with bounds A,B if and only if $A\leq|Z_a g|^2\leq B$, a.e.

(iv) $\mathcal{G}(g,a,b)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|Z_a g|^2=1$, a.e.

As we are interested in the frame property for Gabor systems, we will mostly be using the characterization in (iii). Notice that since we assume $ab=1$ in Proposition 4.4, it follows from (ii) in Theorem 4.2 that being a frame is equivalent to being a Riesz basis.

The Zak transform is not only useful in the case $ab = 1$. For this alternative application we need to assume that the Gabor system $\mathcal{G}(g, a, b)$ is rationally oversampled, which means that

$$ab \in \mathbb{Q}, \quad ab = \frac{p}{q} \quad \gcd(p, q) = 1.$$

Since we have to have $ab \leq 1$ we know that we must have $1 \leq p \leq q$.

For a rationally oversampled Gabor system, the so-called Zibulski-Zeevi matrix is a $p \times q$ matrix defined by

$$\Phi^g(t, \nu) = p^{-\frac{1}{2}} \left((Z_{\frac{1}{b}} g)(t - \ell \frac{p}{q}, \nu + \frac{k}{p}) \right)_{k=0,\dots,p-1;\ell=0,\dots,q-1}, \ a.e.\ t, \nu \in \mathbb{R}.$$

In the Zebulski-Zeevi matrix we have turned the infinite dimensional Gabor system $\mathcal{G}(g, a, b)$ into a finite dimensional vector system. We can use this to determine the frame properties of the infinite dimensional system.

Theorem 4.5. *Let $\mathcal{G}(g, a, b)$ be a rationally oversampled Gabor system and let $A, B > 0$ be given. Then $\mathcal{G}(g, a, b)$ is a Gabor frame if and only if*

$$AI \leq \Phi^g(t, \nu) \left(\Phi^g(t, \nu)\right)^* \leq BI, \ a.e.\ (t, \nu) \in [0, 1]^2. \tag{4.4}$$

The finite system will be of dimension p and we wish to determine if the q columns of the Zibulski-Zeevi matrix constitute a frame for $\mathbb{C}^p$. We can determine this by checking whether (4.4) holds for some $A, B > 0$. If we assume that the singular values are given as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ then this is equivalent to verifying that $\sigma_p \geq \sqrt{A}$ and $\sigma_1 \leq \sqrt{B}$ for *a.e.* $(t, \nu) \in [0, 1]^2$.

Theorem 4.5 looks simple, but it is not always easy to determine the singular values for almost every pair $(t, \nu) \in [0, 1] \times [0, \frac{1}{p}]$. In particular, we will use this method in Subsection 5.6 to investigate the frame properties of a Gabor system numerically by checking (4.4) on a grid of (t, ν) values in $[0, 1]^2$.

The next result shows that the frame set is invariant under translation of the generator function.

Lemma 4.6. *Let $r \in \mathbb{R}$ and let $a, b > 0$, $g \in L^2(\mathbb{R})$, $A, B > 0$. Then*

$$\mathcal{G}(g, a, b) \textit{ is a frame with frame bounds } A, B$$

if and only if

$$\mathcal{G}(T_r g, a, b) \textit{ is a frame with frame bounds } A, B.$$

Proof. Assume that $\mathcal{G}(g, a, b)$ is a frame. Then there exists $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n,m \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 \leq B\|f\|^2$$

for all $f \in L^2(\mathbb{R})$. Consider the frame inequalities for $T_r f$. Then, as $\|T_r f\| = \|f\|$, we have:

$$A\|f\|^2 \leq \sum_{n,m \in \mathbb{Z}} |\langle T_{-r} f, E_{mb} T_{na} g \rangle|^2 \leq B\|f\|^2.$$

Since $(T_{-r})^* = T_r$, we can move the translation operator in the inner product by changing the sign in front of r, and we get

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, T_r E_{mb} T_{na} g\rangle|^2 \leq B\|f\|^2.$$

Then we apply the relation $T_r E_{mb} = e^{-2\pi imbr} E_{mb} T_r$ along with the fact that two translation operators commute with each other. That way we get

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, e^{-2\pi imbr} E_{mb} T_{na}(T_r g)\rangle|^2 \leq B\|f\|^2.$$

Since $e^{-2\pi imbr}$ is a constant, we can take it out of the inner product. Thus we obtain

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |e^{-2\pi imbr}|^2 |\langle f, E_{mb} T_{na}(T_r g)\rangle|^2 \leq B\|f\|^2.$$

Finally, since $|e^{-2\pi imbr}| = 1$ we get

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb} T_{na} T_r g\rangle|^2 \leq B\|f\|^2.$$

This proves that $\mathcal{G}(T_r g, a, b)$ is a frame. To prove the other direction one could choose $f' = T_r f$ and then do similar calculations.

□

This shows that if $\mathcal{G}(g, a, b)$ is a frame, then translating the generator function g will neither affect the frame property nor the frame bounds. This is interesting when it comes to the B-splines as any results proved for the symmetric B-splines B_n will also hold for the cardinal B-splines N_n and vice versa. Thus if it is simpler to prove a result for either of the B-spline types, then we can prove it for that B-spline, and it will automatically hold for the other.

For a Gabor system $\mathcal{G}(g, a, b)$ we consider translations of g along the lattice $a\mathbb{Z}$. If $\mathcal{G}(g, a, b)$ is a frame then one may wonder what happens with the frame properties if we consider translations along a finer lattice for which $a\mathbb{Z}$ is a sublattice.

Lemma 4.7. *Let $g \in L^2(\mathbb{R})$. If $\mathcal{G}(g, a, b)$ is a frame with frame bounds A and B, then $\mathcal{G}(g, \frac{a}{k}, b)$ with $k \in \mathbb{N}$ is also a frame with bounds kA and kB.*

Proof. Let $k = 2$. Assume that $\mathcal{G}(g, 2a, b)$ is a frame with frame bounds A and B. Then

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb} T_{n2a} g\rangle|^2 \leq B\|f\|^2. \tag{4.5}$$

By translating f with one and remembering that $\|T_{-1} f\| = \|f\|$, we obtain the equation

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle T_{-a} f, E_{mb} T_{n2a} g\rangle|^2 \leq B\|f\|^2.$$

Since $(T_{-a})^* = T_a$, we get

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, T_a E_{mb} T_{n2a} g\rangle|^2 \leq B\|f\|^2.$$

We apply the relation $T_a E_{mb} = e^{-2\pi imba} E_{mb} T_a$ to obtain

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |e^{-2\pi imba}||\langle f, E_{mb}T_aT_{n2a}g\rangle|^2 \leq B\|f\|^2.$$

Collecting the two translations and applying $|e^{-2\pi imba}| = 1$ we get

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{(2n+1)a}g\rangle|^2 \leq B\|f\|^2. \tag{4.6}$$

By adding the two inequalities (4.5) and (4.6), we get

$$2A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \leq 2B\|f\|^2.$$

Hence we see that $\mathcal{G}(g, \frac{2a}{2}, b) = \mathcal{G}(g, a, b)$ is a frame with frame bounds $2A$ and $2B$.

This gives the proof for the case where $k = 2$. For $k > 2$ the method is similar. We start with the frame inequality

$$A\|f\|^2 \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{n2a}g\rangle|^2 \leq B\|f\|^2.$$

This can then be translated by $1, 2, \ldots, k-1$ so we obtain k inequalities which when added gives the frame inequality for $\mathcal{G}(g, a, b)$. □

Even if $A, B > 0$ are optimal frame bounds for $\mathcal{G}(g, a, b)$, we cannot guarantee that kA and kB are optimal frame bounds for $\mathcal{G}(g, \frac{a}{k}, b)$.

5 The frame set of B-splines

So far we have studied results about frame sets $\mathcal{F}(g)$ for arbitrary generators $g \in L^2(\mathbb{R})$ and considered methods to determine whether a given Gabor system is a frame. In this section we explore the frame set $\mathcal{F}(B_N)$ for the B-splines. We will both give results that are already known, prove some new results and look at numerical results.

5.1 Known non-frame areas

We start with some negative results, meaning areas where the system $\mathcal{G}(g, a, b)$ is not a frame. A simple result of this kind for the B-splines B_N given in Proposition 5.1.

Proposition 5.1. *$\mathcal{G}(B_N, a, b)$ is not a frame if $a > N$.*

Proof. Since *supp* $B_N = \left[-\frac{N}{2}, \frac{N}{2}\right]$ a translation parameter $a > N$ would mean that union of the supports of the translates $T_{na}B_N,\ n \in \mathbb{Z}$ does not cover the entire real line. Hence, the system can not be complete in $L^2(\mathbb{R})$ and is not a frame.

As a concrete example of a function not in the span of $\mathcal{G}(B_N, a, b)$ where $a > N$ take $f = \chi_{[\frac{N}{2}, \frac{a}{2}]} \in L^2(\mathbb{R})$. For this function we have

$$\langle f, E_{mb}T_{na}B_N \rangle = \int_{-\infty}^{\infty} f(x)\overline{E_{mb}T_{na}B_N(x)}dx = 0,\ \forall n, m \in \mathbb{Z},$$

since the support for the functions do not overlap. Thus the lower frame bound will be violated and $\mathcal{G}(B_N, a, b)$ where $a > N$ is not be a frame. □

For the next part we need a general result for Gabor frames.

Proposition 5.2. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Assume that $\mathcal{G}(g, a, b)$ is a frame with frame bounds A, B, then*

$$bA \leq \sum_{n\in\mathbb{Z}} |g(x - na)|^2 \leq bB, \quad a.e.\ x \in \mathbb{R}, \tag{5.1}$$

and

$$aA \leq \sum_{n\in\mathbb{Z}} |\widehat{g}(\gamma - nb)|^2 \leq aB, \quad a.e.\ \gamma \in \mathbb{R}. \tag{5.2}$$

The result in Corollary 5.3 has previously had more complicated proofs. However, using (5.2) from Proposition 5.2, a very simple proof is possible. The proof stated here is from [7].

Corollary 5.3. *$\mathcal{G}(B_N, a, b)$ is not a frame when $N > 1$ and $b = 2, 3, 4, \ldots$.*

Proof. Let $b \in \mathbb{N} \setminus \{1\}$, $n \in \mathbb{Z}$, and $\gamma = 1$. Then using (3.10) we get

$$\widehat{B_N}(1 - nb) = \left(\frac{\sin\left(\pi\left(1 - nb\right)\right)}{\pi\left(1 - nb\right)}\right)^N = 0.$$

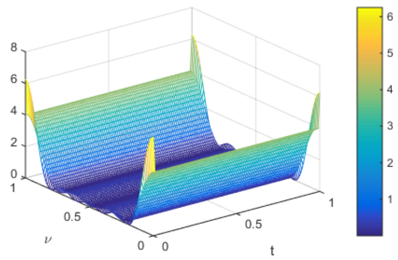
So the lower bound in (5.2) is violated. □

One might wonder whether the method used to prove Corollary 5.3 could be used in situations where $b \notin \mathbb{N} \setminus \{1\}$. However, this is not the case since the proof relies crucially on hitting the zeros of the sine function that are located at $x = n\pi,\ n \in \mathbb{Z}$.

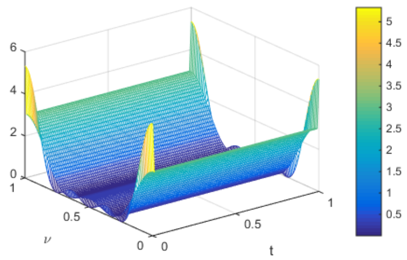
5.2 Zak transform methods at critical sampling ab=1

In Definition 4.3 in Section 4 we introduced the Zak transform. In this section we will use the Zak transform, and particularly the results given in Proposition 4.4 to show known results about the B-splines B_N and their associated Gabor systems $\mathcal{G}(B_N, a, b)$ on the hyperbola $ab = 1$.

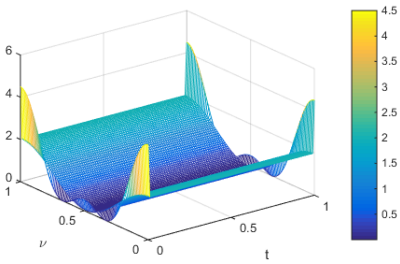
We start by estimating the Zak transform of N_1 and N_2 in Matlab. The Matlab scripts used for these calculations can be found in Appendix A.1. The numerical calculations can be used to give intuition about the behaviour of the Gabor systems before stating the formal results.



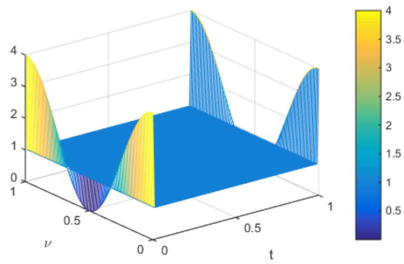
(a) $a = \frac{1}{4}$



(b) $a = \frac{1}{3}$



(c) $a = \frac{1}{2}$



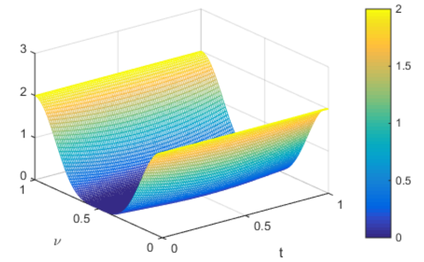
(d) $a = 1$

Figure 3: $|Z_a N_1|^2$ for $a = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$.

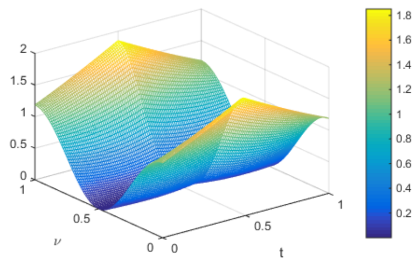
In Figure 3 we see $|Z_a N_1|^2$ for four different values of a. In Figure 3d it appears that for $a = 1$ the Zak transform of N_1 is constantly 1, except for the lines where $t = 0$ and $t = 1$. So we see that $|Z_1 N_1|^2 = 1$, *a.e.* and we have the situation (iv) from Proposition 4.4, meaning that $\mathcal{G}(N_1, 1, 1)$ is an orthonormal basis, and, in particular, it is also a frame. For other values of a we get a function which seems to be constant along lines $\nu = \text{const} \in [0, 1]$. In Figures 3a-3c it appears that the lower bound in (iii) is violated and thus we do not have a frame. The reason we have chosen values of a of the form $a = \frac{1}{n}$, $n \in \mathbb{N}$, is that they are the exact points for which the Zak transform is constant along all lines where ν is constant. For $\frac{1}{M+1} < \nu < \frac{1}{M}$, $M \in \mathbb{N}$, the graph of $(Z_a N_1)(t_0, \nu)$ as a function of ν for a fixed $t_0 \in [0, \frac{1}{a} - [\frac{1}{a}]]$ will look like the graph of for $\left(Z_{\frac{1}{M+1}} N_1\right)(t, \nu)$, $t \in]0, 1[$. Similarly, the graph of $(Z_a N_1)(t_0, \nu)$ as a function of ν for a fixed $t_0 \in]\frac{1}{a} - [\frac{1}{a}], 1]$ will look like the graphs of $\left(Z_{\frac{1}{M}} N_1\right)(t, \nu)$, $t \in]0, 1[$.

We now consider at Figure 4 which shows the Zak transform of N_2 for different values of a. Since N_2 has support on $[0, 2]$, the Zak transform will have more non-zero terms when a is decreased, and the transform will have an equal number of non-zero terms for all t when $a = \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \dots$, that is, $a = 2, 1, \frac{2}{3}, \dots$.

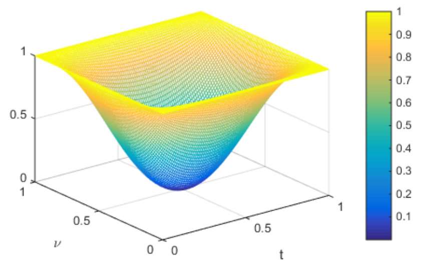
For N_2 it is not as easy to determine a pattern for the Zak transform. However, from the Figures 4a-4d it appears that the Zak transforms of N_2 violate the lower bound of (iii) in Proposition 4.4. Hence, $\mathcal{G}(N_2, a, b)$ is not a frame for $ab = 1$ when $a = \frac{1}{2}, \frac{2}{3}, 1, 2$.



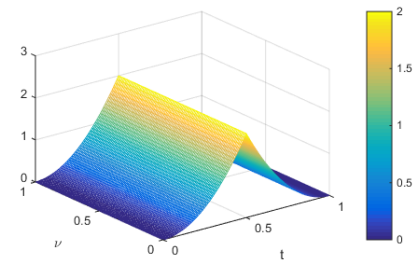
(a) $a = \frac{1}{2}$



(b) $a = \frac{2}{3}$



(c) $a = 1$



(d) $a = 2$

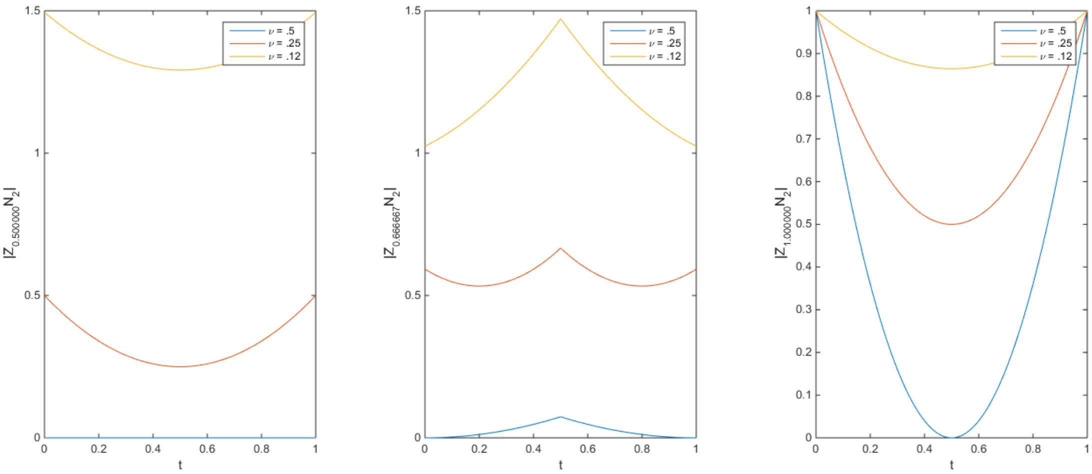
Figure 4: $|Z_a N_2|^2$ for $a = \frac{1}{2}, \frac{2}{3}, 1, 2$.

To examine the number of zeros in the Zak transform, Figure 5 shows the Zak transform of N_2 for three different values of a and fixed values of $\nu \in [0, 1]$. According to [8] the Zak transform of cardinal B-splines for $a = 1$ with $n \geq 2$ will have exactly one zero in $[0, 1[^2$, namely at $(\frac{1}{2}, \frac{1}{2})$ for even n and $(0, \frac{1}{2})$ for odd n. From Figure 5c we can see that this is the case. Figure 5 also shows the Zak transform for $a = \frac{1}{2}$ and $a = \frac{2}{3}$ to see what happens in those cases. We see that zeros still occur on the line $\nu = \frac{1}{2}$. For $a = \frac{2}{3}$ we still only have one zero on $[0, 1[^2$, but for $a = \frac{1}{2}$ we have an infinite number of zeros along the line $\nu = \frac{1}{2}$.

In Figure 3 and Figure 4 we got an indication about the frame properties of the Gabor systems $\mathcal{G}(N_1, a, b)$ and $\mathcal{G}(N_2, a, b)$ for $ab = 1$. These findings can be proved mathematically using properties of the Zak transform, and they can also be extended to B-splines of arbitrary order. The results are summed up in the following theorem.

Theorem 5.4 (Critical sampling). *Let $ab = 1$ and $N \in \mathbb{N}$. Then $\mathcal{G}(B_N, a, b)$ is not a frame unless $N = 1$ and $a = b = 1$.*

Proof. Assume that $ab = 1$. We prove the result for the first B-spline using the cardinal B-spline N_1 since this is simpler. By Lemma 4.6 this implies that the result also holds for the symmetric B-spline B_1. Consider the cardinal the B-spline $N_1 = \chi_{[0,1]}$. For $a > 1$ the conclusion follows from Proposition 5.1.



(a) $a = \frac{1}{2}$ (b) $a = \frac{2}{3}$ (c) $a = 1$

Figure 5: The Zak transform of N_2 for $a = \frac{1}{2}, \frac{2}{3}, 1$ calculated for different values of ν.

Now let for $a = 1$, then the Zak transform is given by

$$(Z_1 N_1)(t, \nu) = \sum_{k \in \mathbb{Z}} \chi_{[0,1]}(t - k) e^{2\pi i k \nu}.$$

For $t \in]0, 1[$ the indicator function will only be non-zero for one k, namely $k = 0$. Hence we only get one term and the Zak transform becomes

$$(Z_1 N_1)(t, \nu) = e^{2\pi i 0 \nu} = 1.$$

For $t = 0$ we get two terms and the Zak transform becomes

$$(Z_1 N_1)(t, \nu) = e^{2\pi i 0 \nu} + e^{2\pi i 1 \nu} = 1 + e^{2\pi i \nu}.$$

This function is continuous in ν and has a zero at $\nu = \frac{1}{2}$. However, since the line $t = 0$ is a set of measure zero, we still have

$$|Z_1 N_1|^2 = 1, \; a.e.$$

Thus, by (iv) in Proposition 4.4, N_1 is an orthonormal basis and thus a frame for $a = b = 1$.

Now let $a < 1$ while still considering N_1. Then we have to look at the Zak transform:

$$(Z_a N_1)(t, \nu) = \sqrt{a} \sum_{k \in \mathbb{Z}} \chi_{[0,1]}(a(t - k)) e^{2\pi i k \nu}.$$

The functions $\chi_{[0,1]}(a(t - k))$ will be translated by the parameter k so the support is translated by 1 when k is increased, while the parameter a scales the functions so each function has support, and thus is 1, on $t \in [k, k + \frac{1}{a}[$. First of all, for $t \in [0, 1[$ this means that we will only get contributions for $k = 0, -1, -2, -3, \ldots$, since the functions translated with a positive k will only have support outside the interval $[0, 1[$. This also explains why there is an equal number of non-zero terms for all $t \in]0,1[$ and $\nu = [0,1]$ when $a = \frac{1}{M}$, $M \in \mathbb{N}$, as this gives support on $t \in [k, k + \frac{1}{(\frac{1}{N})}[= [k, k+N[$. For $\frac{1}{M+1} < a < \frac{1}{M}$, $M \in \mathbb{N}$, we get $M+1$ terms when $t \in [0, \frac{1}{a} - [\frac{1}{a}]]$ and M terms for $t \in [\frac{1}{a} - [\frac{1}{a}], 1[$. The case with $M+1$ terms will be

$$(Z_a N_1)(t,\nu) = \sqrt{a} \sum_{k=0}^{M} e^{2\pi i k \nu}, \ t \in [0, \frac{1}{a} - \left[\frac{1}{a}\right][, \ \nu \in [0,1]$$

and the case with M terms will be

$$(Z_a N_1)(t,\nu) = \sqrt{a} \sum_{k=0}^{M-1} e^{2\pi i k \nu}, \ t \in [\frac{1}{a} - \left[\frac{1}{a}\right], 1[, \ \nu \in [0,1].$$

Both these Zak transforms are continuous in ν since each of the exponential functions are continuous. Hence, the Zak transform is piecewise continuous on $(t,\nu) \in \left[0, \frac{1}{a} - [\frac{1}{a}]\right] \times [0,1]$ and $(t,\nu) \in \left]\frac{1}{a} - [\frac{1}{a}], 1\right] \times [0,1]$. Thus, if we can show that the function has a zero on either of those rectangles, then the lower bound in (iii) in Proposition 4.4 will be violated.

We know that in general

$$\sum_{k=0}^{m-1} e^{2\pi i \frac{k}{m}} = 0.$$

Hence, for $t \in \left[0, \frac{1}{a} - [\frac{1}{a}]\right]$ and $\nu = \frac{1}{M+1}$ we have

$$\sum_{k=0}^{M} e^{2\pi i k \nu} = \sum_{k=0}^{M} e^{2\pi i \frac{k}{M+1}} = 0,$$

Thus we have shown that the Zak transform has zeros on $(t,\nu) \in \left[0, \frac{1}{a} - [\frac{1}{a}]\right] \times [0,1]$ and since it is continuous that means that the lower bound in (iii) in Proposition 4.4 will be violated. Hence, we conclude that N_1 is not a frame for $ab = 1$ and $a < 1$. It is enough to show that the Zak transform has these zeros, but the argument works similarly for $t \in \left]\frac{1}{a} - [\frac{1}{a}]\right]$ and $\nu = \frac{1}{M}$ since we have

$$\sum_{k=0}^{M-1} e^{2\pi i k \nu} = \sum_{k=0}^{M-1} e^{2\pi i \frac{k}{M}} = 0.$$

Now we let $N \geq 2$ and we consider the symmetric B-splines B_N. It is easily seen that the B-splines B_N lie in the space $W(\mathbb{R}) \cap C^0(\mathbb{R})$ for $N \geq 2$. Therefore the Zak transform $Z_a B_N$ will be continuous on $\mathbb{R}^2$. It then follows from [3, Lemma 8.4.2] that $Z_a B_N$ has a zero. Since the Zak transform is continuous, this means that the lower bound in (iii) in Proposition 4.4 will be violated. Hence B_N is not a frame for $N \geq 2$ and $ab = 1$.

□

To summarise our findings so far, we have proved that for $N \geq 2$,

$$\mathcal{F}(B_N) \subset \{(a,b) \in \mathbb{R}_+^2 | ab < 1, \, a < N, \, b \neq 2, 3, \dots\}.$$

As no other non-frame (a,b)-vales for the B-splines of order $N \geq 2$ are known, Gröchenig conjectured the following in [4]:

Conjecture 1 (Gröchenig)**.** *Let $N \geq 2$. Then*

$$\mathcal{F}(B_N) = \{(a,b) \in \mathbb{R}_+^2 | ab < 1,\ a < N,\ b \neq 2, 3, \dots\}.$$

We will investigate this conjecture further in the following subsections. In particular, we will show in Subsection 5.7 that the conjecture is false.

5.3 The painless case

We will now consider some of the cases, where the Gabor systems generated by the B-splines are in fact frames. We will actually show the results for a more general class of compactly supported generators and obtain the results for the B-splines as a special case.

Theorem 5.5. *Suppose that $g \in L^2(\mathbb{R})$ is supported on $[0, N]$. If $a \leq N$ and $b \leq \frac{1}{N}$, then the frame operator S is given by the multiplication operator*

$$Sf(x) = \left(\frac{1}{b} \sum_{k \in \mathbb{Z}} |g(x - ak)|^2 \right) f(x). \tag{5.3}$$

Thus, $\mathcal{G}(g, a, b)$ is a frame with frame bounds A and B if and only if

$$bA \leq \sum_{k \in \mathbb{Z}} |g(x - ak)|^2 \leq bB, \text{ a.e.} \tag{5.4}$$

For continuous generators g with support on an interval I of length $|I| < \infty$, we can show that (5.4) will be satisfied for some $A, B > 0$ if g is positive on the interior of I. The next result is stated without proof in [1]. We give a proof here for completeness.

Theorem 5.6. *Suppose that $g \in L^2(\mathbb{R})$ is a continuous function with support on an interval I of length $|I| < \infty$ and that $g(x) > 0$ on the interior of I. Then $\mathcal{G}(g, a, b)$ is a frame for all $(a, b) \in]0, |I|[\times]0, \frac{1}{|I|}]$. If $a = |I|$ and $b \in]0, \frac{1}{|I|}]$ then $\mathcal{G}(g, a, b)$ is not a frame.*

Proof. Let $g \in L^2(\mathbb{R})$ be a continuous function with support on an interval $I = [\alpha_I, \beta_I]$ of length $|I| < \infty$ and assume that $g(x) > 0$ on the interior of I. Assume that $(a, b) \in]0, |I|[\times]0, \frac{1}{|I|}]$. Then by Theorem 5.5, the Gabor system $\mathcal{G}(g, a, b)$ is a frame if the function $\frac{1}{b} \sum_{k \in \mathbb{Z}} |g(x - ak)|^2$ has finite lower and upper bounds $A, B > 0$. It can easily be seen that an upper bound will be satisfied since g only has support on a bounded interval, and thus the number of non-zero terms in the sum will be finite.

Now consider $g(x_0 - ak)$ for a fixed $x_0 \in \mathbb{R}$. Since $a < |I|$ there exists at least one $k_0 \in \mathbb{Z}$ such that $x_0 + ak_0$ lies in some closed bounded interval J which is a subset of the support I. Since g is continuous, we know that the image of a closed and bounded subset under g will also be mapped into a closed and bounded subset. Thus we know that $g(x) \in [\alpha, \beta]$ for $x \in J$ and some $\alpha, \beta \in \mathbb{C}$. Furthermore, since $g(x) > 0$ on the interior of I we know that $\alpha, \beta > 0$. Thus, we know that at least one of the terms in the sum $\sum_{k \in \mathbb{Z}} |g(x - ak)|^2$ will be positive. Hence, the sum will be strictly greater than zero giving us an optimal lower bound $A > 0$, meaning that $\mathcal{G}(g, a, b)$ is a frame.

Now let $a = |I|$ and let $x = \alpha_I$, then $x+a = \beta_I$. Since g is continuous and $g(x) = 0$ for $x \notin [\alpha_I, \beta_I]$ we know that $g(\alpha_I) = g(\beta_I) = 0$. Furthermore, all other elements of the sum will also be equal to zero since they lie outside the support of g. Thus, $\sum_{k\in\mathbb{Z}} |g(x-ak)|^2 = 0$. Since the sum is continuous in x, this means that the lower bound in (5.4) will be violated. Hence, $\mathcal{G}(g, |I|, b)$ is not a frame for $b \in]0, \frac{1}{|I|}]$. $\square$

Of course the B-splines are a special case of the generators in Theorem 5.6.

Corollary 5.7. *For $N \geq 2$ the B-splines B_N generate Gabor frames for all $(a, b) \in]0, N[\times]0, \frac{1}{N}]$. And for $N \geq 2$ the B-splines B_N do not generate Gabor frames for $a = N$ and $b \in \times]0, \frac{1}{N}]$.*

Since B_1 is not continuous, it will have to be treated separately. However, its frame properties for $(a, b) \in]0, 1]\times]0, 1]$ can also be proved using Theorem 5.5.

Theorem 5.8. *B_1 generates Gabor frames for all $(a, b) \in]0, 1]\times]0, 1]$.*

Proof. Let $(a, b) \in]0, 1]\times]0, 1]$. Then by Theorem 5.5, the Gabor system $\mathcal{G}(B_1, a, b)$ is a frame if the sum $\sum_{k\in\mathbb{Z}} |B_1(x-ak)|^2$ has lower and upper bounds $A, B > 0$. Once again it is clear that the upper bound will be satisfied since there will only be a finite number of non-zero terms in the sum. However, since B_1 is 1 on all of its support, there will always be at least one term in the sum. Hence, the lower bound is satisfied and $\mathcal{G}(B_1, a, b)$ is a frame. $\square$

Note that for B-splines of order $N \geq 2$, the Gabor system $\mathcal{G}(B_N, a, b)$ is not a frame for $a = N$. However, the Gabor system $\mathcal{G}(B_1, a, b)$ is also a frame for $a = 1$.

With the specific expression for the frame operator given in Theorem 5.5, we can examine the optimal frame bounds for the B-splines for $(a, b) \in]0, N[\times]0, \frac{1}{N}]$. Let

$$G(x) = \sum_{k\in\mathbb{Z}} |g(x-ak)|^2.$$

Then $G(x)$ is periodic with period a since

$$G(x+a) = \sum_{k\in\mathbb{Z}} |g(x+a-ak)|^2 = \sum_{k\in\mathbb{Z}} |g(x-a(k-1))|^2 = \sum_{\ell\in\mathbb{Z}} |g(x-a\ell)|^2 = G(x),$$

Therefore we only have to calculate $G(x)$ for $x \in [0, a]$. This will be done for B_1 and B_2. In both cases we find $G(x)$ for some points in the interval $[0, a]$. By finding the maximum and minimum of these values we determine Ab and Bb. Since b is present on both, sides we can find the ratio between A and B as $\frac{Ab}{Bb} = \frac{A}{B}$. This also means that the ratio is independent of b, as long as we stay within the rectangle $(a, b) \in]0, n[\times]0, \frac{1}{n}]$.

Figure 6 shows the ratio $\frac{A}{B}$ for B_1 as a function of a. In this case we get a staircase pattern. This makes sense since we once again add another term to the sum as a each time we, as a increases, get to one of the points $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; each of these terms will be equal to one as $g = \chi_{[0,1]}$. In fact, in all the cases where we hit those specific values of a, we will have $G(x) = \frac{1}{a}$ in all points except $x = ma$, $m \in \mathbb{Z}$. We also see that the ratio decreases as $a \to N = 1$. However, by using the Zak transform of B_1

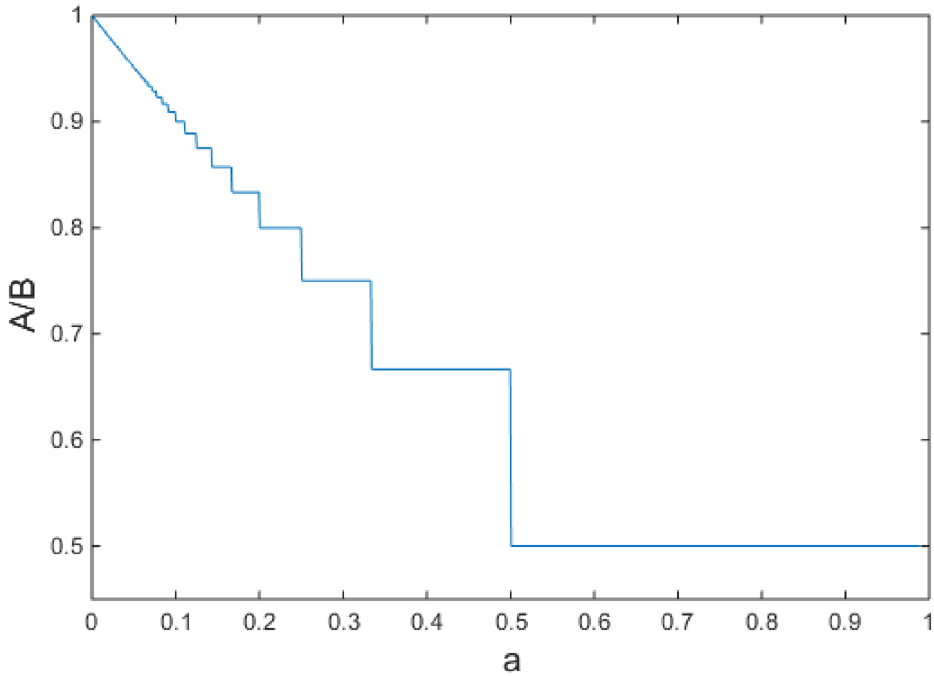


Figure 6: The relation $\frac{A}{B}$ of the frame bounds for B_1 in the case where $a \leq N = 1$ and $b \leq \frac{1}{N} = 1$.

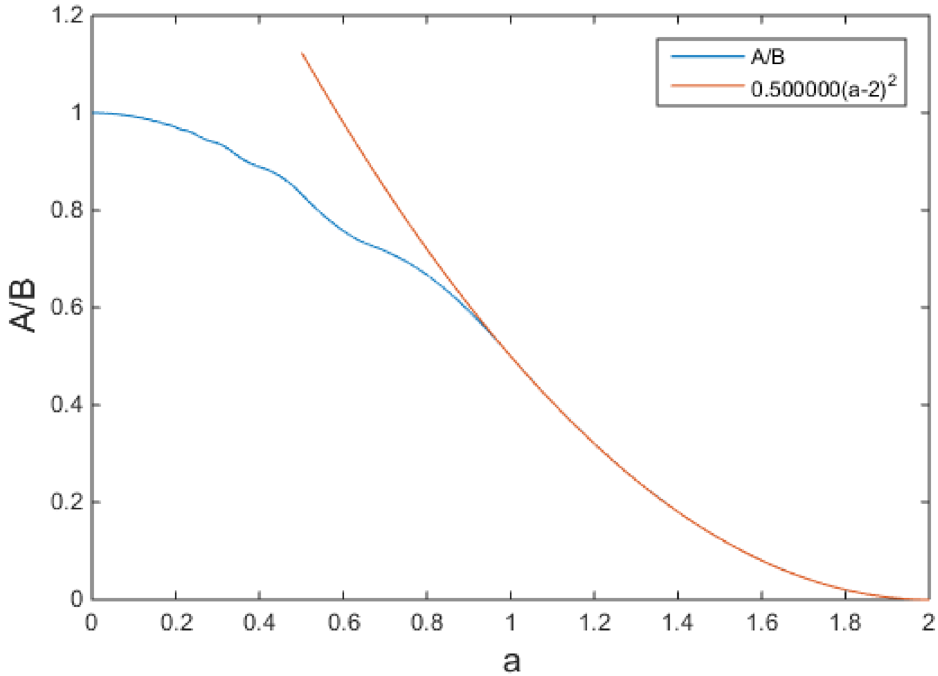


Figure 7: The relation $\frac{A}{B}$ of the frame bounds for B_2 in the case where $a \leq N = 2$ and $b < \frac{1}{N} = \frac{1}{2}$.

in Figure 3, we saw that $\mathcal{G}(B_1, 1, 1)$ is a frame and therefore the ratio does not go to

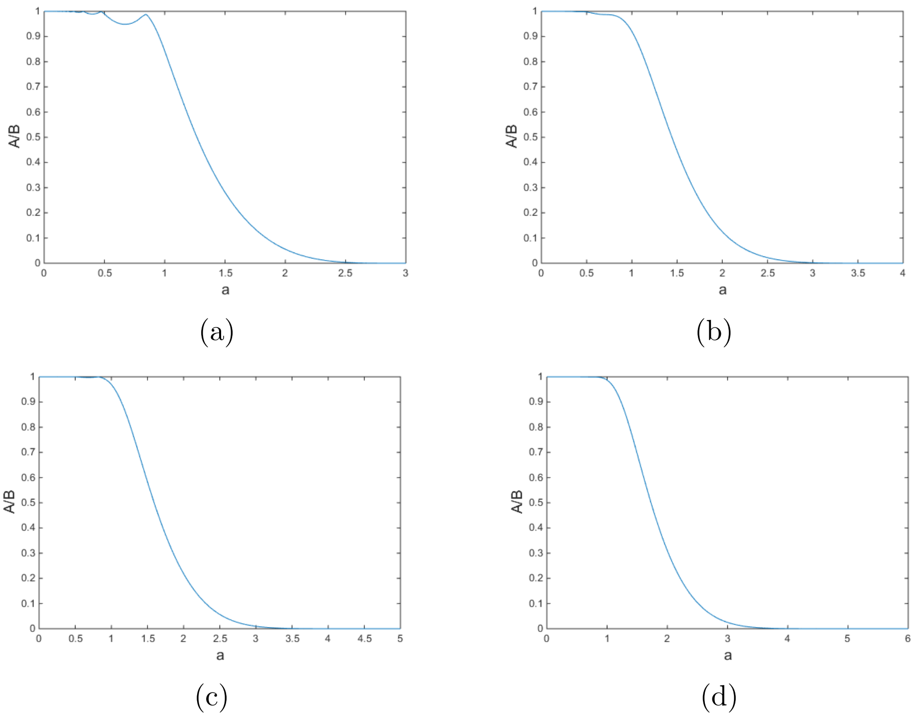


Figure 8: Plots of the ratio $\frac{A}{B}$ for the B-Splines B_3, B_4, B_5 and B_6 in the case where $a \leq N$ and $b < \frac{1}{N}$.

zero.

In Figure 7 we see the same ratio $\frac{A}{B}$ for the B-spline of order 2. Here we see that the ratio starts close to 1 and then the ratio decreases as $a \to N = 2$. This seems reasonable considering the fact that we saw from the Zak transform of B_2 that it was not a frame for $a = 2$ and $b = \frac{1}{2}$. To get an idea of the convergence of the ratio $\frac{A}{B}$ we have plotted the line $C(a - N)^2$ for $C = \frac{1}{2}$. This lies right on top of the graph of $\frac{A}{B}$ for $a > 1$, so this seems to be the rate that this converges with.

We have also plotted the ratio $\frac{A}{B}$ for other B-Splines in Figure 8. Here we can see that the ratio generally appears to stay close to one for $0 < a < 1$. After that it declines in at what looks like a polynomial rate that appears to get higher order as N is increased, and the ratio goes to zero as $a \to N$.

5.4 The dual frame method: Known results

Until now we have proved results for B-splines of any order $N \in \mathbb{N}$. However, in the remainder of this thesis we will focus on the case of B_N with $N \geq 2$. This makes sense as they are different types of functions. For example we have $B_N \in W(\mathbb{R}) \cup C^0(\mathbb{R})$ for $N \geq 2$, whereas $B_1 \notin W(\mathbb{R}) \cup C^0(\mathbb{R})$. Furthermore, the problem of finding the frame set for B_1 has already been solved in [2].

In this subsection we focus on the area of the (a, b)-plane where $\frac{N}{2} \leq a < N$. The result for the B-splines in the part of this area where $b \leq \frac{1}{N}$ was found in Corollary 5.7. However, for certain continuous generators g with support on a symmetric interval around 0, we can also prove the frame properties on the part of that area where $b > \frac{1}{N}$.

The general result for continuous generators g with support on a symmetric interval around zero is given in Theorem 5.9. We apply that theorem to functions that are positive on the interior of their support to obtain Corollary 5.10. The results are given for functions with support on a symmetric interval around zero since this makes the proofs simpler. However, we recall that by Lemma 4.6 these results will also hold for generators supported on other bounded intervals as long as the generator satisfies the rest of the conditions.

First we need to define the functions of interest. Let the set V_α be defined as

$$V_\alpha = \{f \in C(\mathbb{R}) \mid \operatorname{supp} f = [-\alpha, \alpha], f \text{ has a finite number of zeros on } [-\alpha, \alpha]\}$$

From [6] we have the following characterisation of Gabor frames with generators in V_α.

Theorem 5.9 ([6],Theorem 2.1)**.** *Let $g \in V_\alpha$ for some $\alpha > 0$ and assume that $\alpha \leq a < 2\alpha$ and $ab \in \left[\frac{M-1}{M}, \frac{M}{M+1}\right[$ for some $M \in \mathbb{N} \setminus \{1\}$. Let $\kappa \in \{0, 1, \ldots, M-1\}$ be the largest integer for which $(1-ab)\kappa \leq b\alpha$. Then $\mathcal{G}(g, a, b)$ is a Gabor frame if and only if the following conditions are satisfied:*

(i) $|g(x)| + |g(x+a)| > 0, \ x \in [-a, 0]$;

(ii) If $\kappa \neq 0$ and if there exists $n_+ \in \{1, 2, \ldots, \kappa\}$ and $y_+ \in]a - \alpha, \alpha - (1-ab)\frac{n_+}{b}]$ such that $g(y_+) = 0$ and $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$, then

$$g(y_+ + (1-ab)\frac{n_+}{b} - a) \neq 0;$$

(iii) If $\kappa \neq 0$ and if there exists $n_- \in \{1, 2, \ldots, \kappa\}$ and $y_- \in [-\alpha + (1-ab)\frac{n_-}{b}, \alpha - a[$ such that $g(y_-) = 0$ and $\lim_{y \to y_-} |L_{n_-}(y)| = \infty$, then

$$g(y_- - (1-ab)\frac{n_-}{b} + a) \neq 0;$$

(iv) For y_+, y_-, n_+, n_- as in (ii) and (iii),

$$y_+ + (1-ab)\frac{n_+}{b} \neq y_- - (1-ab)\frac{n_-}{b} + a.$$

If we further restrict the function to be positive on $]-\alpha, \alpha[$, then we get the following result.

Corollary 5.10. *Let $g \in V_\alpha$ for some $\alpha > 0$ and assume that $g(x) > 0$ for $x \in]-\alpha, \alpha[$. Furthermore, assume that $\alpha \leq a < 2\alpha$. Then $\mathcal{G}(g, a, b)$ is a Gabor frame. In particular, $\mathcal{G}(B_N, a, b)$ is a frame for $\frac{N}{2} \leq a < N$ and $0 < b < \frac{1}{a}$ for each $N \geq 2$.*

Proof. Let $g \in V_\alpha$ for some $\alpha > 0$ and assume that $g(x) > 0$ for $x \in]-\alpha, \alpha[$. Then as the function does not have any zeros inside the support we do not have to consider any of the requirements $(ii) - (iv)$ of Theorem 5.9. □

We state Theorem 5.9 and Corollary 5.10 without proof as the focus here is on properties of the B-splines. Therefore we will end up giving a proof in the specific case where $g = B_N$ rather than proving the general case. The method used to prove these results is based on finding a function $h \in L^2(\mathbb{R})$ which generates a Bessel sequence $\mathcal{G}(h, a, b)$ that is a dual frame to $\mathcal{G}(g, a, b)$. Hence we need a result that characterises when two functions generate dual frames.

Theorem 5.11. *Let $g, h \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Let $\mathcal{G}(g, a, b)$ and $\mathcal{G}(h, a, b)$ be two Bessel sequences. Then they form dual frames if and only if, for all $n \in \mathbb{Z}$,*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/b - ka)} h(x - ka) = b\delta_{n,0}, \text{ a.e. } x \in [0, a]. \tag{5.5}$$

Note that in order to use this theorem we also need to prove that $\mathcal{G}(h, a, b)$ defines a Bessel sequence. However, in most cases we will define h in a way such that it is bounded and compactly supported. In this case $\mathcal{G}(h, a, b)$ is indeed a Bessel sequence by [1, Corollary 9.1.6].

First we will give an example where the frame property is proven for B_2 with fixed values a, b, where $1 \le a < 2$ and $b > \frac{1}{2}$. We prove that $\mathcal{G}(B_2, a, b)$ is a frame by showing that there exists a dual frame. The structure of the example is inspired by that of [6, Example 2.2]. The example is given in order to give the intuition of the way that duals can be constructed before giving the full proof.

Example 5.12. *Let $g(x) = B_2(x)$. Since $B_2 \in V_1$, we have $\alpha = 1$. $B_2(x) > 0$ for $x \in]-1, 1[$. Thus the conditions for the generator in Corollary 5.10 are satisfied and we know that $\mathcal{G}(B_2, a, b)$ will be a frame for any $\alpha \le a < 2\alpha$. However, we will construct a dual generator which is also the way that the general result is proved. Choose a and b such that $\alpha \le a < 2\alpha$, $ab \le 1$, and $\alpha \le \frac{1}{b} - \frac{a}{2}$. For the B-spline B_2, the values $a = \frac{4}{3}$ and $b = \frac{3}{5}$ would be an example of this since $\alpha = 1$ and $\frac{1}{b} - \frac{a}{2} = \frac{5}{3} - \frac{2}{3} = 1$. Thus, we have $\alpha \le \frac{1}{b} - \frac{a}{2}$. For this choice we have $\frac{(1-ab)}{b} = \frac{\left(1-\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = \frac{1}{3}$. So $\kappa = 3$ is the largest integer for which the inequality $\frac{(1-ab)}{b}\kappa \le \alpha$ holds. Furthermore, we see that $ab = \frac{4}{5} \in \left[\frac{M-1}{M}, \frac{M}{M+1}\right[$ for $M = 5$.*

Then inspired by [6, Lemma 3.3] we set

$$h(x) = 0, \quad x \notin -\left(\bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak + \alpha]\right) \cup [-\alpha, \alpha] \cup \bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak + \alpha]. \tag{5.6}$$

This is done so (5.5) will hold for some values of x and $n \neq 0$.

Now we need to define $h(x)$ on the set given in (5.6). We start by defining h on $[-\alpha, \alpha]$. Since $g(x) > 0$ for $x \in]-\alpha, \alpha[$ and $\frac{a}{2} < \alpha$ we can obtain a bounded function by setting

$$h(x) = \begin{cases} \frac{b}{g(x)}, & x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0, & x \in [-\alpha, \alpha] \setminus [-\frac{a}{2}, \frac{a}{2}] \end{cases} \tag{5.7}$$

This way h and g satisfy (5.5) for $n = 0$.

Finally, we need to define h on the set $\bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak+\alpha]$ and its symmetric counterpart on the negative part of the axis. Due to the support of g we know that only some of *the terms in (5.5) will be non-zero, and thus we only need to check the equations*

$$g(x-\frac{n}{b})h(x)+g(x-\frac{n}{b}+a)h(x+a)=b\delta_{n,0}\quad \textit{for a.e. } x\in[\frac{n}{b}-a,\frac{n}{b}] \tag{5.8}$$

for $n=0,\pm1,\ldots,\pm(M-1)$. *This gives the idea to define* h *on* $[\frac{1}{b},a+\alpha]$ *by*

$$h(x+a)=-\frac{g\left(x-\frac{1}{b}\right)h(x)}{g\left(x-\frac{1}{b}+a\right)},\quad x\in[\frac{1}{b}-a,\alpha].$$

However, $g(x-\frac{1}{b})$ *has support on* $[\frac{1}{b}-\alpha,\frac{1}{b}+\alpha]$ *and* $h(x)$ *has support on* $[-\frac{a}{2},\frac{a}{2}]$. *Thus, since we have chosen* a *and* b *such that* $\alpha\leq\frac{1}{b-\frac{a}{2}}$, *the two supports will not overlap as*

$$\alpha\leq\frac{1}{b-\frac{a}{2}}\Leftrightarrow\frac{a}{2}\leq\frac{1}{b}-\alpha$$

Thus we will have

$$h(x)=0,\quad x\in\left[\frac{1}{b},a+\alpha\right].$$

When we define h *in a similar way for the other two intervals in* $\bigcup_{k=1}^{\kappa}[\frac{k}{b},ak+\alpha]$ *as well as the corresponding intervals on the negative axis, we get* $h(x)=0$ *for* x *in these intervals. Hence, the final definition of* $h(x)$ *will be:*

$$h(x)=\begin{cases}\frac{b}{g(x)}, & x\in[-\frac{a}{2},\frac{a}{2}]\\ 0, & \textit{otherwise}\end{cases} \tag{5.9}$$

Since h *is bounded and compactly supported* $\mathcal{G}(h,a,b)$ *is a Bessel sequence and we can apply Theorem 5.11. Now we need to show that this definition of* h *indeed satisfies the equations (5.5). Since the infinite sum (5.5) is* a*-periodic in* x, *it is enough to show that the equations hold for a.e.* x *in an interval of length* a. *Consider* $x\in[-\frac{a}{2},\frac{a}{2}]$. *Then* $h(x-ka)=0$ *for all* $k\in\mathbb{Z}\setminus\{0\}$. *Hence, we only have to look at one term in the sum (5.5), namely the one where* $k=0$. *Hence, the equations we will have to check are*

$$g\left(x-\frac{n}{b}\right)h(x)=b\delta_{n,0},\quad \textit{a.e. } x\in[-\frac{a}{2},\frac{a}{2}],\ n\in\mathbb{Z}.$$

For $n=0$ *this is easily verified since the equation becomes*

$$g(x)h(x)=b,\quad \textit{a.e. } x\in[-\frac{a}{2},\frac{a}{2}],$$

and we have defined $h(x)=\frac{b}{g(x)}$ *exactly on the interval* $[-\frac{a}{2},\frac{a}{2}]$. *We have already shown that the support of the functions* $g(x-1/b)h(x)$ *and* $h(x)$ *do not overlap. This shows that*

$$g\left(x-\frac{1}{b}\right)h(x)=0,\quad \textit{a.e. } x\in[-\frac{a}{2},\frac{a}{2}].$$

When n *is increased functions* $g\left(x-\frac{n}{b}\right)$ *are translated further away from* $h(x)$ *and thus the equation*

$$g\left(x-\frac{n}{b}\right)h(x)=0,\quad \textit{a.e. } x\in[-\frac{a}{2},\frac{a}{2}]$$

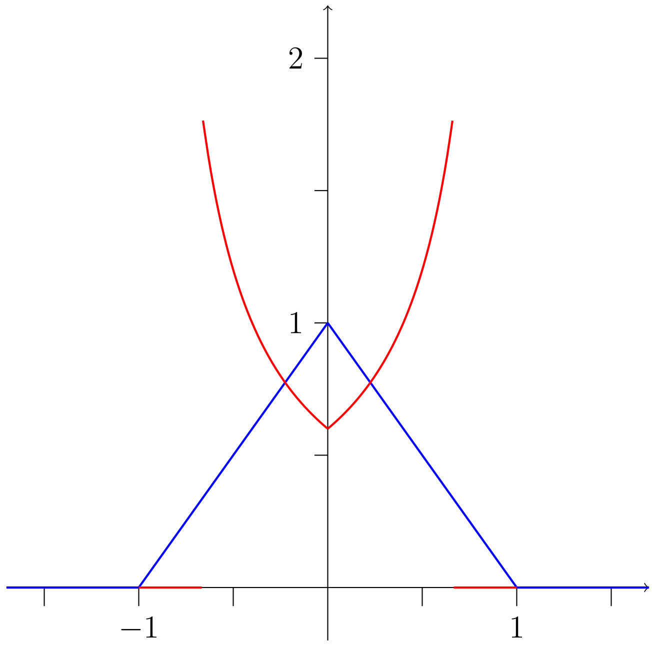


Figure 9: The two functions $g = B_2$ (blue) and h (red) that generate dual frames when $g(x) = B_2(x)$, $a = \frac{4}{3}$ and $b = \frac{3}{5}$.

also holds for all $n \in \mathbb{N}$. A similar argument also holds for the negative values $n \in -\mathbb{N}$. Thus, we have shown that the equations (5.5) are satisfied.

In Figure 9 we have plotted the function $B_2(x)$ along with the function $h(x)$, described in this example, that generates a dual frame to $\mathcal{G}(B_2, \frac{4}{3}, \frac{3}{5})$.

The example gave a very simple dual function. This particular dual will work for any a and b that satisfy the inequalities

$$ab \leq 1, \quad \alpha \leq a < 2\alpha, \quad \text{and} \quad \alpha \leq \frac{1}{b} - \frac{a}{2}.$$

This holds as the specific values were not used when proving that the generators g and h satisfied (5.5). However, if $\alpha \not\leq \frac{1}{b} - \alpha$, then the argument for $h(x)$ becoming zero on $[\frac{1}{b}, a + \alpha]$ fails, and we have to construct the function h to be non-zero on more intervals. Figure 10 shows the curve $\alpha \leq \frac{1}{b} - \frac{a}{2}$ where all points (a, b) below the curve will have simple duals. When going above the curve more non-zero intervals will be added to the dual h and thus its support will be wider.

The method here only shows us that $\mathcal{G}(B_N, a, b)$ is a frame on this area of the (a, b)-plane and does not give any direct frame bounds. However, it would be possible to give estimates on the frame bounds based on the values of the generator function.

The general result for the B-splines on the area of the (a, b)-plane where $\frac{N}{2} \leq a < N$ and $0 < b < \frac{1}{a}$ is given in Theorems 5.13 and 5.14. The theorems also specifies a dual for a B-spline for any point in that area.

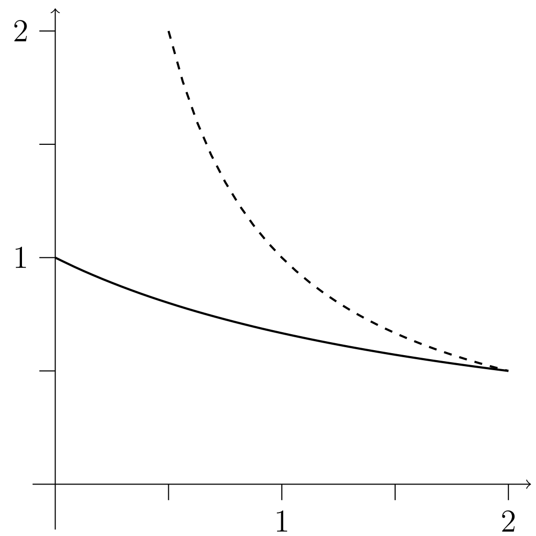


Figure 10: The hyperbola $ab = 1$ (dashed) plotted with the curve $1 = \frac{1}{b} - \frac{a}{2}$ (solid).

Theorem 5.13. *Let $N \geq 2$, $\frac{N}{2} \leq a < N$, $0 < b < \frac{1}{a}$ and $\frac{N}{2} \leq \frac{1}{b} - \frac{a}{2}$. Then the Gabor system $\mathcal{G}(B_N, a, b)$ is a frame and*

$$h(x) = \begin{cases} \frac{b}{B_N(x)} & x \in [-\frac{a}{2}, \frac{a}{2}], \\ 0 & \textit{otherwise.} \end{cases}$$

is a dual generator of $B_N(x)$.

Proof. Assume that $N \geq 2, \frac{N}{2} \leq a < N$, $0 < b < \frac{1}{a}$ and $\frac{a}{2} \leq \frac{1}{b} - \frac{N}{2}$. Define the function h as

$$h(x) = \begin{cases} \frac{b}{B_N(x)}, & x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0, & \textit{otherwise} \end{cases}$$

First of all the function h is well defined since $B_N(x) > 0$ for $x \in [-\frac{a}{2}, \frac{a}{2}]$. Furthermore, h is bounded and compactly supported. Thus, $\mathcal{G}(h, a, b)$ is a Bessel sequence and we can apply Theorem 5.11. Now want to show that h and B_N generate dual frames, because then B_N itself generates a frame. To show that h is a dual function we will show that the equations (5.5) are satisfied for $h(x)$ and $g(x) = B_N(x)$. Since the support of h is on $[-\frac{a}{2}, \frac{a}{2}]$, the functions $h(x - ka)$ are equal to zero for $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $k \neq 0$. Hence we only have to check that

$$h(x)B_N(x - \frac{n}{b}) = \delta_{0,n}b, \quad x \in [-\frac{a}{2}, \frac{a}{2}], \ n \in \mathbb{Z}. \tag{5.10}$$

For $n = 0$ we get

$$h(x)B_N(x) = \frac{b}{B_N(x)}B_N(x) = b, \quad x \in [-\frac{a}{2}, \frac{a}{2}].$$

Thus (5.10) is satisfied for $n = 0$. Looking at the equation for $n = 1$ we get

$$h(x)B_N(x - \frac{1}{b}) = 0, \quad x \in [-\frac{a}{2}, \frac{a}{2}]. \tag{5.11}$$

Looking at the functions individually we see that $h(x)$ has support on $[-\frac{a}{2}, \frac{a}{2}]$ while $B_N(x - \frac{1}{b})$ has support on $[\frac{1}{b} - \frac{N}{2}, \frac{1}{b} + \frac{N}{2}]$. Since a and b have been chosen such that $\frac{a}{2} \leq \frac{1}{b} - \frac{N}{2}$, the support of the functions will not overlap and thus (5.11) holds. We use a similar argument for $n = -1$ by multiplying with minus one on both sides of the inequality to get $\frac{N}{2} - \frac{1}{b} \leq -\frac{a}{2}$. For higher values of $|n|$ we are simply translating the function B_N further away from the support of h. Thus (5.10) is satisfied for all $n \in \mathbb{Z}$. This shows that $\mathcal{G}(B_N, a, b)$ and $\mathcal{G}(h, a, b)$ generate dual frames. Particularly, it shows that $\mathcal{G}(B_N, a, b)$ is a frame. □

The result in Theorem 5.13 only concerns part of the area of the (a, b)-plane where $\frac{N}{2} \leq a < N$ and $0 < b < \frac{1}{a}$. It is included because of the simple dual. However, the full result for (a, b)-values satisfying $\frac{N}{2} \leq a < N$ and $0 < b < \frac{1}{a}$ is given in Theorem 5.14.

Theorem 5.14. *Let $N \geq 2$, $\frac{N}{2} \leq a < N$, $0 < b < \frac{1}{a}$ and $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$. Let κ be the largest integer such that $(1 - ab)\kappa \leq \frac{N}{2}b$ holds. Then the Gabor system $\mathcal{G}(B_N, a, b)$ is a frame and symmetric function h, defined on the positive axis as*

$$h(x) = \begin{cases} \frac{b}{B_N(x)} & x \in [0, \frac{a}{2}], \\ \frac{-B_N(x - \frac{n}{b} - a)h(x-a)}{B_N(x - \frac{n}{b})} & x \in [\frac{n}{b}, \frac{N}{2} + na], \ n = 1, 2, \ldots, \kappa, \\ 0 & x > \textit{otherwise}. \end{cases}$$

Let $h(x) = h(-x)$ for $x < 0$. Then h is a dual generator of $B_N(x)$.

Proof. Assume that $N \geq 2$, $\frac{N}{2} \leq a < N$, $0 < b < \frac{1}{a}$ and $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$. Furthermore, assume that h is defined as stated in the theorem. First of all it is guaranteed that we will not divide by zero for $x \in [0, \frac{a}{2}]$ since $B_N(x) > 0$ for $x \in [-\frac{N}{2}, \frac{N}{2}]$ and $\frac{a}{2} < \frac{N}{2}$. We also want to make sure that we do not divide by zero on any of the other intervals. We know that $B_N(x) > 0$ for $x \in [-\frac{N}{2}, \frac{N}{2}]$. In the denominator we have the functions $B_N(x - \frac{n}{b})$ on $x \in [\frac{n}{b}, \frac{N}{2} + na]$. So the arguments will be in the interval $x - \frac{n}{b} \in [0, \frac{N}{2} + na - \frac{n}{b}]$. Since we know that $\text{supp}B_N = [-\frac{N}{2}, \frac{N}{2}]$, and we have $\frac{N}{2} + na - \frac{n}{b} < \frac{N}{2} + na - na = \frac{N}{2}$, we know that we will not divide by zero.

We wish to prove that h is a dual generator of $g = B_N$. Since h is bounded and compactly supported, we know that $\mathcal{G}(h, a, b)$ is a Bessel sequence. Thus, we can use Theorem 5.11 to prove that $\mathcal{G}(B_N, a, b)$ by showing B_N and h satisfy the equations (5.5). This time we can simplify the equations somewhat because of the bounded support of B_N. In (5.5) we check the equations for $x \in [0, a]$, but the sum is a-periodic so we can actually check it on any interval of length a. In order to simplify the equations we choose to look at the interval $x \in [\frac{n}{b} - a, \frac{n}{b}]$. We can then determine which $k \in \mathbb{N}$ will make $g(x - \frac{n}{b} - ka)$ non-zero. We know that $\text{supp}\, B_N = [-\frac{N}{2}, \frac{N}{2}]$. So looking at the argument of g we wish to determine the $k \in \mathbb{N}$ for which the left end point is within the support of g. The left end point is found by replacing x by $\frac{n}{b} - a$ in the expression $x - \frac{n}{b} - ka$. That way we get the equation

$$\left(\frac{n}{b} - a\right) - \frac{n}{b} - ka = -a\,(k+1) < \frac{N}{2}. \tag{5.12}$$

Similarly, we want the right end point to be in the support of g, and this gives the following equation

$$\left(\frac{n}{b}\right) - \frac{n}{b} - ka = -ka > -\frac{N}{2}. \tag{5.13}$$

We start by determining values of k that satisfy (5.12). Since we assume $\frac{N}{2} \le a$, we have $-a \ge -\frac{N}{2}$. Thus we get

$$-a\,(k+1) \le -\frac{N}{2}\,(k+1) < \frac{N}{2}, \text{ for } k \ge -1.$$

Similarly, we can determine the values of k that satisfy (5.13). To do this we use the assumption that $a < N$, and thus $-a > -N$. This gives

$$-ka > -kN > -\frac{N}{2}, \text{ for } k \le 0.$$

Combining these results we see that for $x \in [\frac{n}{b} - a, \frac{n}{b}]$ the function $g(x - \frac{n}{b} - ka)$ is non-zero for $-1 \le k \le 0$. Hence we can simplify (5.5), and the equations we have to check are

$$h(x)g(x - \frac{n}{b}) + h(x+a)g(x - \frac{n}{b} + a) = \delta_{n,0}b, \quad n \in \mathbb{Z}. \tag{5.14}$$

In (5.14) we still have to check an infinite number of equations. However, we can also reduce this to a finite number. By definition $\operatorname{supp} h \subseteq [-\frac{N}{2} - \kappa a, \frac{N}{2} + \kappa a]$. Using the relations $\frac{N}{2} \le a$ and $\kappa \le M - 1$, we see that

$$\frac{N}{2} + \kappa a \le a + \kappa a \le a + (M-1)a = aM.$$

Thus $\operatorname{supp} h \subseteq [-aM, aM]$.

Now we can determine how far we have to translate g before its support no longer overlaps with the support of h. By comparing $\operatorname{supp} h \subseteq [-aM, aM]$ and $\operatorname{supp} g(x - \frac{n}{b}) = [\frac{n}{b} - \frac{N}{2}, \frac{n}{b} + \frac{N}{2}]$ we can see that the two do not overlap when

$$\frac{n}{b} - \frac{N}{2} > aM.$$

This happens for $n > M - 1$, since this gives

$$\frac{n}{b} - \frac{N}{2} > (M-1)a - \frac{N}{2} \ge (M-1)a - a = aM.$$

Since both functions are symmetric, this also means that the supports of $h(x)$ and $g(x - \frac{n}{b})$ will not overlap when $n < -(M-1)$. Thus it is proved that we only need to consider (5.14) for $n = 0, \pm 1, \pm 2, \ldots, \pm(M-1)$.

For $n = 0$ the equation (5.14) is satisfied since we have

$$h(x)g(x) + h(x+a)g(x+a) = \frac{b}{g(x)}g(x) + 0 \cdot g(x+a) = b, \; x \in]-\frac{a}{2}, \frac{a}{2}].$$

For $n = 1, 2, \ldots, \kappa$, we separate the interval $[\frac{n}{b} - a, \frac{n}{b}]$ into the two cases $[\frac{n}{b} - a, \frac{N}{2} + a(n-1)]$ and $]\frac{N}{2} + a(n-1), \frac{n}{b}[$. We start by looking at $x \in]\frac{N}{2} + a(n-1), \frac{n}{b}[$. Then we will have $x + a \in]\frac{N}{2} + an, \frac{n}{b} + a[\subset]\frac{N}{2} + an, \frac{n+1}{b}[$. By definition of h this means that

$$h(x)g(x - \frac{n}{b}) + h(x+a)g(x - \frac{n}{b} + a) = 0 \cdot g(x - \frac{n}{b}) + 0 \cdot g(x - \frac{n}{b} + a) = 0,$$

for $x \in]\frac{N}{2} + a(n-1), \frac{n}{b}[$ and $n = 1, \ldots, \kappa$. Now we look at $x \in [\frac{n}{b} - a, \frac{N}{2} + a(n-1)]$. Here we have $x + a \in [\frac{n}{b}, \frac{N}{2} + na]$. Thus by definition of h we have

$$
\begin{aligned}
&h(x)g(x - \frac{n}{b}) + \left(-\frac{g(x - \frac{n}{b})h(x)}{g(x - \frac{n}{b} + a)} \right) g(x - \frac{n}{b} + a) \\
&= h(x)g(x - \frac{n}{b}) - g(x - \frac{n}{b})h(x) = 0
\end{aligned}
$$

for $x + a \in [\frac{n}{b}, \frac{N}{2} + na]$ and $n = 1, \ldots, \kappa$. Hence we have shown that g and h satisfy (5.14) for $x \in [\frac{n}{b} - a, \frac{n}{b}]$ and $n = 1, \ldots, \kappa$.

For $n = \kappa, \ldots, (M-1)$, we have $h(x) = h(x+a) = 0$ for $x \in [\frac{n}{b} - a, \frac{n}{b}]$. We can see this by looking at the values of h that are involved. We have $x \in [\frac{n}{b} - a, \frac{n}{b}]$ and $x + a \in [\frac{n}{b}, \frac{n}{b} + a]$. By the definition of κ we have $b\frac{N}{2} < (\kappa+1)(1-ab)$, thus $a\kappa + \frac{N}{2} < \frac{(\kappa+1)}{b} - a$. Hence $\left([\frac{n}{b} - a, \frac{n}{b} + a]\right) \cap \operatorname{supp} h = \emptyset$. Therefore we have

$$
h(x)g(x - \frac{n}{b}) + h(x+a)g(x - \frac{n}{b} + a) = 0 \cdot g(x - \frac{n}{b}) + 0 \cdot g(x - \frac{n}{b} + a) = 0,
$$

for $x \in [\frac{n}{b} - a, \frac{n}{b}]$ and $n = \kappa + 1, \ldots, (M-1)$.

We have now proved that the equations (5.14) hold for $n = 0, 1, 2, \ldots, (M-1)$. Since both g and h are symmetric around the x-axis this means that they also hold for $n = -1, -2, \ldots, -(M-1)$. Thus we have shown that the equations (5.14) are satisfied. Hence h is a dual generator for g, and the Gabor system $\mathcal{G}(B_N, a, b)$ with the given conditions is a frame. □

Using oversampling on the result from Theorem 5.14, we can extend the area of the frame set.

Corollary 5.15. *Let $N \geq 2$ then $\mathcal{G}(B_N, a, b)$ is a frame if there exists a $k \in \mathbb{N}$ such that $\frac{1}{N} < b < \frac{2}{N}$, $\frac{N}{2} \leq ak < \frac{1}{b}$.*

Proof. Let $N \geq 2$ and assume that

$$
\frac{1}{N} < b < \frac{2}{N}, \; \frac{N}{2} \leq ak < \frac{1}{b}.
$$

This is the same as the condition

$$
\frac{1}{N} < b < \frac{1}{a}, \; \frac{N}{2} \leq ak < N.
$$

By Theorem 5.14 this implies that $\mathcal{G}(B_N, ak, b)$ is a frame. Hence, by Lemma 4.7, it implies that $\mathcal{G}(B_N, a, b)$ is also a frame. □

5.5 The dual frame method: New results

From Theorems 5.13 and 5.14 we know that for each $N \geq 2$ the B-splines generate frames for a, b such that $\frac{N}{2} \leq a < N$ and $\frac{1}{N} \leq b < \frac{1}{a}$. With the result in Lemma 4.7, we have shown that this implies that for each $N \geq 2$ the B-splines also generate frames for a, b if there exists a $k \in \mathbb{N}$ such that $\frac{N}{2} \leq ka < N$ and $\frac{1}{N} \leq b < \frac{1}{a}$.

Theorem 5.16. *Let $N \geq 2$, $0 < a < \frac{N}{2}$, $0 < b < \frac{1}{a}$ and $\frac{N}{2} \leq \frac{1}{b} - \frac{a}{2}$. Then the Gabor system $\mathcal{G}(B_N, a, b)$ is a frame and the function*

$$h(x) = \begin{cases} \frac{b}{B_N(x)}, & x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0, & \textit{otherwise} \end{cases}$$

generates a dual frame.

Proof. Let B_N be a B-spline with $N \geq 2$. Assume $0 < a < \frac{N}{2}$, $0 < b < \frac{1}{a}$ and $\frac{N}{2} \leq \frac{1}{b} - \frac{a}{2}$. Define the function h as

$$h(x) = \begin{cases} \frac{b}{B_N(x)}, & x \in [-\frac{a}{2}, \frac{a}{2}] \\ 0, & \textit{otherwise} \end{cases}$$

The Gabor system $\mathcal{G}(h, a, b)$ is clearly a Bessel sequence, and therefore we can apply Theorem 5.11. To show that $\mathcal{G}(B_N, a, b)$ and $\mathcal{G}(h, a, b)$ generate dual frames, we will show that they satisfy the equations (5.5). Since the support of h is only on $[-\frac{a}{2}, \frac{a}{2}]$, we will only have one term in the sum in (5.5) for $x \in [-\frac{a}{2}, \frac{a}{2}]$. Thus the equations we have to check are

$$g\left(x - \frac{n}{b}\right) h(x) = b\delta_{n,0}, \quad \textit{a.e. } x \in [-\frac{a}{2}, \frac{a}{2}],$$

for $n \in \mathbb{Z}$. For $n = 0$ the equation is easily satisfied since

$$g\,(x)\, h(x) = g\,(x)\, \frac{b}{g(x)} = b, \quad \textit{a.e. } x \in [-\frac{a}{2}, \frac{a}{2}].$$

For $n \geq 1$ the support of $h(x)$ and $g(x - \frac{n}{b})$ do not overlap since a and b satisfy the inequality

$$\frac{a}{2} \leq \frac{1}{b} - \frac{N}{2} \leq \frac{n}{b} - \frac{N}{2}.$$

Thus $\mathcal{G}(B_N, a, b)$ and $\mathcal{G}(h, a, b)$ generate dual frames. More importantly it shows that B_N is a frame when $0 < a < \frac{N}{2}$, $\frac{1}{N} < b < \frac{1}{a}$ and $\frac{a}{2} \leq \frac{1}{b} - \frac{N}{2}$. □

To further investigate the area of the (a, b)-plane where we do not have results about the frame property, we have looked at some specific examples that lie above the curve $\frac{a}{2} = \frac{1}{b} - \frac{N}{2}$, that is, (a, b)-values satisfying $\frac{N}{2} > \frac{1}{b} - \frac{a}{2}$. Both the following examples use a similar way of defining the dual h to what we have done previously. In Example 5.17 we get a dual function with bounded support. In Example 5.18 we see a dual function that does not have bounded support, but it still lies in $L^2(\mathbb{R})$.

Example 5.17. *First we look at an example where we have $a = \frac{6}{7}$ and $b = \frac{14}{17}$. With these values of a and b we have $\frac{1}{b} - \frac{N}{2} = \frac{17}{14} - 1 = \frac{3}{14} \ngeq \frac{3}{7} = \frac{a}{2}$. Hence we are outside the area proven in Theorem 5.16. Furthermore, there does not exist a $k \in \mathbb{N}$ such that $\frac{N}{2} \leq ak < \frac{1}{b}$, i.e., $1 \leq \frac{6}{7}k < \frac{17}{14}$. Hence, we are outside the area proved in Theorem 5.14 and Corollary 5.15.*

To generate a dual we first consider satisfying the equation (5.5) for $n = 0$. Therefore we define

$$h(x) = \frac{b}{B_2(x)}, \quad \textit{for } x \in [-\frac{a}{2}, \frac{a}{2}].$$

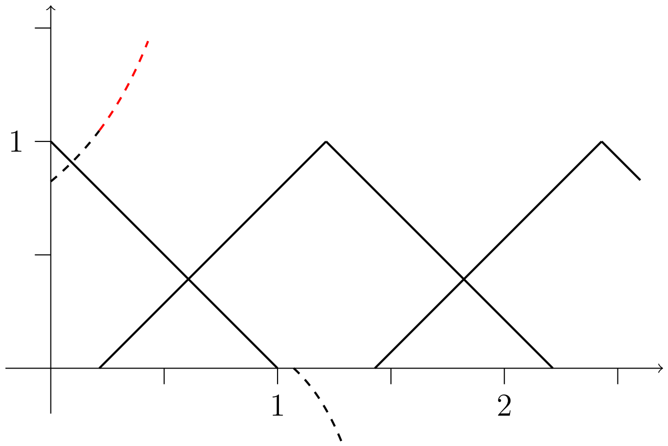


Figure 11: Creation of a dual generator h (dashed) for B_2 with $a = \frac{6}{7}$ and $b = \frac{14}{17}$. The red parts of h are the parts that overlap with more than one of the translates of B_2.

This way we have ensured that

$$\sum_{k\in\mathbb{Z}} h(x-ka)B_2(x-ka) = h(x)B_2(x) = \frac{b}{B_2(x)}B_2(x) = b, \quad \textit{for } x \in [-\frac{a}{2}, \frac{a}{2}].$$

Thus (5.5) will be satisfied for $n = 0$ and all $x \in \mathbb{R}$. However, we now have a piece of h that overlaps with the support of $B_2(x - \frac{1}{b})$ and one that overlaps with the support of $B_2(x + \frac{1}{b})$. Since we will make h symmetric we will focus on the part that overlaps with $B_2(x - \frac{1}{b})$ as shown in Figure 11.

To cancel out this part we define h on some translate of the interval $x \in [\frac{1}{b} - 1, \frac{a}{2}]$ which is where h and $B(x - \frac{1}{b})$ overlap. We want to translate with a multiple of a since all terms in (5.5) get translated by multiples of a. Furthermore we do not want the interval to overlap with the support of $B2(x)$. For the values of a and b used in this example we can translate the interval by a. Thus we define h on

$$x \in [\frac{1}{b} - 1 + a, \frac{a}{2} + a] = [\frac{17}{14} - 1 + \frac{6}{7}, \frac{3}{7} + \frac{6}{7}] = [\frac{15}{14}, \frac{9}{7}].$$

The way we define h on that interval is similar to what we did in the previous examples. When we multiply h by $B_2(x - \frac{1}{b})$ we want to get something that cancels out with the contribution from $B_2(x - \frac{1}{b})h(x)$ on $x \in [\frac{1}{b} - 1, \frac{a}{2}]$. Thus we define h to be

$$h(x) = -\frac{h(x-a)B_2(x - \frac{1}{b} - a)}{B_2(x - \frac{1}{b})}, \quad \textit{for } x \in [\frac{1}{b} - 1 + a, \frac{a}{2} + a] = [\frac{15}{14}, \frac{9}{7}].$$

This way we will get

$$h(x)B_2\left(x - \frac{1}{b}\right) = \begin{cases} h(x)B_2(x - \frac{1}{b}) & x \in [\frac{1}{b} - 1, \frac{a}{2}], \\ -h(x-a)B_2(x - \frac{1}{b} - a) & x \in [\frac{1}{b} - 1 + a, \frac{a}{2} + a], \\ 0 & \textit{otherwise}. \end{cases}$$

So when we look at (5.5) for $n = 1$ we get

$$\sum_{k\in\mathbb{Z}} h(x-ka)B_2(x - \frac{1}{b} - ka) = h(x-ka)g(x - \frac{1}{b} - ka) - h(x-ka)B_2(x - \frac{1}{b} - ka) = 0,$$

for $x \in [\frac{1}{b} - 1 + ka, \frac{a}{2} + ka]$ *and all* $k \in \mathbb{Z}$. *For* $x \notin [\frac{1}{b} - 1 + ka, \frac{a}{2} + ka]$, $k \in \mathbb{Z}$ *we will have*

$$\sum_{k \in \mathbb{Z}} h(x - ka) B_2(x - \frac{1}{b} - ka) = \sum_{k \in \mathbb{Z}} 0 = 0.$$

Thus the equation (5.5) is satisfied for $x \in \mathbb{R}$ *and* $n = 1$. *Since* B_2 *is symmetric, we can define* $h(x) = h(-x)$ *for* $x \in [-\frac{a}{2} - a, -\frac{1}{b} + 1 - a]$, *and then (5.5) is also satisfied for* $x \in \mathbb{R}$ *and* $n = -1$.

We can see that $\frac{a}{2} + a = \frac{9}{7} \leq \frac{20}{14} = \frac{2}{b} - 1$)*.Hence the function* h *does not overlap with* $B_2(x - \frac{n}{b}$ *for* $n = \pm 2, \pm 3, \ldots$. *Thus (5.5) is satisfied for* $x \in \mathbb{R}$ *and* $n = \pm 2, \pm 3, \ldots$. *Since* $\mathcal{G}(h, a, b)$ *is a Bessel sequence, this means that the functions* B_2 *and* h *generate dual frames and thus* $\mathcal{G}(B_2, \frac{6}{7}, \frac{14}{17})$ *is a frame.*

Example 5.18. *Now we look at another case where* $a = \frac{4}{7}$ *and* $b = \frac{7}{8}$. *Once again we are outside the are considered in Theorem 5.16 since* $\frac{1}{b} - \frac{N}{2} = \frac{8}{7} - 1 = \frac{1}{7} \ngeq \frac{2}{7} = \frac{a}{2}$. *Furthermore, there does not exist a* $k \in \mathbb{N}$ *such that* $\frac{N}{2} \leq ak < \frac{1}{b}$, *i.e.,* $1 \leq \frac{4}{7}k < \frac{7}{8}$. *Hence, we are outside the areas proved in Theorem 5.14 and Corollary 5.15.*

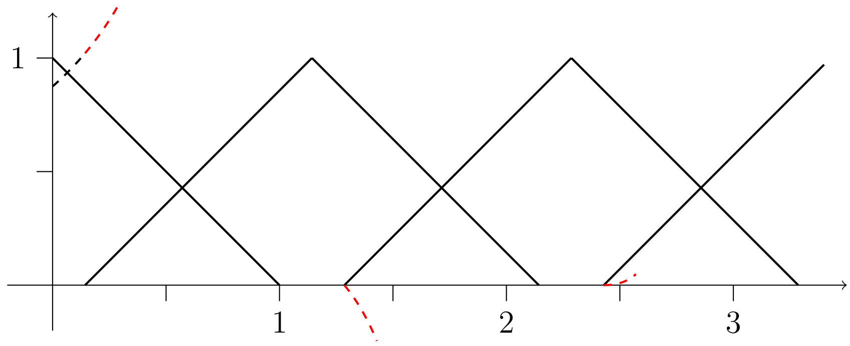


Figure 12: Creation of a dual generator for B_2 with $a = \frac{4}{7}$ and $b = \frac{7}{8}$. The red parts of h are the parts that overlap with more than one of the translates of B_2

First we start like we did in Example 5.17 by defining h *on the interval* $x \in [-\frac{a}{2}, \frac{a}{2}]$ *as* $h(x) = \frac{b}{B_2(x)}$. *Just as before this ensures that*

$$\sum_{k \in \mathbb{Z}} h(x - ka) B_2(x - ka) = h(x) g(x) = \frac{b}{B_2(x)} B_2(x) = b, \quad \textit{for } x \in [-\frac{a}{2}, \frac{a}{2}].$$

Thus (5.5) will be satisfied for $n = 0$ *and all* $x \in \mathbb{R}$.

As indicated by the red part on the figure there is a part of h *that overlaps with* $B_2(x - \frac{1}{b})$ *and therefore there will also be a part on the negative axis that overlaps with* $B_2(x + \frac{1}{b})$. *To cancel these out we need to define* h *on an interval that is a translation of the overlap, i.e.,* $x \in [\frac{1}{b} - \frac{N}{2} + ka, \frac{a}{2} + ka], /k \in \mathbb{N} \setminus \{0\}$. *Furthermore,* k *needs to be chosen such that*

$$\left[\frac{1}{b} - \frac{N}{2} + ka, \frac{a}{2} + ka\right] \cap \left[-\frac{N}{2}, \frac{N}{2}\right] = \emptyset, \tag{5.15}$$

and

$$\left[\frac{1}{b} - \frac{N}{2} + ka, \frac{a}{2} + ka\right] \subset \left[\frac{1}{b} - \frac{N}{2}, \frac{1}{b} + \frac{N}{2}\right]. \tag{5.16}$$

If we choose $k = 1$ then (5.15) does not hold since

$$\frac{1}{b} - \frac{N}{2} + ka = \frac{8}{7} - 1 + \frac{4}{7} = \frac{5}{7} \leq 1 = \frac{N}{2}.$$

And thus $\left[\frac{1}{b} - \frac{N}{2} + a, \frac{a}{2} + a\right] \cap \left[-\frac{N}{2}, \frac{N}{2}\right] \neq \emptyset$ *and we need to choose a larger k. For $k = 2$ the two conditions (5.15) and (5.16) both hold. In general we do not have to try k's until we find a suitable one. A sufficiently large k can be found by setting*

$$k = \lceil \left(\frac{N}{2} - \left(\frac{1}{b} - \frac{N}{2} \right) \right) /a \rceil.$$

We then define h as

$$h(x) = -\frac{h(x-2a)B_2(x - \frac{1}{b} - 2a)}{B_2(x - \frac{1}{b})}, \quad \text{for } x \in \left[\frac{1}{b} - 1 + 2a, \frac{a}{2} + 2a\right] = \left[\frac{9}{7}, \frac{10}{7}\right].$$

This way we will get

$$h(x)B_2\left(x - \frac{1}{b}\right) = \begin{cases} h(x)B_2(x - \frac{1}{b}) & x \in [\frac{1}{b} - 1, \frac{a}{2}], \\ -h(x-2a)B_2(x - \frac{1}{b} - 2a) & x \in [\frac{1}{b} - 1 + 2a, \frac{a}{2} + 2a], \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

$$\sum_{k \in \mathbb{Z}} h(x - ka)B_2(x - \frac{1}{b} - ka) = h(x-ka)g(x - \frac{1}{b} - ka) - h(x-ka)g(x - \frac{1}{b} - ka) = 0,$$

for $x \in [\frac{1}{b} - 1 + ka, \frac{a}{2} + ka]$. *If we have* $x \notin [\frac{1}{b} - 1 + ka, \frac{a}{2} + ka]$ *then*

$$\sum_{k \in \mathbb{Z}} h(x - ka)B_2(x - \frac{1}{b} - ka) = \sum_{k \in \mathbb{Z}} 0 = 0.$$

Thus, the equation (5.5) is satisfied for $n = 1$ and $x \in \mathbb{R}$. If we define

$$h(x) = h(-x), \quad \text{for } x \in \left[-\left(\frac{a}{2} + 2a\right), -\left(\frac{1}{b} - 1 + 2a\right)\right] = \left[-\frac{10}{7}, \frac{9}{7}\right],$$

then, due to the symmetry of $B_2(x)$, (5.5) is also satisfied for $n = -1$ and $x \in \mathbb{R}$.

As we can see in Figure 12 the new parts of h overlap with the functions $B_2(x - \frac{2}{b})$ and $B_2(x + \frac{2}{b})$. Therefore we need to define h on another interval to cancel out its contribution in the equation (5.5) with $n = 2$. Notice, that we have $(1/b)/a = \frac{8}{7}\frac{7}{4} = 2$. This means that when we defined h on $\left[\frac{1}{b} - 1 + 2a, \frac{a}{2} + 2a\right]$ *then that interval starts exactly at the left starting point of the support of $B_2(x - \frac{2}{b})$. So to cancel out the contribution that h gives to (5.5) with $n = 2$, we will have to translate the interval* $\left[\frac{1}{b} - 1 + 2a, \frac{a}{2} + 2a\right]$ *by another $2a$ and define h similarly to the way we did on* $\left[\frac{1}{b} - 1 + 2a, \frac{a}{2} + 2a\right]$. *The translated interval then has its left end point in exactly* *the same point as the left end point of the support of $B_2(x-\frac{3}{b})$. Thus to satisfy (5.5) for all $n \in \mathbb{Z}$ we will need to define h on the positive axis as*

$$h(x) = \begin{cases} \frac{b}{B_2(x)}, & x \in [0, \frac{a}{2}], \\ -\frac{h(x-2a)B_2(x-\frac{n}{b}-2a)}{B_2(x-\frac{n}{b})} & x \in \left[\frac{n}{b}-1+2a, \frac{a}{2}+2an\right],\ n \in \mathbb{N}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then h is defined on the negative axis as $h(x) = h(-x)$. This means that the support of h will not be bounded. However, the condition that we have on h is that it must lie in $L^2(\mathbb{R})$. We know that on the interval $\left[\frac{n}{b}-1+2a, \frac{a}{2}+2an\right]$ we will have $B_2(x-\frac{n}{b}-2a) \leq \frac{1}{7}$ and $B_2(x-\frac{n}{b}) \geq 1-\frac{2}{7}=\frac{2}{7}$. Thus we have

$$-\frac{h(x-2a)B_2(x-\frac{n}{b}-2a)}{B_2(x-\frac{n}{b})} \leq -h(x-2a)\frac{(1/7)}{(2/5)} = -\frac{1}{5}h(x-2a),$$

for $x \in \left[\frac{n}{b}-1+2a, \frac{a}{2}+2an\right]$. Therefore $h(x) \to 0$ as $x \to \pm\infty$. To prove that $h \in L^2(\mathbb{R})$ we must show that $\int_{-\infty}^{\infty}|h(x)|^2dx < \infty$. Since h is an even function we have

$$\int_{-\infty}^{\infty}|h(x)|^2dx = 2\int_0^{\infty}|h(x)|^2dx.$$

Furthermore, we can split the interval so we get

$$2\int_0^{\infty}|h(x)|^2dx = 2\left(\int_0^{\frac{a}{2}}|h(x)|^2dx + \sum_{n=1}^{\infty}\int_{\frac{n}{b}-1+2a}^{\frac{a}{2}+2an}|h(x)|^2dx\right).$$

On $[0, \frac{a}{2}]$ we have a bounded function on a bounded interval and therefore we can find a constant $0 < C$ such that $|h(x)| \leq C$, for all $x \in [0, \frac{a}{2}]$. Then we will also have $|h(x)| \leq \left(\frac{1}{5}\right)^n C$ for $x \in \left[\frac{n}{b}-1+2a, \frac{a}{2}+2an\right]$. Thus we get

$$\begin{aligned} 2\left(\int_0^{\frac{a}{2}}|h(x)|^2dx + \sum_{n=1}^{\infty}\int_{\frac{n}{b}-1+2a}^{\frac{a}{2}+2an}|h(x)|^2dx\right) &\leq 2\left(\int_0^{\frac{a}{2}}C^2dx + \sum_{n=1}^{\infty}\int_{\frac{n}{b}-1+2a}^{\frac{a}{2}+2an}\left(\left(\frac{1}{5}\right)^n\right)^2dx\right) \\ &= 2\left(\frac{a}{2}C^2 + \sum_{n=1}^{\infty}\frac{1}{7}\left(\frac{1}{5}\right)^{2n}\right) = aC^2 + \frac{2}{7}\sum_{n=1}^{\infty}\left(\frac{1}{25}\right)^n \\ &= aC^2 + \frac{2}{7}\frac{1}{1-1/25} < \infty. \end{aligned}$$

Thus we have shown that h lies in $L^2(\mathbb{R})$. In order to use Theorem 5.11 we should also show that h is a Bessel Sequence. We will not do that here, but if h is a Bessel sequence then B_2 and h generate dual frames since they satisfy (5.5). Particularly this would show that the Gabor system $\mathcal{G}(B_2, \frac{4}{7}, \frac{7}{8})$ is a frame.

Examples 5.17 and 5.18 prove that two specific points outside the known area of the frame set of B_N, $N \geq 2$. For now we have not found a general way of producing dual functions outside the known areas of the frame set. However, it does seem like it might be possible to generate duals for B_N in a way similar to what has been done in this subsection as long as $a < \frac{N}{2}$ and $\frac{1}{N} < b < \frac{2}{N}$. Figure 13 shows the two functions $B_2(x)$ and $B_2(x-\frac{n}{b})$ when $\frac{1}{N} < b < \frac{2}{N}$. In general it will hold for all $N \geq 2$ that the condition $\frac{1}{N} < b < \frac{2}{N}$ ensures that at most two translates of the B-spline B_N will overlap for any given point $x \in \mathbb{R}$.

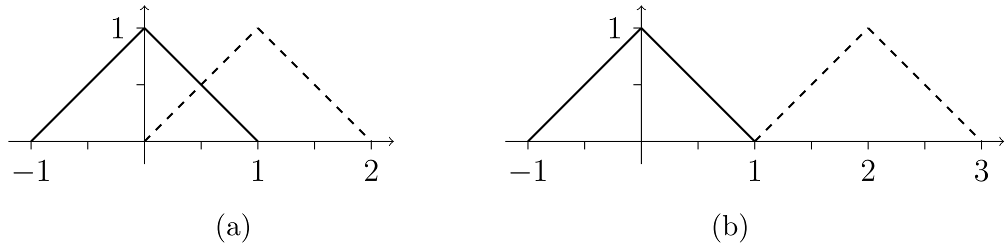


Figure 13: $B_2(x)$ (solid) and $B_2(x - \frac{1}{b})$ (dashed) for $b = \frac{2}{N}$ and $\frac{1}{N}$.

5.6 Numerical methods

There are still areas of the (a, b)-plane that have not been proven to be in frame set $\mathcal{F}(B_N)$. In Subsection 5.1 we introduced the conjecture by Gröchenig that the frame set of the B-splines for $N \geq 2$ consists of all the points $(a, b) \in \mathbb{R}^2_+$ that avoid the known obstructions. It is interesting to investigate whether this conjecture seems to hold. We can do this numerically by using the Zibulski-Zeevi matrix and the results in Theorem 4.5. In this analysis we will restrict our attention to B_2 as it has the simplest form.

Our approach is as follows. We choose several rational values of a and b such that $ab < 1$. When a and b are rational their product ab will also be rational. For the plots in Figure 14 we have chosen $a = 0.1, 0.2, 0.3, 0.4, 0.5$. For each value of a we choose b such that it starts from 0.1 and is increased in steps of 0.1 until it reaches a value such that $ab \geq 1$ or $b = 5$. For each point (a, b) we calculate the Zibulski-Zeevi matrix on a grid spanned by 25 evenly spaced points along both the t- and the ν-axis. For each Zibulski-Zeevi matrix we calculate the singular vales and record the smallest one. Once the results are found for all points on the (t, ν) grid, we find the smallest of them all and that is stored for the point (a, b). For each point (a, b) Figure 14 shows the minimum value of σ_p of the Zibulski-Zeevi matrix on a 25×25 grid in (t, ν). This value is an estimate of $\sqrt{A}$. Hence, if it goes to zero it also indicates that $A = 0$, and thus that $\mathcal{G}(B_2, a, b)$ is not a frame in such a point.

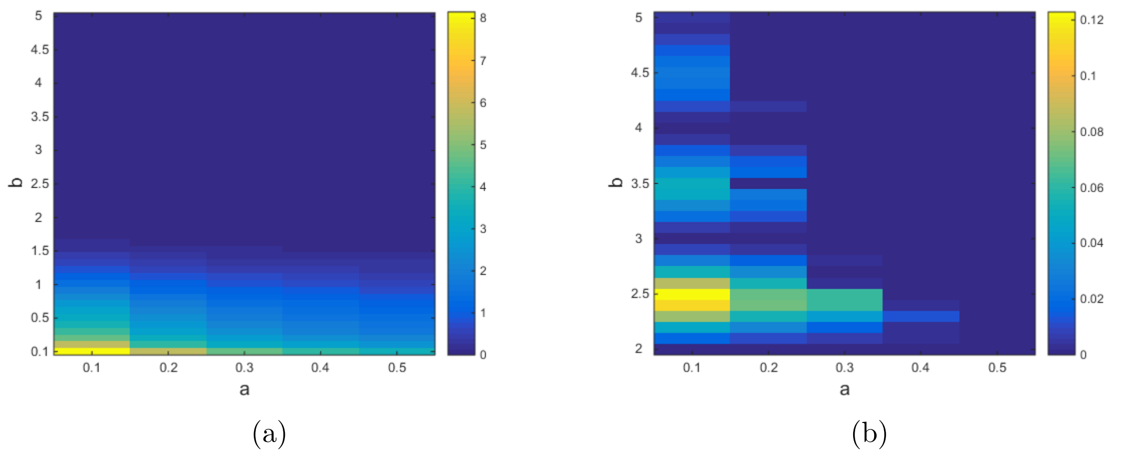
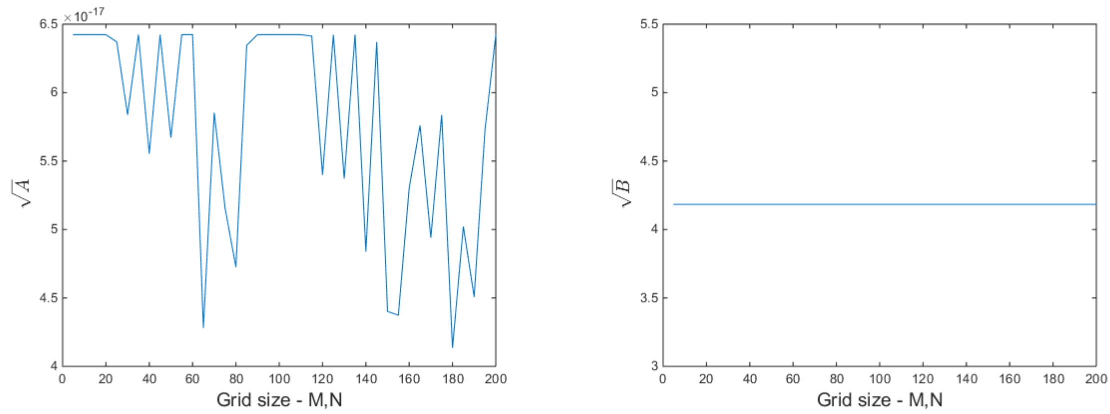


Figure 14: Plots of the estimate of $\sqrt{A}$.

Figure 14a shows the full result for $(a, b) \in [0, 0.5] \times [0, 5]$. However, the values for $b > 1.5$ are smaller compared to those where $b < 1.5$ which is why they appear to be zero in this plot. A log plot of the values did not look good, therefore Figure 14b shows the same results but zooms in on the a smaller area where $(a, b) \in [0, 0.5] \times [2, 5]$, and we can see that there are indeed non-zero values here.

From Theorem 4.5 we know that if the smallest singular value we find is greater than zero then the Gabor system $\mathcal{G}(B_N, a, b)$ is a frame. So the results in Figure 14 mostly seem to agree with the conjecture as most values are positive except for the lines $b = 2, 3, 4, 5, 6, 7$. However, the point $(a, b) = (\frac{7}{2}, \frac{1}{5})$ does stand out as it has value zero up to machine precision. If the conjecture holds we would expect it to be positive as $ab = \frac{7}{2}\frac{1}{5} = \frac{7}{10} < 1$ and it is not in any of the known non-frame areas. Therefore we will examine this point more carefully.

First of all we look at the specific point with different grid sizes to see if the result could be affected by the grid being to coarse. In Figure 15 we see the results of increasing the grid size.



(a) (b)

Figure 15: Plots of the smallest and largest singular values as a function of the grid size.

In Figure 15 we see that the values of both the first and the pth singular value are the same for the different grid sizes. Indeed σ_p seems to be equal to zero up to machine precision for all grid sizes. In Figure 16 we have plotted the value of σ_p for the Zibulski-Zeevi matrix with $(t, \nu) \in [0, \frac{1}{p}] \times [0, 1]$ and a grid size of 200. The reason we can consider $(t, \nu) \in [0, \frac{1}{p}] \times [0, 1]$ rather than $(t, \nu) \in [0, 1]^2$ is down to the 1-periodicity of the Zak transform. From the figure we can see why the size of the smallest value of σ_p does not change with the grid size. The smallest of σ_p is found at the edge of the grid where $t = 0$ or $t = 1$ and those points will always be included in the grid. Also, it is worth noting that if we have already found a place where the smallest value of σ_p across the grid was zero then that will still be there when we increase the grid size.

The grid used in Figure 14 was very coarse. So it could be interesting to look at a finer grid. However, as we increase the number of points then the matrices for which we need to find the singular values generally get larger as well. Therefore we focus on the line where $b = 3.5$ and calculate the smallest singular values for different values of a. The result of this can be seen in Figure 17.

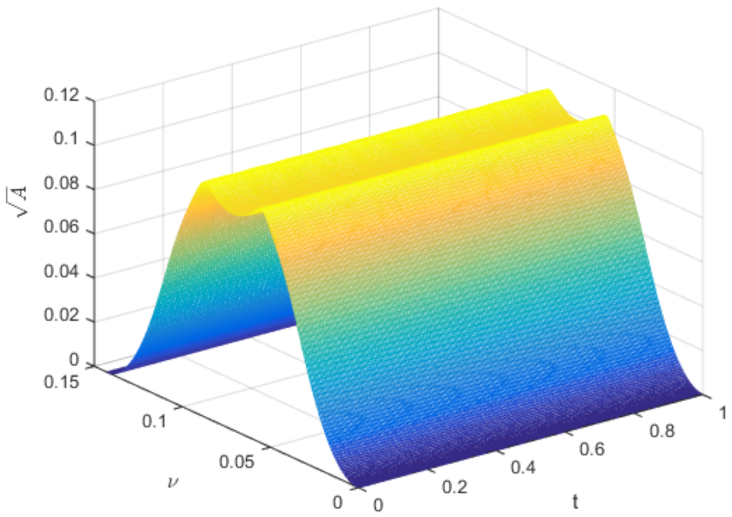
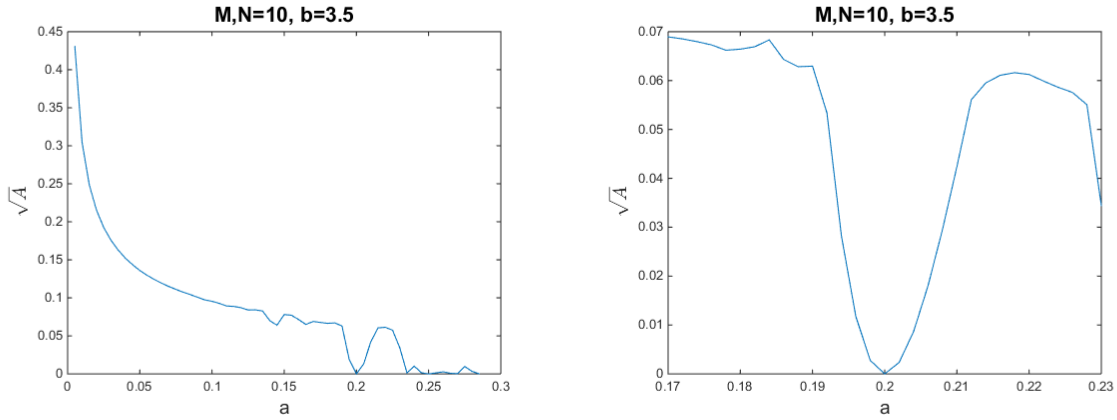


Figure 16: A plot of σ_p for $(t, \nu) \in [0, \frac{1}{p}] \times [0, 1]$ with 200 points in both the t and the ν direction.



(a)

(b)

Figure 17: Plots of $\sqrt{A}$ for $b = 3.5$ and $0 < a < \frac{1}{b}$.

In Figure 17a we see that the minimal value of σ_p found on the (t, ν) grid decreases in a seemingly smooth way and flattens out until we get close to the point $a = \frac{1}{5}$. Here the value decreases quicker until it reaches zero in $a = \frac{1}{5}$ and starts increasing again. The de- and increase is quicker than in the first part of the plot, but it still seems smooth. Figure 17b shows a smaller interval just around $a = \frac{1}{5}$ with more points than there was in that area in Figure 17a. Here we see that smallest value of σ_p, and thus the frame bound, seems to be zero in the point $a = \frac{1}{5}$ and non-zero on the interval just around $a = \frac{1}{5}$. There may also be some points where the smallest value of σ_p is zero for $a > \frac{1}{5}$, but it is not clear from Figure 17a.

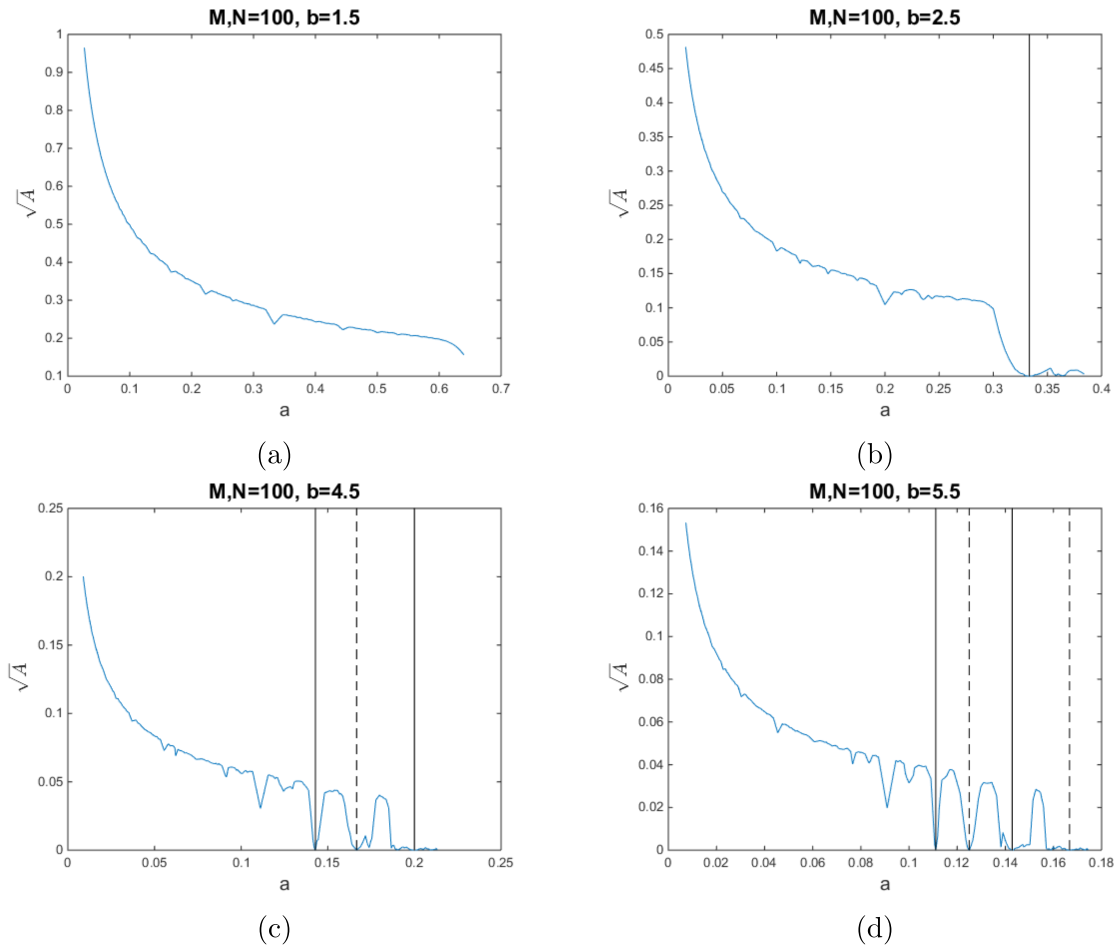


Figure 18: Plots of the estimate of $\sqrt{A}$ based on the Zibulski-Zeevi matrix for $b = 1.5, 2.5, 4.5, 5.5$ and $0 < a < \frac{1}{b}$.

Since we found a zero for $b = \frac{7}{2}$ it would be interesting to see if we also get zeros for other values of b. In Figure 18 we plotted the smallest singular value found for other fixed b's of the form $b = \frac{2m+1}{2}$ for $m \in \mathbb{N}$ and $0 < a < \frac{1}{b}$.

Figure 18a shows the value of the smallest singular value for $b = 1.5$ and $0 < a < \frac{1}{b}$. We see that there does not seem to be any points where the smallest singular value and thus the lower frame bound goes to zero.

However, in Figure 18b we have $b = 2.5$ and the $\sqrt{A}$ estimate, which we will refer to as the plot in the following, does seem to drop to zero. We have plotted the estimate of $\sqrt{A}$ and the vertical line on the plot is the line $a = \frac{1}{3}$. The plot appears to go to zero in the point $a = \frac{1}{3}$. At the higher values of a the values are lower, and it is not clear whether they are all non-zero.

In Figure 18c we have $b = 4.5$, and the plot also appears to drop to zero. We have plotted the estimate of $\sqrt{A}$ and from left to right the vertical lines on the plot are $a = \frac{1}{7}$, $a = \frac{1}{6}$ (dashed) and $a = \frac{1}{5}$. Here the lower frame bound appears to be zero in all the points indicated by vertical lines. Once again the vales of A for the higher values of a are lower than the rest and it is difficult to determine if there are more zeros there.

In Figure 18d the plot does seem to drop to zero. We have plotted the estimate of $\sqrt{A}$ and from left to right the vertical lines on the plot are $a = \frac{1}{9}$, $a = \frac{1}{8}$ (dashed), $a = \frac{1}{7}$ and $a = \frac{1}{6}$ (dashed). Here the lower frame bound also appears to be zero in all the points indicated by vertical lines.

We wish to further investigate these points that appear not to be in the frame set. We start with the point $b = 2.5 = \frac{5}{2}$ and $a = \frac{1}{3}$. This gives $ab = \frac{5}{6}$ and we wish to see whether all points along this hyperbola are non-frame points. Therefore we tried to calculate the lower frame bound for $ab = \frac{5}{6}$ and different values of b which gave the plot in Figure 19. Here we see that the lower frame bound appears to be zero on some, but not all, of the hyperbola $ab = \frac{5}{6}$. The non-frame points all seem to have b-values in a symmetric interval around $b = \frac{5}{2}$.

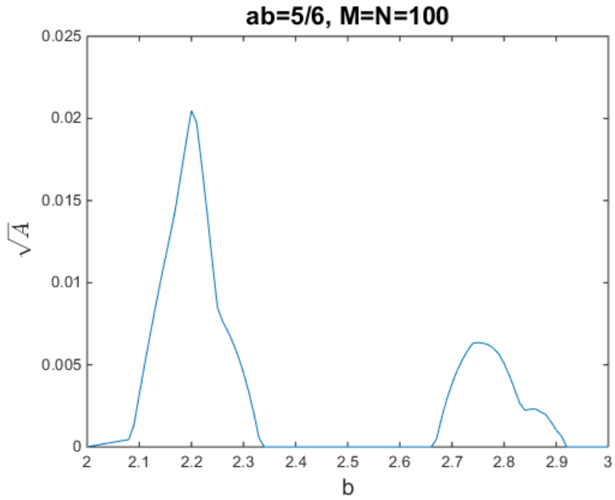
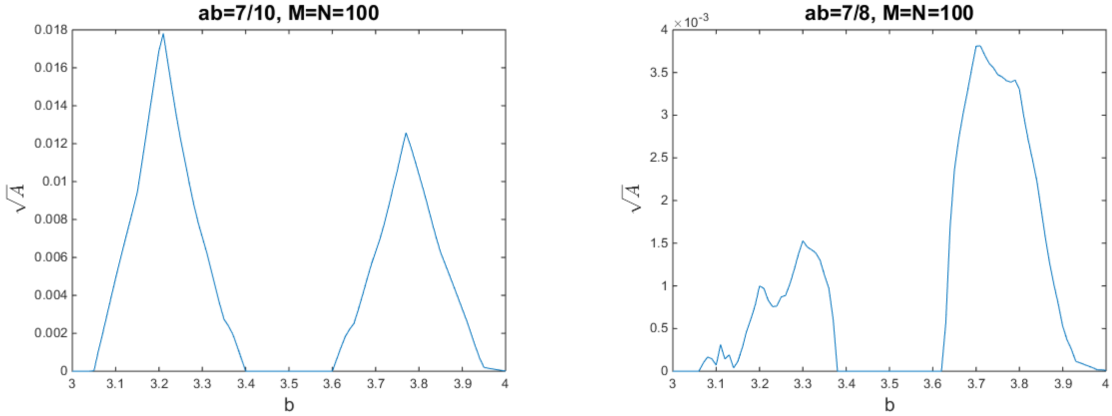


Figure 19: Plot of $\sqrt{A}$ for $ab = \frac{5}{6}$ and $b \in [2, 4]$.



(a) (b)

Figure 20: Plots of the estimate of $\sqrt{A}$ based on the Zibulski-Zeevi matrix for $b \in [3, 4]$ and $ab = \frac{7}{10}, \frac{7}{8}$.

In Figure 18 and Figure 17, we saw that the lower bound seemed to be zero in some points when we had $b = 2.5, 3.5, 4.5$. Now we will examine the two zeros that we have for $b = 3.5$ further. For $b = 3.5$ the two points that appear to not be in the frame set are $a = \frac{1}{5}$ and $\frac{1}{4}$ which give $ab = \frac{7}{10}$ and $ab = \frac{7}{8}$, respectively. We examine the frame bound for points on these hyperbolas with b close to 3.5 and the result is seen in Figure 20. Again we see that there are non-frame points on the hyperbolas $ab = \frac{7}{10}$ and $ab = \frac{7}{8}$ when b is in some symmetric interval around 3.5.

5.7 New non-frame (a, b)-values

We will now prove that Conjecture 1 is false by proving that the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$ is not a frame. We will also state conjectures about further (a, b)-values that do not belong to the frame set of B_2.

Theorem 5.19. *The Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$ is not a frame.*

Proof. Let $a = \frac{1}{3}$ and $b = \frac{5}{2}$ and consider the Gabor system $\mathcal{G}(B_2, a, b)$. We wish to prove that this system is not a frame showing that the Zibulski-Zeevi matrix does not have full rank and thus the lower frame bound in Theorem 4.5 is violated. In Theorem 4.5 it is stated that the lower frame condition needs to hold for a.e. point $(t, \nu) \in [0, 1]^2$. However, since $B_2 \in W(\mathbb{R}) \cup C^0(\mathbb{R})$, we know that the Zak transform is continuous. Therefore, each entry of the Zibulski-Zeevi matrix will be continuous and the singular values of the matrix depend continuously on the entries. Hence, if we prove that the rank is not full in one point, and thus that $\sigma_p = 0$, then we know that the lower frame bound will be violated.

First we remind ourselves of the Zibulski-Zeevi matrix for a rationally oversampled Gabor system $\mathcal{G}(g, a, b)$ with $ab = \frac{p}{q}$ where $\gcd(p, q) = 1$:

$$\Phi^g(t, \nu) = p^{-\frac{1}{2}} \left(\left(Z_{\frac{1}{b}} g \right) (t - \ell \frac{p}{q}, \nu + \frac{k}{p}) \right)_{k=0,\dots,p-1; \ell=0,\dots,q-1}$$

We consider the Zibulski-Zeevi matrix for the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$ in the point $(t, \nu) = (0, 0)$. Since $ab = \frac{1}{3}\frac{5}{2} = \frac{5}{6}$, we get the following 5×6 Zibulski-Zeevi matrix

$$\Phi^{B_2}(0, 0) = 5^{-\frac{1}{2}} \left(\left(Z_{\frac{2}{5}} B_2 \right) (-\ell \frac{5}{6}, \frac{k}{5}) \right)_{k=0,\dots,4; \ell=0,\dots,5}$$

The reduced row echelon form of $\Phi^{B_2}(0, 0)$ is:

$$\Phi^{B_2}(0, 0) \xrightarrow{\text{Gauss-Jordan elimination}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The reduced row echelon form is obtained by Gauss-Jordan elimination. Since the computation and the notation is cumbersome, we perform these algebraic manipulations in Maple in Appendix A.3.

We see that the Zibulski-Zeevi matrix does not have full rank in the point $(t, \nu) = (0, 0)$. Hence, by Theorem 4.5, the lower frame bound is violated and the Gabor system $\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$ is not a frame. □

In the proof we used Maple to calculate the reduced row echelon form of the Zibulski-Zeevi matrix. To double check we also let Maple find a vector in the null space:

$$v = \begin{pmatrix} 0 \\ 5e^{-\frac{4}{5}\pi i} - 3e^{-\frac{2}{5}\pi i} + 3e^{\frac{2}{5}\pi i} - 5e^{\frac{4}{5}\pi i} \\ -3e^{-\frac{4}{5}\pi i} - 5e^{-\frac{2}{5}\pi i} + 5e^{\frac{2}{5}\pi i} + 3e^{\frac{4}{5}\pi i} \\ 3e^{-\frac{4}{5}\pi i} + 5e^{-\frac{2}{5}\pi i} - 5e^{\frac{2}{5}\pi i} - 3e^{\frac{4}{5}\pi i} \\ -5e^{-\frac{4}{5}\pi i} + 3e^{-\frac{2}{5}\pi i} - 3e^{\frac{2}{5}\pi i} + 5e^{\frac{4}{5}\pi i} \end{pmatrix}$$

We also find row operations that will lead to a row of zeros. The first row operations are replacing row two, $R2$, and row three, $R3$, with $R2 - R5$ and $R3 - R4$, respectively. Then the new second row, $R2$, is normalised by dividing with its second element. Similarly, the new third row, $R3$, is normalised by dividing with its second element. Finally, we perform the row operation $R2 - R3$ which yields a row of zeros showing that the Zibulski-Zeevi matrix does not have full rank. Interestingly, the same row operations on the Zibulski-Zeevi matrices of other Gabor systems $\mathcal{G}(B_2, a, b)$ with $\frac{5}{2} - \frac{1}{6} \leq b \leq \frac{5}{2} + \frac{1}{6}$ and $ab = \frac{5}{6}$ also yields a row of zeros.

We have further investigated some of the points from Figure 18 that did not appear to be in the frame set for B_2. The points were investigated both by using Matlab plots like the ones in Figure 19 and Figure 20 as well as using the Maple sheet in Appendix A.3. Based on these investigations we pose Conjecture 2.

Conjecture 2. *The Gabor system $\mathcal{G}(B_2, a_0, b_0)$ is not a frame for the points*

$$a_0 = \frac{1}{2m+1}, \; b_0 = \frac{2n+1}{2}, \; n, m \in \mathbb{N}, \; n > m, \; a_0 b_0 < 1. \tag{5.17}$$

Furthermore, the Gabor system $\mathcal{G}(B_2, a, b)$ is not a frame along the hyperbolas

$$ab = \frac{2n+1}{2\,(2m+1)}, \quad \text{with } b \in \left[b_0 - a_0 \frac{k}{2}, b_0 + a_0 \frac{k}{2}\right] \tag{5.18}$$

for all a_0 and b_0 defined by (5.17).

Conjecture 2 only deals with the non-frame points where a has an odd denominator. Based on the numerical analysis in Subsection 5.6 it seems that there are also non-frame points for values of a with an even denominator.

Conjecture 3. *The Gabor system $\mathcal{G}(B_2, a, b)$ is not a frame for*

$$a = \frac{1}{2m}, \; b = \frac{2n+1}{2}, \; n, m \in \mathbb{N}, \; n > m, \; ab < 1. \tag{5.19}$$

There also appears to be an interval along the hyperbolas corresponding to the points in Conjecture 3, where the Gabor systems $\mathcal{G}(B_2, a, b)$ are not frames. However, we have not been able to determine expressions for these interval b needs to be in.

In both conjectures above we have the condition $n > m$. There was not time to investigate this further. For example it would have been a good idea to estimate the frame bounds for non-integer values of b that were not of the form $\frac{2n+1}{2}$, $n \in \mathbb{N}$. However, from what we have seen so far it would appear that the Gabor system $\mathcal{G}(B_2, a, b)$ is a frame as long as $ab \leq \frac{1}{2}$ and we avoid the usual obstructions.

5.8 Summary of results

We summarise the now known results on the frame set of B-splines B_N with $N \geq 2$ in Proposition 5.20. The results here are only results that give us areas or curves in the (a, b)-plane. There are some points outside these areas that are known to be in the frame set.

Proposition 5.20. *Let $N \in \mathbb{N} \setminus \{1\}$, and consider $a, b > 0$ such that $ab < 1$. Then the following hold*

(i) $\mathcal{G}(B_N, a, b)$ *is not a frame if* $a \geq N$.

(ii) $\mathcal{G}(B_N, a, b)$ *is not a frame if* $b = 2, 3, \dots$.

(iii) $\mathcal{G}(B_N, a, b)$ *is a frame if* $a < N$, $b \leq \frac{1}{N}$.

(iv) $\mathcal{G}(B_N, a, b)$ *is a frame if there exists a* $k \in \mathbb{N}$ *such that*

$$\frac{1}{N} < b < \frac{2}{N}, \ \frac{N}{2} \leq ak < \frac{1}{b}$$

.

(v) $\mathcal{G}(B_N, a, b)$ *is a frame if* $b \in \{1, \frac{1}{2}, \dots, \frac{1}{N-1}\}$.

(vi) $\mathcal{G}(B_N, a, b)$ *is a frame if* $a = \frac{k}{p}$ *for some* $k = 1, \dots, N-1$, $p \in \mathbb{N}$, *and* $b < \frac{1}{k}$.

(vii) $\mathcal{G}(B_N, a, b)$ *is a frame if* $a < N$, *and* $\frac{1}{b} - \frac{N}{2} \geq \frac{a}{2}$.

(i) is Proposition 5.1. (ii) is Corollary 5.3. (iii) is Corollary 5.7. (iv) is Corollary 5.15. The results (v) and (vi) have not been studied in this thesis, for the interested reader we refer to [9]. (v) is by Kloos and Stöckler [9] who also proved (vi) for $p = 1$, the case with $p \in \mathbb{N}$ is an oversampling of the case with $p = 1$. (vii) is our new result from Theorem 5.16.

Furthermore, we proved in Theorem 5.19 that

$$\mathcal{G}(B_2, \frac{1}{3}, \frac{5}{2})$$

is not a frame which proved that Conjecture 1 is not true.

Figure 21 shows the known frame set for the B-splines of order 2 and 3, including non-frame areas. The new non-frame point for B_2 is also included in Figure 21a. For $N = 2$ we see that the vertical lines from (vi) overlap with the lines of the yellow areas that are due to (iv). However, for $N = 3$ these lines lie in different places, and thus add something to the frame set.

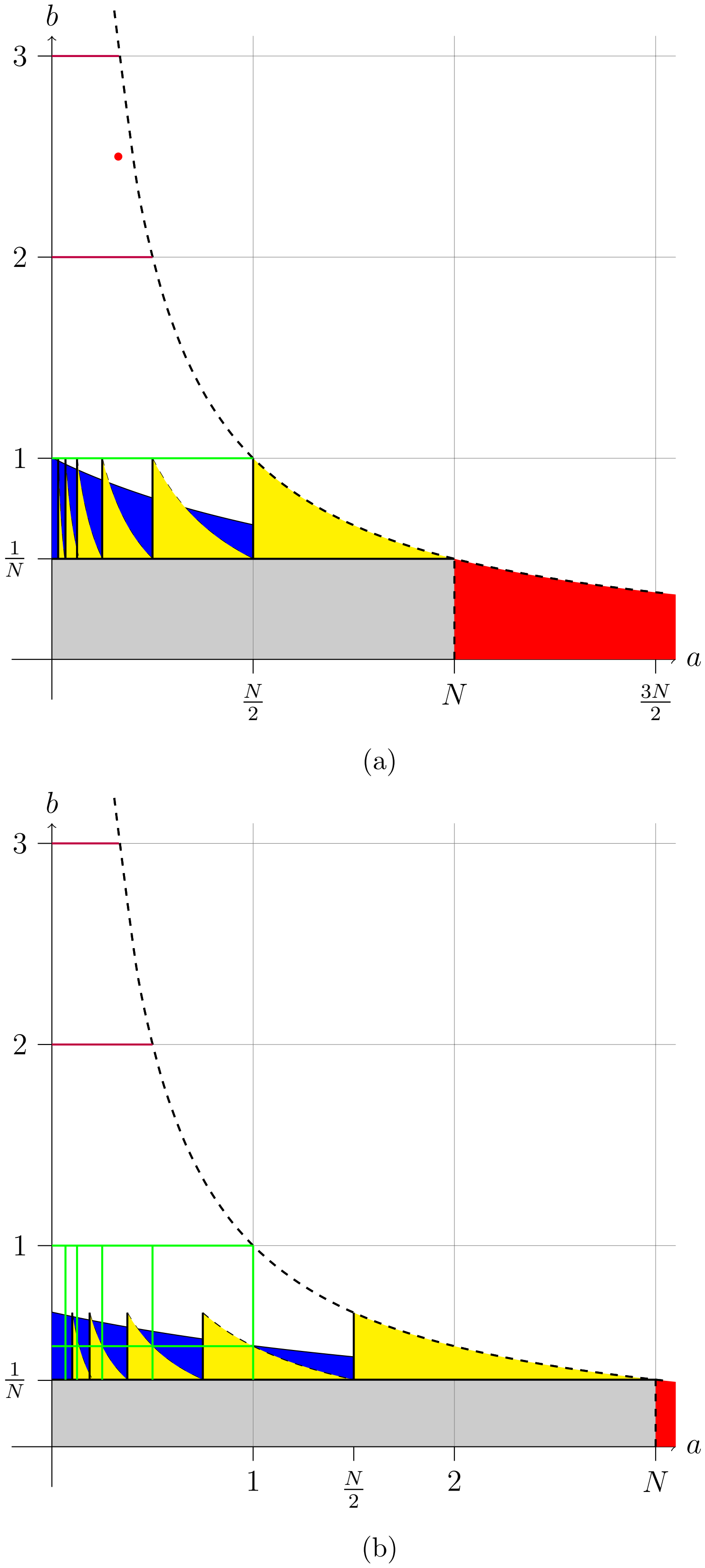


Figure 21: The frame set of B_N for $N = 2$ (a) and $N = 3$ (b). Red is non-frame area. White below the curve $ab = 1$ is unknown. All other colors indicate frames. The gray area is Corollary 5.7 (painless case), yellow is Corollary 5.15, green is [9] and blue is Theorem 5.16.

5.9 Possibilities for future work

Now we look at some of the subjects in this thesis that could give basis for further work. There are three main subjects that would be interesting to study further. Those are the dual frame method, numerical computations using the Zibulski-Zeevi matrix and further studies of Conjecture 2 and Conjecture 3.

We used the dual frame method used to prove the new part of the frame set for the B-splines (Theorem 5.16. Though there was not enough time to go further with this, it does seem like it might be worth trying to extend the method to prove the frame properties for all (a, b)-values satisfying $0 < a < \frac{N}{2}$ and $\frac{1}{N} < b < \frac{2}{N}$.

For the numerical computations it would be good to have a look at the code could be optimized further. With an improved computation time, it would be easier to consider finer grids in the (a, b)-plane. It would also make it possible to increase the grid size in t and ν, hence making the results more stable, with less numerical errors. One way to make the code more efficient could be re-writing it in such a way that we could use Matlabs fft function to calculate sums of exponentials. As we have seen our numerical methods provide good tools to help one find possible non-frame points to study analytically.

Finally, it would be interesting to see if a general proof could be found for Conjecture 2 and Conjecture 3 other than proving it pointwise by showing that the Zibulski-Zeevi matrix does not have full rank. It would also be interesting to see whether similar results hold for B-splines of order $N \geq 3$.

A Appendix

A.1 Calculations of the Zak transform for N_1 and N_2

```
% Function that calculates the value of the indicator function on an
% interval [a,b] for given points x.
% Input:
%   x - a vector of points to evaluate the indicator function in.
%   a - the left end point of the support of the indicator function.
%   b - the right end point of the support of the indicator function.
% Output:
%   chi - the value of the indicator function on [a,b] in the points x.
function chi = indicator(x,a,b)

chi = (x >= a) & (x <= b);
```

```
% Function that calculates the value of N2 for given points x.
% Input:
%   x - a vector of points to evaluate N2 in.
% Output:
%   res - the value of N2 in the points x.
function res = N2(x)
% Calculate N2 for x.
res=max(1-abs(x-1),0);
```

```
% Function that calculates the value of the Zak transform  of N1 with
% parameter a in the point (t,nu).
% Input:
%   a  - the parameter of the Zak transform.
%   t  - the point t where we calculate the Zak transform.
%   nu - the point nu where we calculate the Zak transform.
% Output:
%   res - the value of teh Zak transform of N1 in the point (t,nu).
function res = zakN1(a,t,nu)
% Finds the smallest k such that a*(t-k)<=1.
k_min = ceil(t-2/a);
% Finds the largest k such that a*(t-k)>=0.
k_max = floor(t);
% These are the k's that will contribute in the sum.
k = k_min:k_max;
% The non zero terms of the sum.
vec = indicator(a*(t-k),0,1).*exp(2*pi*1i*k*nu);
% Calculate the Zak transform.
res = sqrt(a)*sum(vec);
```

```
% Function that calculates the value of the Zak transform  of N2 with
% parameter a in the point (t,nu).
% Input:
%   a  - the parameter of the Zak transform.
```

```
%   t  — the point t where we calculate the Zak transform.
%   nu — the point nu where we calculate the Zak transform.
% Output:
%   res — the value of teh Zak transform of N2 in the point (t,nu).
function res = zakN2(a,t,nu)
% Finds the smallest k such that a*(t-k)<=2.
k_min = ceil(t-2/a);
% Finds the largest k such that a*(t-k)>=0.
k_max = floor(t);
% These are the k's that will contribute in the sum.
k = k_min:k_max;
% The non zero terms of the sum.
vec = N2(a*(t-k)).*exp(2*pi*1i*k*nu);
% Calculate the Zak transform.
res = sqrt(a)*sum(vec);
```

```
% The number of grid points in t and nu.
N=100;
% The points that we use on the t and nu axes.
t = linspace(0,1,N);
nu = linspace(0,1,N);
% A grid of t and nu in [0,1]^2.
[T,NU] = meshgrid(t,nu);
% The vales of a for which we wish to calculate the Zak transform.
a = [1/4,1/3,1/2,1];
% We go through all values of a.
for k = 1:length(a)
    zak = zeros(N,N);
    % We calculate the Zak transform in all points of the (t,nu) grid.
    for ii = 1:N
        for jj = 1:N
            zak(ii,jj) = abs(zakN1(a(k),T(ii,jj),NU(ii,jj)));
        end
    end
    % For each a we plot |Z_a N_1|^2.
    FigHandle = figure('Position', [100, 100, 500, 300]);
    mesh(T,NU,zak.^2)
    xlabel('t')
    ylabel('\nu')
    colorbar
end
```

```
% The number of grid points in t and nu.
N=101;
% The points that we use on the t and nu axes.
t = linspace(0,1,N);
nu = linspace(0,1,N);
% A grid of t and nu in [0,1]^2.
[T,NU] = meshgrid(t,nu);
% The vales of a for which we wish to calculate the Zak transform.
a = [1/2,2/3,1,2];
% We contruct an array to save the Zak transform for all values of a.
zak_g = zeros(N,N,length(a));
% We go through all values of a.
for k = 1:length(a)
    zak = zeros(N,N);
    % We calculate the Zak transform in all points of the (t,nu) grid.
    for ii = 1:N
        for jj = 1:N
            zak(ii,jj) = abs(zakN2(a(k),T(ii,jj),NU(ii,jj)));
        end
    end
    % For each a we plot |Z_a N_1|^2.
    FigHandle = figure('Position', [100, 100, 500, 300]);
    mesh(T,NU,zak.^2)
    xlabel('t')
    ylabel('\nu')
    colorbar
    % We save the values of the Zak transform.
    zak_g(:,:,k) = zak;
end

for k = 1:(length(a)-1)
    % We plot the Zak transform for nu=.5, .25, .12 and t in [0,1].
    FigHandle = figure('Position', [100, 100, 400, 600]);
    plot(t,zak_g(51,:,k).^2)
    hold on
    plot(t,zak_g(26,:,k).^2)
    plot(t,zak_g(13,:,k).^2)
    hold off
    legend('\nu = .5','\nu = .25','\nu = .12','Location','northeast')
    xlabel('t')
    ylabel(sprintf('|Z_{%f}N_2|',a(k)))
end
```

A.2 Calculations of frame bounds

```
% The points a in [0,1[ where we calculate the frame bounds for N_1.
k = 1:986; a = k./997;
N = length(a);
% Initialise vectors to save the frame bounds.
A = zeros(1,N);
B = zeros(1,N);
% For each a we calculate the frame bounds A and B.
for ii = 1:N
    % We calculate the sum of |N1(x-ak)|^2 for k from -ceil(1/a) to
    % ceil(1/a). This includes all non-zero terms.
    x = linspace(0.001,a(ii)-0.001,1000);
    gSum = indicator(x,0,1);
    for jj = 1:ceil(1/a(ii));
        gSum = gSum + indicator(x+a(ii)*jj,0,1);
        gSum = gSum + indicator(x-a(ii)*jj,0,1);
    end
    % What we calculate here is actually Ab and Bb, but since we divide
    % the two later the b's cancel out.
    A(ii) = min(gSum);
    B(ii) = max(gSum);
end
% Calculates the rate A/B.
rate = A./B;
% Plots the rate A/B as a function of a.
FigHandle = figure('Position', [100, 100, 600, 400]);
plot(a,rate)
xlabel('a','FontSize',14)
ylabel('A/B','FontSize',14)
ylim([0.45,1])
```

```
% The points a in [0,1[ where we calculate the frame bounds for N_2.
k = 1:2000; a = k./1000;
N = length(a);
% Initialise vectors to save the frame bounds.
A = zeros(1,N);
B = zeros(1,N);
% For each a we calculate the frame bounds A and B.
for ii = 1:N
    % We calculate the sum of |N2(x-ak)|^2 for k from -ceil(2/a) to
    % ceil(2/a). This includes all non-zero terms.
    x = linspace(0,a(ii),1000);
    gSum = N2(x).^2;
    for jj = 1:ceil(2/a(ii));
        gSum = gSum + N2(x+a(ii)*jj).^2;
        gSum = gSum + N2(x-a(ii)*jj).^2;
    end
    % What we calculate here is actually Ab and Bb, but since we divide
    % the two later the b's cancel out.
    A(ii) = min(gSum);
    B(ii) = max(gSum);
end
% Calculates the rate A/B.
rate = A./B;
% Plots the rate A/B as a function of a with a 2nd order polynomial.
FigHandle = figure('Position', [100, 100, 600, 400]);
c=1/2;
plot(a,rate)
hold on
plot(a(501:2000),c*(a(501:2000)-2).^2)
xlabel('a','FontSize',14)
ylabel('A/B','FontSize',14)
legend('A/B',sprintf('%f(a-2)^2', c))
```

The plots of the rate A/B for N_n, $n = 3, 4, 5, 6$ uses the same outline as the two pieces of code above. All that is changed is that we need to have $a \in [0, n]$ and we need to use the right B-spline. For this I have made a function that calculates the value of any B-spline N_n of order $n \geq 2$ in a point x. The formula used in this Matlab function is [1, Theorem 6.1.3].

```
% Function that calculates the value of N_n for some n >= 2 and given
% points x.
% Input:
%   x - a vector of points to evaluate the indicator function N_n in.
%   n - the order of the B-spline.
% Output:
%   res - the value of N_n in the points x.
function res = MyBSplines(x,n)
res = zeros(1,length(x));
for j = 0:n
    cond = ((x-j) > 0);
    res = res + cond.*(-1)^j.*nchoosek(n,j).*(x-j).^(n-1);
end
res = 1/factorial(n-1)*res;
```

A.3 Calculations of the Zibulski-Zeevi matrix

A function to calculate the Zak transform of B_2.

```
% Function that calculates the value of the Zak transform  of B2 with
% parameter a in the point (t,nu).
% Input:
%   a  - the parameter of the Zak transform.
%   t  - the point t where we calculate the Zak transform.
%   nu - the point nu where we calculate the Zak transform.
% Output:
%   res - the value of teh Zak transform of B2 in the point (t,nu).
function res = zakB2(a,t,nu)
% Finds the smallest k such that a*(t-k)<=1.
k_min = floor(t-1/a);
% Finds the largest k such that a*(t-k)>=-1.
k_max = ceil(t+1/a);
% These are the k's that will contribute in the sum.
k = k_min:k_max;
% The non zero terms of the sum.
vec = B2(a*(t-k)).*exp(2*pi*1i*k*nu);
% Calculate the Zak transform.
res = sum(vec);
```

This is a function that calculates the Zibulski-Zeevi matrix foor tthe second B-spline and given parameters t, ν, b, p and q.

```
% Function that calculates the Zibulski-Zeevi matrix for B2 with
% parameter b and ab=p/q in the point (t,nu).
% Input:
%   t, nu - the point t where we evaluate the Zibulski-Zeevi matrix.
%   b,p,q - parameters of the Zibulski-Zeevi matrix.
% Output:
%   zz - the Zibulski-Zeevi matrix in the point (t,nu).
function zz = ZZmatrix(t,nu,b,p,q)
% Initialise the Zibulski-Zeevi matrix.
zz = zeros(p,q);
% Go through each point of the Zibulski-Zeevi matrix.
for ii = 1:p
    for jj = 1:q
        zz(ii,jj) = zakB2(1/b,t-(jj-1)*p/q,nu+(ii-1)/p);
    end
end
zz = zz/sqrt(p);
```

This script calculates the singular values on the grid used in Figure 14.

```
% Define the (t,nu) grid.
N = 25; M = 25;
t = linspace(0,1,N);nu = linspace(0,1,M);
[T,NU] = meshgrid(t,nu);
% Define the (a,b) grid.
Amax = 5; Bmax = 70;
aq = 10; bq = 10;
% Initialise.
lowerBound = zeros(Bmax,Amax);
% Run through the (a,b) grid.
for ap = 1:Amax
    for bp = 1:min(Bmax,(aq*bq/ap))
        % Calculate the parameters.
        b = bp/bq;
        k = gcd(ap*bp,aq*bq);
        p = ap*bp/k; q = aq*bq/k;
        % Initialise.
        minvals = zeros(M,N);
        % Run through the (t,nu) grid.
        for ii = 1:N
            for jj = 1:M
                % Calculate the Zibulski-Zeevi matrix.
                zz = ZZmatrix(T(ii,jj),NU(ii,jj),b,p,q);
                % Calculate the singular vales of the Zibulski-Zeevi matrix.
                [U,S,V] = svd(zz);
                % Save the singular value(s) in a vector.
                if p == 1
                    v = S(1,1);
                else
                    v = diag(S);
                end
                % Find the smallest singular value.
                minvals(ii,jj) = min(v);
            end
        end
        % Save the smallest of all singular values on the (t,nu) grid.
        lowerBound(bp,ap) = min(min(minvals));
    end
end
% Plot the values of the estimate of sqrt(A) on the (a,b) grid
figure(1)
imagesc(flipud(lowerBound));colorbar
xticklabels = 0.1:0.1:0.5;
xticks = linspace(1, size(lowerBound, 2), numel(xticklabels));
set(gca, 'XTick', xticks, 'XTickLabel', xticklabels)
xlabel('a','FontSize',14)
yticklabels = [0.1,0.5:.5:5];
yticks = linspace(1, size(lowerBound, 1), numel(yticklabels));
set(gca, 'YTick', yticks, 'YTickLabel', flipud(yticklabels(:)))
ylabel('b','FontSize',14)
```

This script calculates the smallest singular values in the point (1/5,7/2) for varying grid sizes.

```
% Parameters.
a=1/5; b=7/2;p=7; q=10;
% Different grid sizes.
gridSizes = 5:5:200;
% Initialise.
minSingVal = zeros(length(gridSizes),1);
maxSingVal = zeros(length(gridSizes),1);
% Run through the different grid sizes.
for kk = 1:length(gridSizes)
    % Dedfine the (t,nu) grid.
    N = gridSizes(kk); M = ceil(N/1);
    t = linspace(0,1,N); nu = linspace(0,1/p,M);
    [T,NU] = meshgrid(t,nu);
    % Initialise.
    minvals = zeros(M,N);
    maxvals = zeros(M,N);
    % Run through the (t,nu) grid.
    for ii = 1:M
        for jj = 1:N
            % Calculate the Zibulski-Zeevi matrix.
            zz = ZZmatrix(T(ii,jj),NU(ii,jj),b,p,q);
            % Calculate the singular vales of the Zibulski-Zeevi matrix.
            [U,S,V] = svd(zz);
            % Save the singular values in a vector.
            v = diag(S);
            % Find the smallest and largest singular values.
            minvals(ii,jj) = min(v);
            maxvals(ii,jj) = max(v);
        end
    end
    % Save the smallest and largest singular values on the current (t,nu)
    % grid.
    minSingVal(kk) = min(min(minvals));
    maxSingVal(kk) = max(max(maxvals));
end
% Plot the results
figure(1)
plot(gridSizes,minSingVal);
xlabel('Grid size - M,N','FontSize',14);
ylabel('$$\sqrt{A}$$','Interpreter','latex','FontSize',14);
figure(2)
plot(gridSizes,maxSingVal);
xlabel('Grid size - M,N','FontSize',14);
ylabel('$$\sqrt{B}$$','Interpreter','latex','FontSize',14);
figure(3)
mesh(T,NU,minvals)
xlabel('t'); ylabel('\nu')
zlabel('$$\sqrt{A}$$','Interpreter','latex')
```

This script estimates $\sqrt{A}$ for a fixed value of b and a chosen such that $ab = \frac{p}{q}$ with $\gcd(p,q) = 1$, $p \leq 24$ and $q \leq 25$.

```
% Defines the (t,nu) grid.
N = 100; M = 100;
t = linspace(0,1,N);nu = linspace(0,1,M);
[T,NU] = meshgrid(t,nu);
% Defines p and q such that we go through all fractions of the type p/q,
% where gcd(p,q)=1 and p<=24, q<=25.
ps = [ones(24,1);
    2*ones(12,1);
    3*ones(14,1);
    4*ones(11,1);
    5*ones(16,1);
    6*ones(7,1);
    7*ones(16,1);
    8*ones(9,1);
    9*ones(11,1);
    10*ones(6,1);
    11*ones(13,1);
    12*ones(5,1);
    13*ones(12,1);
    14*ones(5,1);
    15*ones(7,1);
    16*ones(5,1);
    17*ones(8,1);
    18*ones(3,1);
    19*ones(6,1);
    20*ones(2,1);
    21*ones(3,1);
    22*ones(2,1);
    23*ones(2,1);
    24];
qs = [(2:25)';
     (3:2:25)';
    4;5;7;8;11;13;14;16;17;19;20;22;23;25;
    5;7;9;11;13;15;17;19;21;23;25;
    6;7;8;9;11;12;13;14;16;17;18;19;21;22;23;24;
    7;11;13;17;19;23;25;
    8;9;10;11;12;13;15;16;17;18;19;20;22;23;24;25;
    9;11;13;15;17;19;21;23;25;
    10;11;13;14;16;17;19;20;22;23;25;
    11;13;17;19;21;23;
    12;13;14;15;16;17;18;19;20;21;23;24;25;
    13;17;19;23;25;
    14;15;16;17;18;19;20;21;22;23;24;25;
    15;17;19;23;25;
    16;17;19;21;22;23;24;
    17;19;21;23;25;
    18;19;20;21;22;23;24;25;
    19;23;25;
    20;21;22;23;24;25;
    21;23;
    22;23;25;
    23;25;
```

```
    24;25;
    25;];
% Sorts the fractions in increasing order.
[ab,I] = sort(ps./qs);
ps = ps(I);qs = qs(I);
% Initialise.
NN = length(ps);
lowerBound = zeros(NN,1);
% Set the value of b.
b = 9/2;
% Run though all vales of ab.
for kk = 1:NN
    % Define p and q.
    p = ps(kk); q = qs(kk);
    % Initialise.
    minvals = zeros(M,N);
    % Run through (t,nu) grid.
    for ii = 1:N
        for jj = 1:M
            % Calculate the Zibulski-Zeevi matrix.
            zz = ZZmatrix(T(ii,jj),NU(ii,jj),b,p,q);
            % Calculate the singular vales of the Zibulski-Zeevi matrix.
            [U,S,V] = svd(zz);
            % Save the singular value(s).
            if p == 1
                v = S(1,1);
            else
                v = diag(S);
            end
            % Find the smallest singular value
            minvals(ii,jj) = min(v);
        end
    end
    % Save the smallest of all singular values on the (t,nu) grid.
    lowerBound(kk) = min(min(minvals));
end
% Plot the results
plot(ab/b,lowerBound)
set(gca,'fontsize',18)
title('M,N=100, b=4.5','FontSize',20)
xlabel('a','FontSize',18);
ylabel('$$\sqrt{A}$$','Interpreter','latex','FontSize',18)
hold on
plot([1/7,1/7],[0,0.5],'k')
plot([1/6,1/6],[0,0.5],'k--')
plot([1/5,1/5],[0,0.5],'k')
```

This script estimates $\sqrt{A}$ for fixed $ab = \frac{p}{q}$ and varying values of b.

```
% Define the (t,nu) grid.
N = 100; M = N;
t = linspace(0,1,N);nu = linspace(0,1,M);
[T,NU] = meshgrid(t,nu);
% Define parameters.
p=5; q=6;
bs = 2:0.01:3;
% Initialise.
minvals = zeros(N,M);
K = length(bs);
lowerBound = zeros(1,K);
% Run through the different values of b.
for kk = 1:K
    b = bs(kk);
    for ii = 1:N
        for jj = 1:M
            % Calculate the Zibulski-Zeevi matrix.
            zz = ZZmatrix(T(ii,jj),NU(ii,jj),b,p,q);
            % Calculate the singular vales of the Zibulski-Zeevi matrix.
            [U,S,V] = svd(zz);
            % Save the singular value(s) in a vector.
            if p == 1
                v = S(1,1);
            else
                v = diag(S);
            end
            % Find the smallest singular value.
            minvals(ii,jj) = min(v);
        end
    end
    % Save the smallest of all singular values on the (t,nu) grid.
    lowerBound(kk) = min(min(minvals));
end
% Plot the results.
plot(bs,lowerBound);
xlabel('b','FontSize',14);
ylabel('$$\sqrt{A}$$','Interpreter','latex','FontSize',14)
title('ab=9/14, M=N=100','FontSize',16);
```

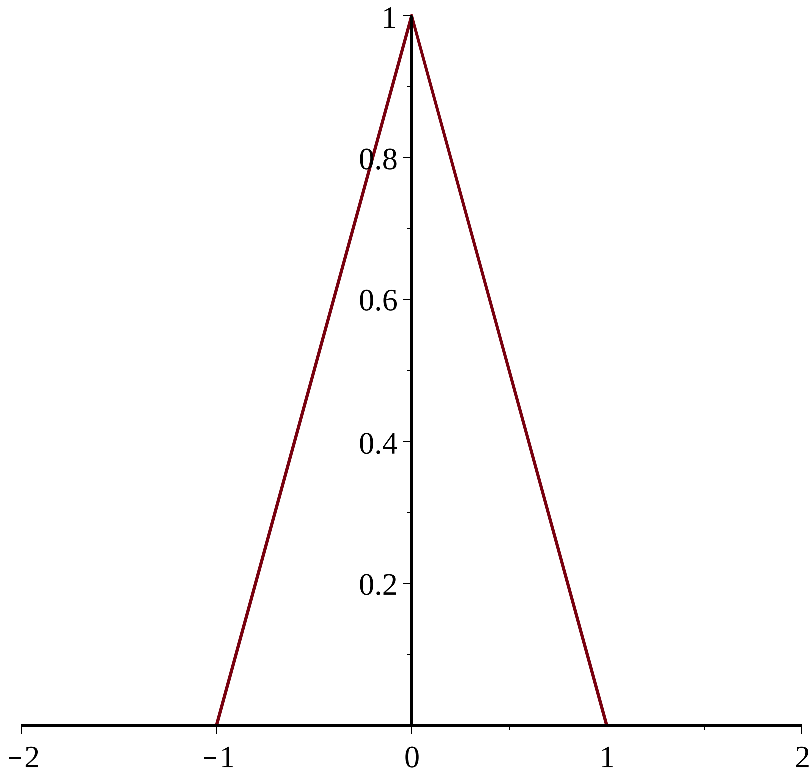
```
> restart:with(LinearAlgebra):
> interface(rtablesize=15);
```

$$10 \tag{1}$$

```
> B2:=x->piecewise(-1<=x and x<0, 1+x,0<=x and x<1,1-x,0);
```

$$B2 := x \rightarrow piecewise(-1 \leq x \textbf{ and } x < 0, 1+x, 0 \leq x \textbf{ and } x < 1, 1-x, 0) \tag{2}$$

```
> plot(B2,-2..2);
```




```
> Zak := proc(lambda,t,nu)
    local min_k, max_k;

    min_k := ceil(t-1/lambda);
    max_k := floor(t+1/lambda);
    sqrt(lambda)*sum(B2(lambda*(t-k))*exp(2*Pi*I*k*nu),k=min_k..
  max_k);
> end proc;
```

Zak := **proc**(lambda, *t*, nu) **(3)**

local *min_k*, *max_k*;

min_k := ceil(*t* − 1 / lambda);

max_k := floor(*t* + 1 / lambda);

sqrt(lambda) * (*sum*(*B2*(lambda * (*t* − *k*)) * exp(2 * I * Pi * *k* * nu), *k* = *min_k*..*max_k*))

end proc

```
> b:=5/2; a:=1/b*5/6; a*b;
```

$$b := \frac{5}{2}$$

$$a := \frac{1}{3}$$

$$\frac{5}{6} \tag{4}$$

```
> p:=numer(a*b);
```

$$p := 5 \tag{5}$$

```
> q:=denom(a*b);
```

$$q := 6 \tag{6}$$

```
> t:='t':nu:='nu':
> t:=0:nu:=0:
> Entries:=(k,l)->Zak(1/b,t-(l-1)*p/q,nu+(k-1)/p);
```

$$Entries := (k, l) \to Zak\left(\frac{1}{b}, t - \frac{(l-1)\,p}{q}, \nu + \frac{k-1}{p}\right) \tag{7}$$

```
> Z:=1/sqrt(p)*Matrix(p,q,Entries):
> Rank(evalf(Z));
```

$$4 \tag{8}$$

```
> ReducedRowEchelonForm(Z);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

```
> NullSpace(Transpose(Z));
```

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ \dfrac{-5\,e^{-\frac{2}{5} I\pi} + 5\,e^{\frac{2}{5} I\pi} + 3\,e^{\frac{4}{5} I\pi} - 3\,e^{-\frac{4}{5} I\pi}}{3\,e^{-\frac{2}{5} I\pi} - 3\,e^{\frac{2}{5} I\pi} + 5\,e^{\frac{4}{5} I\pi} - 5\,e^{-\frac{4}{5} I\pi}} \\ -\dfrac{-5\,e^{-\frac{2}{5} I\pi} + 5\,e^{\frac{2}{5} I\pi} + 3\,e^{\frac{4}{5} I\pi} - 3\,e^{-\frac{4}{5} I\pi}}{3\,e^{-\frac{2}{5} I\pi} - 3\,e^{\frac{2}{5} I\pi} + 5\,e^{\frac{4}{5} I\pi} - 5\,e^{-\frac{4}{5} I\pi}} \\ 1 \end{bmatrix} \right\} \tag{10}$$

```
> evalf(Z);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0.2961967334 + 0.\,I, 0.1408959466 - 0.2458332466\,I, -0.1408959469 - 0.2458332465\,I, -0.2961967340 + 0.\,I, -0.1408959469 + 0.2458332466\,I, 0.1408959465 + 0.2458332465\,I], [0.04321452108 + 0.\,I, -0.02775886191 - 0.003234088118\,I, 0.02775886157 - 0.003234088118\,I, -0.04321452136 + 0.\,I, 0.02775886157 + 0.003234088124\,I, -0.02775886190 + 0.003234088118\,I], [0.04321452108 + 0.\,I, -0.02775886190 + 0.003234088118\,I, 0.02775886157 + 0.003234088124\,I, -0.04321452136 + 0.\,I, 0.02775886157 - 0.003234088118\,I, -0.02775886191 - 0.003234088118\,I], [0.2961967334 + 0.\,I, 0.1408959465 + 0.2458332465\,I, -0.1408959469 + 0.2458332466\,I, -0.2961967340 + 0.\,I, -0.1408959469 - 0.2458332465\,I, 0.1408959466 - 0.2458332466\,I]] \tag{11}$$

```
> Z1:=RowOperation(Z,[2, 5],-1):evalf(%);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0., 1.\,10^{-10} - 0.4916664931\,I, 0. - 0.4916664931\,I, 0., 0. + 0.4916664931\,I, -1.\,10^{-10} + 0.4916664931\,I], [0.04321452108 + 0.\,I, -0.02775886191 - 0.003234088118\,I, 0.02775886157 - 0.003234088118\,I, -0.04321452136 + 0.\,I, 0.02775886157 + 0.003234088124\,I, -0.02775886190 + 0.003234088118\,I], [0.04321452108 + 0.\,I, -0.02775886190 + 0.003234088118\,I, 0.02775886157 + 0.003234088124\,I, -0.04321452136 + 0.\,I, 0.02775886157 - 0.003234088118\,I, -0.02775886191 - 0.003234088118\,I], [0.2961967334 + 0.\,I, 0.1408959465 + 0.2458332465\,I, -0.1408959469 + 0.2458332466\,I, -0.2961967340 + 0.\,I, -0.1408959469 - 0.2458332465\,I, 0.1408959466 - 0.2458332466\,I]] \tag{12}$$

```
> Z2:=RowOperation(Z1,[3, 4],-1):evalf(%);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0., 1.\ 10^{-10} - 0.4916664931\ I, 0. - 0.4916664931\ I, 0., 0. + 0.4916664931\ I, -1.\ 10^{-10} + 0.4916664931\ I], [0., -1.\ 10^{-11} - 0.006468176236\ I, 0. - 0.006468176242\ I, 0., 0. + 0.006468176242\ I, 1.\ 10^{-11} + 0.006468176236\ I], [0.04321452108 + 0.\ I, -0.02775886190 + 0.003234088118\ I, 0.02775886157 + 0.003234088124\ I, -0.04321452136 + 0.\ I, 0.02775886157 - 0.003234088118\ I, -0.02775886191 - 0.003234088118\ I], [0.2961967334 + 0.\ I, 0.1408959465 + 0.2458332465\ I, -0.1408959469 + 0.2458332466\ I, -0.2961967340 + 0.\ I, -0.1408959469 - 0.2458332465\ I, 0.1408959466 - 0.2458332466\ I]] \quad (13)$$

```
> Z3:=RowOperation(Z2,2,1/Z2(2,2)): evalf(%);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0., 1., 1.000000000 - 2.033899023\ 10^{-10}\ I, 0., -1.000000000 + 2.033899023\ 10^{-10}\ I, -1.000000000 + 0.\ I], [0., -1.\ 10^{-11} - 0.006468176236\ I, 0. - 0.006468176242\ I, 0., 0. + 0.006468176242\ I, 1.\ 10^{-11} + 0.006468176236\ I], [0.04321452108 + 0.\ I, -0.02775886190 + 0.003234088118\ I, 0.02775886157 + 0.003234088124\ I, -0.04321452136 + 0.\ I, 0.02775886157 - 0.003234088118\ I, -0.02775886191 - 0.003234088118\ I], [0.2961967334 + 0.\ I, 0.1408959465 + 0.2458332465\ I, -0.1408959469 + 0.2458332466\ I, -0.2961967340 + 0.\ I, -0.1408959469 - 0.2458332465\ I, 0.1408959466 - 0.2458332466\ I]] \quad (14)$$

```
> Z4:=RowOperation(Z3,3,1/Z3(3,2)): evalf(%);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0., 1., 1.000000000 - 2.033899023\ 10^{-10}\ I, 0., -1.000000000 + 2.033899023\ 10^{-10}\ I, -1.000000000 + 0.\ I], [0., 1., 1.000000001 + 1.546030851\ 10^{-9}\ I, 0., -1.000000001 - 1.546030851\ 10^{-9}\ I, -1.000000000 + 0.\ I], [0.04321452108 + 0.\ I, -0.02775886190 + 0.003234088118\ I, 0.02775886157 + 0.003234088124\ I, -0.04321452136 + 0.\ I, 0.02775886157 - 0.003234088118\ I, -0.02775886191 - 0.003234088118\ I], [0.2961967334 + 0.\ I, 0.1408959465 + 0.2458332465\ I, -0.1408959469 + 0.2458332466\ I, -0.2961967340 + 0.\ I, -0.1408959469 - 0.2458332465\ I, 0.1408959466 - 0.2458332466\ I]] \quad (15)$$

```
> Z5:=RowOperation(Z4,[2, 3],-1): evalf(%);
```

$$[[0.7353910522, 0.7165348712, 0.6976786906, 0.6788225098, 0.6976786906, 0.7165348712], [0., 0., -1.\,10^{-9} - 1.749420753\,10^{-9}\,I, 0., 1.\,10^{-9} + 1.749420753\,10^{-9}\,I, 0. + 0.\,I], [0., 1., 1.000000001 + 1.546030851\,10^{-9}\,I, 0., -1.000000001 - 1.546030851\,10^{-9}\,I, -1.000000000 + 0.\,I], [0.04321452108 + 0.\,I, -0.02775886190 + 0.003234088118\,I, 0.02775886157 + 0.003234088124\,I, -0.04321452136 + 0.\,I, 0.02775886157 - 0.003234088118\,I, -0.02775886191 - 0.003234088118\,I], [0.2961967334 + 0.\,I, 0.1408959465 + 0.2458332465\,I, -0.1408959469 + 0.2458332466\,I, -0.2961967340 + 0.\,I, -0.1408959469 - 0.2458332465\,I, 0.1408959466 - 0.2458332466\,I]] \quad (16)$$

```
> simplify(Z5(2,1..5));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (17)$$

```
> simplify(Z5);
```

$$\Big[\Big[\frac{13}{25}\sqrt{2}, \frac{38}{75}\sqrt{2}, \frac{37}{75}\sqrt{2}, \frac{12}{25}\sqrt{2}, \frac{37}{75}\sqrt{2}, \frac{38}{75}\sqrt{2}\Big], \Big[0, 0, 0, 0, 0, 0\Big], \Big[0, 1, 1, 0, -1, -1\Big], \Big[-\frac{1}{25}\sqrt{2}\left(-2\cos\left(\frac{2}{5}\pi\right) + 6\cos\left(\frac{1}{5}\pi\right) - 5\right), \frac{2}{75}\sqrt{2}\left(5 - 2(-1)^{1/5} + 7(-1)^{4/5} - 4(-1)^{3/5} + (-1)^{2/5}\right), \frac{1}{75}\sqrt{2}\left(5 + 11(-1)^{4/5} - 13(-1)^{3/5} + 7(-1)^{2/5} - (-1)^{1/5}\right), -\frac{4}{25}\sqrt{2}\left(-2\cos\left(\frac{2}{5}\pi\right) + \cos\left(\frac{1}{5}\pi\right)\right), \frac{1}{75}\sqrt{2}\left(5 + (-1)^{4/5} - 7(-1)^{3/5} + 13(-1)^{2/5} - 11(-1)^{1/5}\right), \frac{2}{75}\sqrt{2}\left(5 - (-1)^{3/5} + 4(-1)^{2/5} - 7(-1)^{1/5} + 2(-1)^{4/5}\right)\Big], \Big[-\frac{1}{25}\sqrt{2}\left(-6\cos\left(\frac{2}{5}\pi\right) - 5 + 2\cos\left(\frac{1}{5}\pi\right)\right), \frac{2}{75}\sqrt{2}\left(5 - 2(-1)^{3/5} + 7(-1)^{2/5} + 4(-1)^{4/5} - (-1)^{1/5}\right), \frac{1}{75}\sqrt{2}\left(5 + 11(-1)^{2/5} + 13(-1)^{4/5} - 7(-1)^{1/5} - (-1)^{3/5}\right), -\frac{4}{25}\sqrt{2}\left(-\cos\left(\frac{2}{5}\pi\right) + 2\cos\left(\frac{1}{5}\pi\right)\right), \frac{1}{75}\sqrt{2}\left(5 + (-1)^{2/5} + 7(-1)^{4/5} - 13(-1)^{1/5} - 11(-1)^{3/5}\right), \frac{2}{75}\sqrt{2}\left(5 + (-1)^{4/5} - 4(-1)^{1/5} - 7(-1)^{3/5} + 2(-1)^{2/5}\right)\Big]\Big] \quad (18)$$

References

[1] O. Christensen. *Frames and Bases*. Birkhäuser, 2008.

[2] Xin-Rong Dai and Qiyu Sun. The abc-Problem for Gabor Systems, 2013.

[3] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, 1989.

[4] K. Gröchenig. The Mystery of Gabor Frames. *Journal of Fourier Analysis and Applications*, 2014.

[5] E. Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley & Sons, Limited, 1989.

[6] R.Y Kim O.Christensen, O.H. Kim. On Gabor frames generated by sign-changing windows and B-splines, 2015.

[7] V. Del Prete. Estimates, Decay Properties, and Computation of the Dual Function for Gabor Frames. *Journal of Fourier Analysis and Applications*, 1999.

[8] I.J. Schoenberg. *Cardinal Spline Interpolation*. Society for Industrial and Applied Mathematics, 1973.

[9] J. Stöckler T. Kloos. Zak transforms and Gabor frames of totally positive functions and exponential B-splines. *Journal of Approximation Theory*, 2014.