STOCHASTIC MODELING, CONTROL AND OPTIMIZATION OF DISTRICT HEATING SYSTEMS

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Madsen, H., O. P. Palsson, K. Sejling and H. T. Søgaard (1992): *Models and Methods for Optimization of District Heating Systems, Part II: Models and Control Methods.* The Institute of Mathematical Statistics and Operations Research, The Technical University of Denmark, Lyngby.

Preface

This thesis has been prepared at The Institute of Mathematical Statistics and Operations Research (IMSOR), The Technical University of Denmark, in partial fulfillment of the requirements for the degree of Ph.D. in Engineering.

The thesis deals with various aspects concerning modeling, optimal control and optimal operation of district heating systems. The district heating systems considered are of the type commonly applied in Denmark, i.e., the district heat comes from fossil-fired combined heat and power plants.

The thesis consists of a summary report and a collection of 9 research papers written during the period 1990-1993, and elsewhere published. The report contains a general introduction to district heating followed by summaries which link the papers and the obtained results to each other.

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Papers Summarized in the Thesis

[A1] Jonsson, G., O. P. Palsson and K. Sejling (1992): "Modeling and Parameter Estimation of Heat Exchangers - A Statistical Approach," The ASME transaction: *Journal of Dynamic Systems, Measurements and Control*, Vol. 114, pp. 673–679.

[A2] Jonsson, G. and O. P. Palsson (1991): "Use of Empirical Relations in the Parameters of Heat-Exchanger Models," *Industrial & Engineering Chemistry Research*, Vol. 30, no. 6, pp. 1193–1199.

[A3] Jonsson, G. and O. P. Palsson (1993): "An Application of Extended Kalman Filtering to Heat Exchanger Models," To be published in the ASME transaction: *Journal of Dynamic Systems, Measurements and Control.*

[B1] Palsson, O. P., H. Madsen and H. T. Søgaard (1992): "Predictor-based Optimal Control of Supply Temperature in District Heating Systems," In *Proceedings of the IFAC Symposium on Control of Power Plants and Power System*, pp. 81–85, March 9–11, Munich, Germany.
Also available as IMSOR research report no. 21/1991.

[B2] Palsson, O. P., H. Madsen and H. T. Søgaard (1993): "Generalized Predictive Control for Non-Stationary Systems," In *Preprints of the 12th World Congress of IFAC*, July 19th–23rd 1993, in Sydney, Vol. 2, pp. 17–20 (reduced version). The paper has been conditionally accepted for publication in *Automatica.*
Also available as IMSOR technical report no. 3/1993.

[B3] Palsson, O. P., H. Madsen and H. T. Søgaard (1993): "Application of Predictive Control in District Heating Systems," To appear in Proceedings of the Institution of Mechanical Engineers (IMechE), Part A: *Journal of Power and Energy.*
Also available as IMSOR technical report no. 4/1993.

[B4] Madsen, H., K. Sejling, H. T. Søgaard and O. P. Palsson (1993): "On Flow and Supply Temperature Control in District Heating Systems," *Submitted for publication.*
Also available as IMSOR technical report no. 10/1993.

[C1] Palsson, O. P. and H. F. Ravn (1993): "Scenario Analysis and the Progressive Hedging Algorithm – Simple Numerical Examples," IMSOR technical report no. 11/1993.

[C2] Palsson, O. P. and H. F. Ravn (1993): "Stochastic Heat Storage Problem – Solved by the Progressive Hedging Algorithm," *Submitted for publication.*
Also available as IMSOR technical report no. 12/1993.

Summary

The present thesis consists of 9 research papers published in the period 1991 - 1993 together with a summary report. The thesis discusses different aspects in connection with stochastic modeling, control and optimization of district heating systems. The main part of the papers have their background in research projects at The Institute of Mathematical Statistics and Operations Research, entitled "Models and Methods for Optimization of District Heating Systems" and "Operational Optimization of District Heating Systems".

The thesis divides into three parts, where the first part deals with dynamic modeling of heat exchangers, together with estimation of the parameters in these models. The second part is concerned with control of supply temperatures from district heating plants, in particular prediction based controllers for non-stationary systems. The third part treats operational optimal loading and unloading of a heat storage tank, which is connected to a combined heat and power (CHP) extraction plant, and examines the stochastic optimization method applied.

The summary report is initiated by a general description of district heating systems and the relevant components, and a brief description of district heating systems in the Nordic countries. This is followed by summaries of the papers from the three previously mentioned parts.

In papers **[A1]**, **[A2]**, and **[A3]** modeling of heat exchangers and the parameter estimation in these models are described. The distributed system which the heat exchanger constitutes, is described by a number of sections (compartments). The models are established as a set of differential equations, which approximate the energy conservation in each section, and where process and observation noise are introduced. This leads to stochastic state space models in continuous time of the outlet temperature from the heat exchanger. The models are made applicable over the whole operation area by using the structure of known empirical relations for the total heat transfer coefficient thereby making it both massflow and temperature dependent. Carrying out the parameter estimation in the time-continuous model formulation means that the parameters are directly physically interpretable, and can therefore be compared with parameters obtained from dimensions of the heat exchanger, physical properties, etc. Models where the influence of the intermediate metal is considered are described in **[A1]**. In this paper the estimation procedure, which minimizes a score function consisting of the sum of the squared one-step prediction residuals of the outlet temperatures, with respect to the parameters, is described. The one-step predictions are obtained by using a standard Kalman filter. In **[A2]** the main stress is attached to the incorporation of empirical relations, in order to describe the massflow dependence of the heat transfer coefficient, the score function in this paper consists of the multi-step prediction error. Paper **[A3]** discusses the incorporation of the temperature dependence in the heat transfer coefficient, leading to a model non-linear in the states. In this case an extended Kalman filter is applied.

Optimal control of supply temperature in district heating systems is the main subject of **[B1]** – **[B4]**. In district heating systems where the heat is produced at a CHP plant it is most economical to keep the supply temperature as low as possible, thereby minimizing the heat losses from the pipes but also reducing the production cost. This is due to the fact that the ratio between electricity and heat is increased with decreasing supply temperature, and since the electricity is more valuable than heat, a more economical operation is achieved. This minimization is subject to some restrictions, such as that the system is to be supplied by the necessary amount of heat, and the consumers require some minimum ambient air dependent supply temperature. The general control concept is described in **[B4]**. Papers **[B1]** – **[B3]** discuss controllers for controlling the supply temperature. In **[B1]** a linear-quadratic controller is investigated, which is based on a cost function that keeps the supply temperature in the network close to the desired temperature, but penalizes large changes in the supply temperature from the district heating plant. In **[B2]** a modified version of the Generalized Predictive Controller is proposed, the modifications consider, e.g., the non-stationarities seen in district heating systems. This modified controller is applied in connection with supply temperature control in [B3].

A recently developed method for solving stochastic optimization problems is described in [C1]. The method is based on scenario analysis where each scenario is assigned with a given probability. The method is called Progressive Hedging Algorithm. The method is tested and analyzed by simple numerical examples. Paper [C2] illustrates how the method described in [C1] can be used in order to find an operational optimal loading and unloading of a heat storage combined with a CHP extraction plant. In this paper the future power production is assumed known with some uncertainty, described by a number of scenarios, but the future heat consumption is assumed known. In the paper the Progressive Hedging Algorithm is connected with a receding horizon idea, meaning that the problem can be solved on-line.

Sammenfatning (Summary in Danish)

Den nærværende afhandling indeholder 9 publicerede (tidskrifts)artikler fra perioden 1991–1993 samt en sammendragsrapport. Afhandlingen diskuterer forskellige aspekter i forbindelse med stokastisk modellering, regulering og optimering af fjernvarmesystemer. Størstedelen af artiklerne tager udgangspunkt i forskningsprojekter, udarbejdet på Instituttet for Matematisk Statistik og Operationsanalyse med titlerne "Modeller og metoder til optimering af fjernvarmesystemer" og "Optimeret drift af fjernvarmesystemer".

Afhandlingen kan opdeles i tre dele, hvor den første del omhandler dynamisk modellering af varmevekslere samt estimation af parametrene i disse modeller. Anden del omhandler regulering af fremløbstemperaturer fra fjernvarmeværker, specielt prædiktionsbaserede regulatorer for ikke-stationære systemer. Den tredje del omhandler en driftoptimal op- og afladning for en varmeakkumulatortank, som er forbundet med et udtags kraftvarmeværk, samt en undersøgelse af den anvendte stokastiske optimeringsmetode.

Sammendragsrapporten indledes med en beskrivelse af fjernvarmesystemer generelt og de relevante komponenter, samt en kort beskrivelse af fjernvarmesystemer i de nordiske lande. Dette efterfølges af sammenfatninger af artikler fra de tre førnævnte dele.

I artikler **[A1]**, **[A2]** og **[A3]** beskrives modellering af varmevekslere og estimation af parametrene i disse modeller. Det distribuerede system, som varmeveksleren udgør, beskrives ved et antal sektioner. Opstilling af et sæt differentialligninger, som approksimerer energibevarelsen i hver sektion og indførelse af støj på proces og målinger, fører til en stokastisk tidskontinuert tilstandsbeskrivelse for varmevekslerens udløbstemperaturer. Modellerne er gjort anvendelige over hele varmevekslerens arbejdsområde ved at bruge strukturen fra kendte empiriske formler til bestemmelse af varmevekslerens totale varmeoverføringskoefficient og derved at gøre den både massestrøms- og temperaturafhængig. Gennemføring af parameterestimationen i den tidskontinuerte modelformulering, betyder at parametrene er umiddelbart fysisk fortolkbare og kan derfor sammenlignes med parametre opnået v.h.a. varmevekslerens dimensioner, fysiske parametre, etc. I artikel **[A1]** beskrives modeller hvor indflydelsen af det mellemliggende metal betragtes. I denne artikel beskrives estimationsproceduren, som går ud på at minimere en kostfunktion bestående af kvadratsummen af et-trins-prædiktionsresidualerne med hensyn til modelparametrene. Et-trins-prædiktionsfejlen fås v.h.a. et standard Kalman filter. I artikel **[A2]** lægges hovedvægten på inddragelsen af empiriske formler, som beskriver massestrømsafhængigheden af varmeoverføringskoefficienten. I denne artikel består kostfunktionen af kvadratsummen af multistep-prædiktionsfejl. I artikel **[A3]** diskuteres indførelsen af temperaturafhængigheden i varmeoverføringskoefficienten, hvilket fører til en ikke-lineær tilstandsbeskrivelse. I dette tilfælde anvendes et udvidet (extended) Kalman filter.

I artiklerne **[B1]** – **[B4]** er hovedemnet optimal regulering af fremløbstemperaturer i fjernvarmesystemer. I fjernvarmesystemer hvor varmen er produceret på et kraftvarmeværk er det mest økonomisk at holde fremløbstemperaturen fra værket så lav som muligt, derved minimeres varmetabet fra ledninger, men ligeledes formindskes produktionsomkostningerne. Dette skyldes at forholdet mellem el og varme forøges ved lavere fremløbstemperatur, og dersom el har større værdi end varme opnås en mere økonomisk drift. Denne minimering er dog begrænset af, at systemet skal forsynes med den nødvendige varme og at forbrugerne har krav på en mindstetemperatur på fremløbsvandet, som afhænger af udetemperaturen. I artikel **[B4]** beskrives det benyttede regulerings-koncept generelt. Artiklerne **[B1]** - **[B3]** diskuterer regulatorer for regulering af fremløbstemperaturer. Artikel **[B1]** omhandler en lineær-kvadratisk regulator, som baseres på en kostfunktion som holder fremløbstemperaturen ude i nettet tæt på det ønskede, men samtidig straffer store ændringer i fremløbstemperaturen fra fjernvarmeværket. Artikel **[B2]** foreslår en udvidet udgave af den generelle prædiktiv kontrol, udvidelserne tager bl.a. hensyn til de ikke-stationariteter, som forekommer i fjernvarmesystemer. Denne regulator er undersøgt i forbindelse med regulering af fremløbstemperaturer i **[B3]**.

I artikel [C1] beskrives en relativt nyudviklet metode til løsning af stokastiske optimeringsproblemer. Metoden baseres på scenarioanalyse, hvor hver scenarium tildeles en vis sandsynlighed. Denne metode kaldes Progressive Hedging Algorithm. Metoden er afprøvet og analyseret v.h.a. simple numeriske eksempler. I artikel [C2] illustreres, hvorledes metoden beskrevet i [C1] kan anvendes ved at finde et driftoptimalt op- og afladnings forløb for en varmeakkumulatortank, som er forbundet med et udtagskraftvarmeværk. I denne artikel antages den fremtidige el-produktion at være forbundet med en usikkerhed, som beskrives v.h.a. scenarier; men den fremtidige varmeefterspørgsel antages at være kendt. I artiklen anvendes Progressive Hedging Algoritmen i forbindelse med en rullende horisont, hvilket betyder, at problemet bedre kan løses on-line.

Contents

1 Introduction

The issues of the present thesis are dynamic modeling, optimal predictive control and stochastic optimization in connection with district heating systems. The modeling focuses on models for heat exchangers, the control part investigates controllers suitable for controlling the supply temperature in the district heating systems and the stochastic optimization part, in addition to the controllers, discusses optimal management of heat storage tanks.

The purpose of this research is to formulate and investigate controllers suitable for controlling district heating systems by modeling the relevant system components and using suitable optimization criterion and solution methods. The ultimate goal is to improve the operation of district heating systems, hopefully leading to benefits in economic as well as environmental terms.

The thesis consists of a summary report and a collection of 9 research papers (the papers will, in the summary report, be referred to by the notation given on pages ix and x):

The first three papers, **[A1]**, **[A2]**, and **[A3]**, discuss continuous in time dynamic models of heat exchangers, where the model parameters are estimated by statistical methods. A well known empirical relation, from the heat transfer theory, is used to improve the models. The papers **[B1]**, **[B2]**, **[B3]**, and **[B4]** are devoted to the control of supply temperature in district heating systems. Finally, papers **[C1]** and **[C2]** are about a recently developed method to solve stochastic optimization problems. The method called Progressive Hedging Algorithm, is described in **[C1]** and applied for obtaining an optimal operation of a heat storage tank in **[C2]**.

The thesis starts with a brief description of district heating systems. This is followed by a summary of the research papers. Finally, the conclusions are drawn.

2 District Heating Systems

This section consists of a brief description and comparison of the district heating systems in the Nordic countries and a description of the type of district heating systems, which this thesis takes its origin in. The main components are especially described. First a definition of district heating is given (Andersen and Brydov, 1987):

> **Definition of District Heating:** District heating may be defined as space and water heating of a number of buildings from a central plant. The heat produced in this plant is delivered to the consumers as hot water through an insulated, double pipeline system. The heated water is carried in the forward pipe distribution system and having given up its heat, the cooler water returns to the plant in the other pipe for reheating.

2.1 District Heating in the Nordic Countries

Due to the geographical location of the Nordic countries, the need for space and water heating is evident, especially in the colder part of the year.

Therefore, in the last decades the utilization and development of district heating has been progressive, and the Nordic countries have taken the lead, e.g., in utilizing district heat from CHP plants and from geothermal sources. The district heating in these countries is, though, quite different. A brief description of the district heating in these countries is found in the sequel.

District Heating in Denmark Approximately 44% of the total heat requirement in Denmark is supplied by district heating, of which almost 60% is derived from cogeneration of heat and power, see Gerlach (1991), and Nordvärme (1988). The CHP plants are mostly fired with coal, oil and natural gas.

The district heating systems in Denmark are operated at rather low temperatures and pressures compared to the systems in other European (and Scandinavian) countries. The maximum design temperature in a Danish system is 120 °C, but most plants are operated below 100 °C and typically about 80 °C (Andersen and Brydov, 1987). This implies several advantages. Standard preinsulated and inexpensive pipelines may be used as well as rather uncomplicated equipment. Furthermore, the low temperature allows an efficient utilization of waste heat from industries, and in addition a simple hot water boiler may be installed, instead of traditional steam boilers which are widely used abroad, see Andersen and Brydov (1987). Moreover, the low temperature allows installations of pressureless heat storage (or accumulation) tanks.

The pressure is kept at a low level in order to reduce demands on the various components. This enables direct connections to building installations to be made without the need for heat exchangers. The great majority of consumer installations are connected directly, only 10-15% of the all consumers are supplied by indirect systems. This reduces the installation costs to a minimum.

District Heating in Iceland The district heating systems in Iceland utilize the geothermal water to be found at several warm (hot) areas around the country, e.g., under the capital city Reykjavik.

The warm water is pumped from the boreholes to storage tanks. The water leaves the boreholes at constant temperature. From the tanks the water is pumped to the district pumping stations, and from these district stations the water is pumped to the consumers through either single or double pipe distribution systems.

The difference between single and double pipe systems is that in single pipe system the water is wasted through the city drains, after having given up it's heat in the houses, whereas in a double pipe system it is returned to the district stations and mixed with high temperature supply water, maintaining a set temperature of about 80 °C.

A considerable thermal energy remains in the water after it has been used for heating, but owing to its low temperature (25 – 30 °C) it is difficult to find use for it. The consumers are normally permitted to use it in any way they wish, i.e., by heating small greenhouses, swimming-basins and melting snow off pavements and parking spaces at their homes.

The district heating prices in Iceland are low compared with prices in other Nordic countries. The main costs are in the drilling of the boreholes and in the distribution systems and the operation costs are mainly due to the pumping of the water in the system.

Around 86% of the total space and water heating in Iceland is due to district heating (Nordvärme, 1988), mostly based on geothermal sources. The remaining part of the heating is mainly performed by electric power.

District Heating in Finland District heating accounts for around 40% of the Finnish domestic heating, where around 58% of the total district heating is from CHP (Nordvärme, 1988).

Coal accounts for nearly 50% of the district heat and thermal power generation (in 1986). Other important sources of energy for heating are oil and peat. The peat resources in Finland are extremely large.

Most of the Finnish district heating consumers are supplied indirectly. This is mainly due to high operating temperature (120 °C, with the supply temperature at 120-75 °C) and high pressure (10 or 12 bar) which again is due to major differences in geographical levels. Other advantages of the indirect system include easier control of pressure conditions in the network, high reliability and low leakage, and low pressure level, i.e., lower price in consumers' heating systems.

District Heating in Norway Due to the low prices (and large supply) of electrical power (and oil), utilization of district heating in Norway is limited. Only about 4% of the space and water heating is by district heating.

Most systems are based on moderate temperatures (up to 120 °C) and utilize heat exchangers to interface the distribution system with the consumer. A system operating at lower temperatures (below 80 °C) and on heat pumps, has been developed. Waste heat recovery from industry, waste incineration, oil or electrical boilers are the main heat sources.

District Heating in Sweden District heating accounts for 27% of the total heating demand in Sweden (Nordvärme, 1988). Due to large amounts of electrical energy from hydro and nuclear power plants, CHP is being utilized only to a limited extent. This is though assumed to increase in the future.

The district heat is produced in boilers, from different sources, such as, coal, oil, refuse, biomass, and electric power. The use of heat pumps is also significant (10% of the total district heating production is due to various low-grade heat sources).

The supply temperature and pressure are higher than in Denmark, the maximum temperature is around 150 °C. Therefore, all Swedish district heating systems operate with an indirect distribution system, with a heat exchanger between the distribution system and the consumer unit, see IEA (1983). Furthermore, the boiler stations and the district heating network are often connected by heat exchangers. The pressure in the flow pipes for the indirect system is usually around 8–10 bar, and always below 16 bar.

2.2 Simplified Description of District Heating Systems

The Nordic district heating systems are quite different as seen in the previous description. The main part of the present research is related to the Danish approach, i.e., CHP plants and direct installation. In the following a simplified description of district heating systems is given.

In general a district heating system can be divided into three subsystems:

- Production.
- Transportation.
- Consumption.

Production There exists many types of heat producers. One of the most economically and environmentally advantageous is the combined heat and power plant, which will be briefly described below. Other heat producers are heat-only plants fired with, e.g., oil, gas, coal or refuse.

Transportation The transportation consists usually of two parts: a transmission network (pipelines) and a distribution network, which are connected by heat exchanger stations. Each part consists of insulated double pipelines.

Sometimes the heat exchanger stations are omitted, though. The purposes of dividing the transportation in a transmission and a distribution part are several, such as, different companies can operate on each side, and it allows higher temperature and pressure on the transmission side and thus it is easier to sustain the necessary pressure difference in the network.

The district heating pipes and their water content (and the surrounding media) provide a great heat capacity and therefore influence the dynamic behavior of the system to a large extent.

Due to the temperature difference between the district heating water and the surrounding media some heat is lost from the network. Losses are also due to leakage.

Furthermore, considerable time-delay is induced by the long distance in the transmission and the distribution network. The time-delay is time-varying mainly due to the variation in the massflow.

Consumption The heat is consumed by consumers of various sizes, from large consumers such as schools, hospitals and industrial factories to small single family houses.

Obviously, the consumption of these consumers is different, due to different diurnal, week and annual behavior.

The consumers in Denmark are usually directly connected to the distribution network, due to the low temperature and pressure. The opposite is the case in Sweden and Finland where the consumer installations (consisting of small heat exchangers) are frequently used.

2.2.1 Combined Heat and Power

The fossil fired power plants are of three kinds:

- a condensing power plant (this is not CHP),
- a back pressure power plant,
- an extraction power plant.

The last two types produce both electric power and heat, hence they are called CHP plants. The general principles are briefly described in the following: (see e.g., Horlock (1987) for a thorough description).

Condensing Power Plant In condensing power plants, pressurized water is pumped into a boiler, where it is vaporized. Generally, the fuel is coal or oil. From the boiler the steam is led to the turbine, normally at a pressure of about 180 bars and a temperature of 540 °C. In the turbine, thermal energy of the steam is converted into mechanical energy and a generator converts the mechanical energy into electricity. After having given off its energy to the turbine the steam is condensed in a condenser, which is cooled with large quantities of (sea) water. The condensate is then pumped back into the boiler. This process is rather inefficient as only 40% of the fired energy is utilized.

Back Pressure Power Plant In order to improve the efficiency a process where the heat may be utilized is considered. Such a process is called a combined heat and power generation process. By this process the heat removed from the condenser is used for district heating. The steam is condensed at pressure and temperature conditions which allow district heating water to a temperature of 100–120 °C.

Although some power is lost by this process, the overall conversion efficiency of the fuel energy to useful heat or power can be as high as 85%.

This process is called back pressure process, it has the drawback that the electric power and the heat is generated in a fixed ratio.

Extraction Power Plant The extraction power plant has a considerably higher flexibility between the generation of power and heat. The district heating water is heated by means of steam extracted from the turbine. The greater the steam extracted for hot water production, the smaller power generation. This means the larger the production of hot district heating water, the smaller the loss through the cooling water in the condenser.

Supply Temperature A low supply temperature is of significant importance in relation to a CHP system, especially in extraction plants, as the electricity output decreases as the extraction temperature (i.e., the district heating supply temperature) increases, resulting in greater fuel consumption to provide the same energy output, see e.g., IEA (1983).

2.2.2 Heat Storage Tank (Heat Accumulator)

For the reason that demands for heat and power never occur simultaneously, installation of heat storage tanks at CHP plants can be advantageous. By means of the heat storage it is possible to equalize the production. Due to the low temperature and pressure in the Danish district heating systems pressure-less tanks are normally adequate.

When the heat storage is loaded (charged), warm supply water is pumped in at the top and cold return water is discharged from the bottom. The opposite holds when the tank is discharged. The supply water at the top and the return water at the bottom are separated by a natural thermic boundary layer (of 1/2-2 m) (Dyrelund and Brandon, 1993).

The tank may also serve other purposes, such as covering peak loads, back-up reserve for short term fault situations, supplying static pressure for the network and as an expansion tank.

2.2.3 Heat Exchangers

Heat exchangers play an important role in district heating systems. In district heating systems there are two main utilities of heat exchangers. Firstly, large heat exchangers are often used for separating the transmission and distribution networks, e.g., to maintain a pressure difference between different parts of the network. Secondly, small heat exchangers are used to interface the consumer and the distribution system. This is normally the case in Sweden and Finland.

Moreover, the condensers in the CHP plants and the radiators in the consumer buildings may also be considered as heat exchangers.

3 Heat Exchanger Models (Summary of Papers [A1], [A2] and [A3])

3.1 Introduction

The papers [A1], [A2] and [A3] are devoted to modeling of heat exchangers and (statistical) estimation of the parameters in these models. A heat exchanger is a distributed system in which the dynamics may be described by physical laws concerning mass, energy and momentum. One important property is that the dynamic response depends upon the operating points of the massflows and temperatures. For instance, transport lags and time constants depend on massflow and the heat transfer coefficient is a function of both temperature and massflow. This has to be accounted for in the model if it has to be valid over a wide range of operating conditions, which implies either a non-linear model or a linear model with different parameter sets for various operating intervals.

As an approximation of the distributed system lumped parameter models are proposed in which the heat exchanger is described by a number of sections or compartments. The known physical laws concerning the conservation of energy and mass are applied in order to formulate state space models continuous in time.

The parameterization in the models is very important since the non-linearity of the heat transfer process is mainly handled by the parameters. The main reason for the non-linearity is the heat transfer dependence upon massflows and temperatures. For this reason a number of empirical relations showing this dependence have been developed for the heat transfer coefficients, see e.g., Kakac *et al.* (1983) and Palen (1986). Different relations apply to different types of heat exchangers. For a given type of heat exchanger, however, it is expected that the empirical relations will only give a rough indication of the true heat transfer coefficients. A certain adjustment of the relations is therefore required if an accurate model of a specific heat exchanger is required.

In **[A1]**, **[A2]**, and **[A3]** it is shown how the structure of the empirical relations of the heat transfer coefficients are incorporated in the parameters of the model in order to cope with massflow and temperature dependence. Some of the parameters in the empirical relations are also estimated thereby adjusting the formulas to the actual heat exchanger.

The parameters are estimated by minimizing a score function with respect to the parameters. The score function is based on one-step or multistep prediction errors, where the one-step predictions are obtained by a standard or an extended Kalman filter.

Measurements from a small counter flow plate heat exchanger were used in **[A1]**, **[A2]**, and **[A3]**. Recently obtained model results from the same heat exchanger are to be found in Sejling (1993). Similar models are proposed for a large shell and tube heat exchanger in Jonsson and Holst (1989) and Palsson (1989).

3.2 The Models and the Empirical Relations

The general continuous time (and for the time being deterministic) model can be written in state space form as

$$\frac{d}{dt}\underline{T} = \boldsymbol{A}(\underline{\dot{m}}, \underline{T}(t))\underline{T}(t) + \boldsymbol{B}(\underline{\dot{m}}, \underline{T}(t))\underline{T}_{\text{in}}(t) \tag{1}$$

where the states, $\underline{T}(t)$, are the temperature in each section, or the temperature in each section and the temperature of the metal between the sections (in the case the metal effect is included in the model, see **[A1]**). The model inputs, $\underline{T}_{\text{in}}(t)$, are the inlet temperatures on the hot and the cold side. The massflow, $\underline{\dot{m}}(t)$ enter the model through the state matrix, $\boldsymbol{A}(\underline{\dot{m}}, \underline{T}(t))$, and the input matrix, $\boldsymbol{B}(\underline{\dot{m}}, \underline{T}(t))$.

As an example a model with two sections on each side (and neglecting the

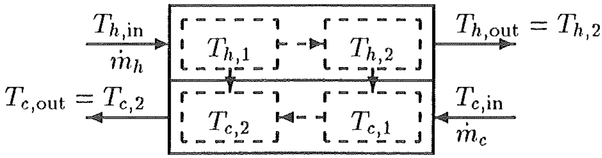

Fig. 1 Schematic diagram of a counter-flow heat exchanger model with two sections on each side.

effect of the metal) is given (see also Fig. 1):

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & 0 & \frac{\alpha}{2\tau_h} & \frac{\alpha}{2\tau_h} \\ (1-\frac{\alpha}{2})/\tau_h & -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} & 0 \\ \frac{\beta}{2\tau_c} & \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c & 0 \\ \frac{\beta}{2\tau_c} & 0 & (1-\frac{\beta}{2})/\tau_c & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix}$$

$$\times \begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & 0 \\ 0 & \frac{\alpha}{2\tau_h} \\ 0 & (1-\frac{\beta}{2})/\tau_c \\ \frac{\beta}{2\tau_c} & 0 \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{2}$$

where the parameters are

$$\alpha(t) = \frac{A_h U(t)}{\dot{m}_h(t) c_h}, \quad \beta(t) = \frac{A_c U(t)}{\dot{m}_c(t) c_c}, \quad \tau_h(t) = \frac{M_h}{\dot{m}_h(t)}, \quad \tau_c(t) = \frac{M_c}{\dot{m}_c(t)} \tag{3}$$

A is the heat transfer area, U is the overall heat transfer coefficient, M is the mass of the water in each section and c is the specific heat for the water. Note that α and β are dimensionless, and τ_h and τ_c are in the units of time. This model is in fact the one found to be sufficient in **[A1]**, **[A2]** and **[A3]**.

Two main versions of the models are considered:

- a model where the effect of the intermediate metal is neglected (see Eq. (2)), **[A1]**, **[A2]** and **[A3]**,
- a model where the effect of the metal is included. Hence, the temperature of the metal between the two media is incorporated in the model as a model state, **[A3]**.

In addition different numbers of sections are considered.

As indicated in Eq. (1) the parameters may be functions of the massflow and the temperature which is due to the fact that the convection heat transfer coefficient is a function of the massflow and the temperature. This is confirmed by the following empirical relation of the heat transfer coefficients which applies to both sides of the considered heat exchanger, see e.g., Palen (1986) or Kakac *et al.* (1983)

$$\mathrm{Nu} = C\mathrm{Pr}^x\mathrm{Re}^y \tag{4}$$

Nusselt (Nu), Prandtl (Pr), and Reynolds (Re) numbers are dimensionless numbers, containing the velocity of the water and some physical properties of the water. C is a constant and x and y are exponents. The empirical relation in Eq. (4) is the general relation for computing the convection heat transfer. Due to the fact that the Reynolds number depends on the velocity and that some of the physical properties (mainly the thermal conductivity and the dynamic viscosity) depend on the temperature, the convection heat transfer coefficient is found to be a function of the mass flow and the temperature.

Hence, three versions of the parameterization are considered and compared:

1. Constant heat transfer, **[A1]**, **[A2]** and **[A3]**.
2. Massflow dependent heat transfer, **[A1]**, **[A2]** and **[A3]**.
3. Massflow and temperature dependent heat transfer, **[A3]**, where the temperature dependence is
   (a) either according to the empirical relation in Eq. (4)
   (b) or assumed linear.

In the first two cases the model is linear in the states and non-linear in the parameters, while in the third case the model is non-linear in the states and non-linear in the parameters. The exponent y is estimated in Case 2 and 3 and the exponents y and x or the slope b (in Case 3 (b)) are estimated along with the other parameters.

As shown in Eq. (3) the parameters are time-varying, mainly due to their direct dependence on the massflow, but also due to the massflow and the temperature dependence of the heat transfer coefficient. Hence, the parameters are scaled in order to relate them to some reference conditions (*). Depending on the parameterization considered, the scaling factors are time-varying functions of the actual massflow and temperature, the reference massflow and temperature as well as the exponents y, x and/or the slope b. This is illustrated, for the general case, by the expression (5).

$$\text{parameter}(t) = \text{parameter}_* \times f(\underline{\dot{m}}(t), \underline{\dot{m}}_*, \underline{T}(t), \underline{T}_*, y, x, b) \tag{5}$$

The reference values are chosen from some typical operating conditions of the heat exchanger. The reference (or the nominal) parameters, parameter$_*$, are then estimated along with the exponents and/or the slope.

3.3 The Estimation Methods - Kalman and Extended Kalman Filtering

The heat exchanger model as given by Eq. (1) is time continuous and deterministic. To allow for variations between the model and the true temperatures, it is appropriate to add a noise term. The model is then described by a stochastic differential equation as

$$d\underline{T}(t) = \boldsymbol{A}(\underline{\dot{m}})\underline{T}(t)dt + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}}(t)dt + d\underline{w}_c(t) \tag{6}$$

where $\underline{w}_c(t)$ is assumed to be a Wiener process with the incremental covariance $\boldsymbol{Q}'(t)dt$.

The model for the measurements is assumed to be

$$\underline{T}_{\text{out}}(t) = \boldsymbol{C}\underline{T}(t) + \underline{v}(t) \tag{7}$$

with $\{\underline{v}(t)\}$ a Gaussian white noise sequence with zero mean and covariance $\boldsymbol{R}(t)$. The matrix $\boldsymbol{C}$ is constant in the present case (all entries are 0 or 1).

In the case where the heat transfer coefficient is not temperature dependent, i.e., the model is linear in the states, a standard discrete Kalman filter, see e.g., Gelb (1974), is used to the state estimation. Kalman filtering in this context is discussed in detail in **[A1]**. In this case the continuous time model is discretized. On the other hand, in the case where the heat transfer coefficient is also assumed temperature dependent, the model is non-linear in the states and therefore a continuous-discrete version of an extended Kalman filter has been used for the state estimation, **[A3]**. This version of the extended Kalman filter is quite similar to the standard discrete Kalman filter. The difference is that the state vector and the covariance matrix can not be propagated in the same manner due to the non-linearity in the model, see **[A3]**.

The aim of the parameter estimation is to determine the parameter set which minimizes some score function. In **[A1]** – **[A3]** the score function is based one the least squares principle,

- either on squared one-step ahead prediction errors, **[A1]** and **[A3]**,
- or on squared multistep prediction errors, **[A2]**.

To obtain the one-step prediction errors, a standard or an extended (in the case of temperature dependence) Kalman filter is used for the state estimation. In the multistep case the prediction of the model is simulated or propagated from one time point to another without any correction of the model states as in the case when the Kalman filter is applied. Maximum Likelihood type score function is applied on related heat exchanger models in Sejling (1993).

3.4 The Main Results

The following conclusions can be drawn based on the results presented in [A1], [A2], and [A3]:

- There are no obvious advantages of including the effect of the metal between the two media. The modest reduction in the score function does not compensate for the increase in the number of parameters and higher model order, [**A1**].
- The model performance is drastically improved when the heat transfer is assumed massflow dependent instead of being constant, [**A1**], [**A2**] and [**A3**].
- The steady state behavior is improved when heat transfer also is assumed temperature dependent, [**A3**].
- The estimated values of the parameters α and β compare nicely with values determined from steady state conditions of the heat exchanger and from its geometry, [**A1**] and [**A2**].
- The parameters τ_h and τ_c are larger than expected from physical consideration, mostly due to the heat capacity in the metal and the time-delay which are not modeled explicitly, [**A1**] and [**A2**].
- The obtained values of the Reynolds number exponent, y, [**A1**], [**A2**] and [**A3**], and of the Prandtl number exponent x, [**A3**], are reasonably close to what the literature suggests.
- The temperature dependence of the heat transfer may advantageously be approximated by a linear function, [**A3**].
- The parameter estimates obtained by using the multistep prediction based score function, [**A2**], compare nicely with those obtained by using the one-step prediction based score function, [**A1**]. This indicates a good modeling of the process.
- Comparison of the model performance when using statistically obtained parameter values versus physically obtained parameter values

Table 1 Comparison of the papers.

Description		[A1]	[A2]	[A3]
Model:	No metal effect	✓	✓	✓
	Metal effect	✓		
	Different no. of sections	✓	✓	✓
Parameters:	$\{\alpha_*, \beta_*, \tau_{h,*}, \tau_{c,*}\}$	✓	✓	✓
	$\{\alpha_{m*}, \beta_{m*}, \tau_{h,*}, \tau_{c,*}, \tau_{mh*}, \tau_{mc*}\}$	✓		
	$\{\alpha_*, \beta_*, \tau_{h,*}, \tau_{c,*}, y\}$	✓	✓	✓
	$\{\alpha_{m*}, \beta_{m*}, \tau_{h,*}, \tau_{c,*}, \tau_{mh*}, \tau_{mc*}, y\}$	✓		
	$\{\alpha_*, \beta_*, \tau_{h,*}, \tau_{c,*}, y, x\}$			✓
	$\{\alpha_*, \beta_*, \tau_{h,*}, \tau_{c,*}, y, b\}$			✓
Estimation:	Multistep prediction errors		✓	
	One-step prediction errors	✓		✓
	- Kalman Filter	✓		✓
	- Extended Kalman Filter			✓

shows that a more accurate description of the dynamics is obtained by using statistical methods, **[A2]**.

- Based on the data used, from the small plate heat exchanger, it is shown that using two sections on each side and letting the heat transfer coefficient be massflow dependent gives a satisfactory description of the dynamics of the heat exchanger, **[A1]**, **[A2]** and **[A3]**.

Finally, a comparison of the papers is found in Table 1.

3.5 Discussion

The benefit of the present research depends on the situation at hand. If the heat exchanger model is to be used in a simulation study of the (whole) district heating system, where the sampling frequency is relatively large compared to the time constants of the heat exchanger, dynamic models of the heat exchanger are of minor importance, and a steady state description of the heat exchanger may be sufficient. The method can, however, still be used to estimate the massflow and temperature dependence of the heat transfer coefficients. On the other hand, if the intention is to, e.g., study the control of the consumer installations (where indirect system is used), e.g., in order to optimize the cooling of the primary water or minimize variations in the temperature of the secondary water, these models may be found useful, due to the low model order and the massflow and temperature dependence. These models can also be used for detection purposes, i.e., by comparing the model outputs and the observations, or by monitoring the heat transfer parameters (α and β), in order to detect fouling of the heat transfer surfaces.

4 Control of the Supply Temperature (Summary of Papers [B1], [B2], [B3] and [B4])

4.1 Introduction to Supply Temperature Control

In general, the potential control variables, to control the heat supply in district heating systems are the supply temperature from the plant and the flow of the water circulating in the system. I.e., in order to fulfill the heat requirements of the consumers, it can either be done by adjusting the supply temperature of the water, or the mass flow, or both.

In district heating systems, where the heat is produced at CHP plants, the simplest and most widely used method is the central temperature control, see Oliker (1980). In this case, the required amount of heat is provided to the consumer by varying the supply temperature at the power plant as a function of the current outdoor air temperature. Such functions (control curves) can be found, e.g., in Madsen *et al.* (1992), Horlock (1987), and Oliker (1980). This means that the flow rate of water circulated in the district heating system is almost constant during the heating period, see e.g., Oliker (1980). This control strategy must obviously be very cautious, i.e., frequently the supply temperature must be higher than necessary, since the ambient air temperature is the only meteorological factor considered, and the control curves do not take into account the dynamic characteristic of the district heating system and the diurnal variation of the heat consumption.

The saving potential in district heating systems lies either in lowering the supply temperature from the plant, hence lowering the heat losses in the transmission and the distribution network and lowering the production costs, or lowering the mass flow, which lowers the pumping costs. The savings in the production costs are due to the fact, that a decrease in the supply temperature implies an increase in the ratio of the power to heat output, and as electricity is more valuable than heat a more profitable operation is achieved, see e.g., IEA (1983).

If the heat is cogenerated at a Combined Heat and Power (CHP) plant, as is the case in the present study, the savings achieved by lowering the supply temperature are much greater than the pumping cost savings that could be achieved. Therefore, the pumping cost is neglected[1] in the present study and the emphasis is on lowering the supply temperature. However, this can not be done without some restrictions, which are:

- The total heat requirement for all consumers is supplied at any time.
- Each individual consumer is guaranteed some minimum supply temperature at any time.
- The short time changes in the supply temperature from the district heating plant are kept small.

This control concept has been the subject for extensive research activities at IMSOR during the past years, see e.g., Madsen *et al.* (1992) and Madsen *et al.* (1990). See **[B4]** for a general description of the control concept.

The first two restrictions are coped with by a flow sub-controller, see Madsen *et al.* (1992), Søgaard (1993), where use is made of models for prediction of the heat load, see Sejling (1993). The objective of the flow sub-controller is to keep the mass flow circulating in the system near the upper nominal (physical) limit, hence minimizing the supply temperature. The last two restrictions are coped with by several supply temperature sub-controllers which keep the supply temperature at several representative points in the distribution network near some ambient air temperature dependent minimum supply temperature and restrain the variation in the supply temperature from the plant.

An overall controller then selects the highest supply temperature among optimal supply temperatures calculated both from the flow sub-controller and from the supply temperature sub-controllers. This temperature is used as the recommended supply temperature from the plant in the next time interval.

[1] Studies where the pumping cost is considered in the operational optimization can, e.g., be found in Benonysson (1991).

The supply temperature sub-controllers are the subject in **[B1]**, **[B2]**, and **[B3]**. The subject of the different papers are as follows:

- The supply temperature sub-controller proposed in **[B1]** is a Linear Quadratic Gaussian (LQG) type controller for non-stationary systems.
- Paper **[B2]** presents modifications of the classical Generalized Predictive Controller (GPC). The modifications are made in order to cope with the embedded non-stationarity in the district heating system and in order to be able to use an ARMAX model instead of an ARIMAX model as in the classical case.
- The modified GPC controller, described in **[B2]**, is studied in a district heating context **[B3]**.

This section briefly summarizes the controllers and the simulation results presented in **[B1]**, **[B2]** and **[B3]**.

4.2 The Predictive Controllers

The controllers investigated in the present study are all members of (model-based) predictive controllers. These controllers are reasonable choices due to the relatively large delay in the district heating systems. In the following a brief description of these controllers is given.

The Model The controllers considered are all based on predictions of the future outputs. The predictions are obtained by using statistical transfer function models. The transfer function models considered in **[B1]**, **[B2]** and **[B3]** belong to the single-input single-output ARMAX, with *time-varying* parameters, which describes the relation between the supply temperature from the district heating plant (control) and the supply temperature in the district heating network (output). The model used in the simulation studies is originally identified in Søgaard (1988) and the parameters used in the present study are estimated from real data (from the first month of 1990) from the district heating system in Esbjerg, Denmark. The variation of the parameters is shown in [**B1**].

The Prediction Traditionally the predictions are found, in this context, by solving the Diophantine[2] Equation, see e.g., Clarke *et al.* (1987). Due to the time-variation of the parameters this procedure cannot be applied in the present study. Instead the predictions are found as the conditional expectation, which is briefly described below.

In [**B2**] and [**B3**] it is shown how the optimal j-step prediction of the output, $\hat{y}_{t+j|t}$ (for *time-varying* ARMAX models) can be found as the conditional expectation of y_{t+j}, conditioned on observations known at time t. Furthermore, it is shown how the general j-step prediction can be written as a time-varying impulse response function

$$\begin{aligned} \hat{y}_{t+j|t} &= h_{j,t+j}u_t + \cdots + h_{1,t+j}u_{t+j-1} + v_{j,t} \\ &= \sum_{i=1}^{j} h_{i,t+j}u_{t+j-i} + v_{j,t} \end{aligned} \tag{8}$$

where $v_{j,t}$ contains all the known information (at time t), $u_t, \ldots, u_{t+j-1}$ are the (unknown) present and future controls, and the coefficients $h_{i,t+j}$ $(i = 1, 2, \ldots)$ are the weights of the time-varying impulse response function. I.e., the j-step predictor is given as a sum of the unknown controls times the time-varying impulse response weights, plus a part which is known at time t.

Considering Eq. (8) and introducing a minimum and maximum prediction horizon N_1 and N_2 and a control horizon N_u, it is seen that the j-step predictions, j running from N_1 up to N_2, can be written as a linear matrix expression:

$$\hat{\boldsymbol{y}}_t = \boldsymbol{H}_t \boldsymbol{u}_t + \boldsymbol{v}_t \tag{9}$$

where $\hat{\boldsymbol{y}}_t$ is a $(N_2 - N_1)$-dimensional vector containing the predictions, $\boldsymbol{u}_t$ is a N_u-dimensional vector containing the controls, $\boldsymbol{v}_t$ is a $(N_2 - N_1)$-dimensional

[2] The same equation is called the Aryabhatta Equation in Vidyasagar (1985) and the Van Compernolle Equation in Bitmead *et al.* (1990).

vector containing the known part of the prediction, and $\boldsymbol{H}_t$ is a $(N_2 - N_1) \times N_u$-dimensional matrix containing the time-varying impulse response weights.

Only the k-step prediction (k = time-delay) is needed in **[B1]**, where it is written as

$$\hat{y}_{t|t+k} = \alpha_t u_t + \beta_t \tag{10}$$

here α_t is the first weight of the impulse response and β_t is the known part of the prediction.

The Cost Functions The purpose of the controllers is twofold: to keep the output (the supply temperature at a representative point in the distribution network, see **[B4]**) close to a reference output (the ambient air temperature dependent minimum, see **[B4]** and **[B3]**), and to keep the variation in time of the control signal (the supply temperature from the plant) reasonably small, since rapidly changing the supply temperature from the plant is not desirable as it implies wearing of the pipelines and causes an undesirable operation of the plant.

In order to satisfy these requirements the following cost function is proposed in **[B1]**

$$J = E\left[(y_{t+k} - y^{\text{ref}}_{t+k|t})^2 + \lambda(\Delta u_t)^2\right] \tag{11}$$

where λ is a penalty parameter, i.e., it puts more or less penalty on Δu. Note that for $\lambda = 0$ the Minimum Variance controller is obtained. The behavior of this controller depends seriously on accurate estimates of the time-delay and the model order.

In **[B2]** and **[B3]** the GPC cost function considered is written as

$$J = E\left[\sum_{j=N_1}^{N_2} (y_{t+j} - y^{\text{ref}}_{t+j|t})^2 + \sum_{j=1}^{N_u} \lambda_{j,t}(\Delta u_{t+j-1})^2\right] \tag{12}$$

In addition the controller is subject to the constraint that the control signal is constant after the control horizon, i.e., $\Delta u_{t+i-1} = 0$, for $i > N_u$. This constraint influences the optimization only through the dimension of $\boldsymbol{H}_t$ and $\boldsymbol{u}_t$.

In fact a more general type of cost function is proposed in **[B2]**

$$J = E\left[\sum_{j=N_1}^{N_2}(y_{t+j} - y^{\text{ref}}_{t+j|t})^2 + \sum_{j=1}^{N_u}\lambda_{j,t}(\bar{u}_{t+j-1})^2\right] \tag{13}$$

where $\bar{u}_t$ is a filtered control signal, with $\bar{u}_t = \Delta u_t$ as an important special case.

The Controllers It is easily shown that also in the time-varying case the predictions and the prediction errors are uncorrelated. This implies that the output can be decomposed into two independent parts, and the cost function can be decomposed into a deterministic and a stochastic part.

The control laws are obtained by minimizing the resulting cost function. This is done by setting the derivative of the cost function, with respect to the control, to zero.

In **[B1]** this gives the optimal control, u_t, directly, but in **[B2]** and **[B3]** this gives an optimal control vector, where only the first element of the vector is of interest (implemented), viz., u_t, (leading to a receding horizon control).

4.3 The Simulation Results Obtained

The performance of the proposed controllers is studied by simulation. The following conclusions can be drawn based on the simulation results presented in **[B1]**, **[B2]**, and **[B3]**:

- For adequate values of the penalty, the variance of the control changes decreases without noticeably higher output variance, **[B1]** and **[B3]**.

- A "small" value of the penalty parameter reduces the amplitude of the control signal noticeably, without any visible change in the output, **[B1]** and **[B2]**.
- Filtering can be used to eliminate the offset, **[B2]**.
- By increasing the control horizon, N_u, the control becomes more active for low values of λ, **[B3]**.
- It turns out that the difference between various values of N_u vanishes for higher λ, **[B3]**.
- A controller with $N_u = 1$ is not interesting, **[B3]**.
- Violations of the Δu constraint can be coped with by tuning the penalty parameter λ, **[B3]**.
- The "best" controller, for the particular model, is obtained with $N_u =$ 2, 3, or 4 and for small non-zero λ values, **[B3]**.
- Higher savings can be achieved by increasing the service parameter α, but this can result in more frequent complaints from the consumers, **[B3]**.
- The proposed modified GPC controller is well suited for controlling the supply temperature in district heating systems, **[B3]**.

4.4 Discussion

In this study no emphasis is put on optimizing the value of the penalty parameter. The only conclusion drawn is that the penalty parameter should be "small", implying a smaller variation of the plant supply temperature, without aggravating the other constraint, viz., keeping the supply temperature in the network close to the reference temperature.

For the previously described controllers it has not been taken into account that the controller is intended to serve as an integrated part in a global control strategy. I.e., the control (the supply temperature from the plant),

u_t, recommended by the particular controller is not necessarily the same as recommended by the overall controller (the highest among several). In order to cope with this further extensions are suggested.

The extended GPC cost function is written as

$$J = E\left[\sum_{j=N_1}^{N_2}(y_{t+j} - y^{\text{ref}}_{t+j|t})^2 + \sum_{j=1}^{N_u}\lambda_{j,t}\Delta u^2_{t+j-1} + \sum_{j=1}^{N_u}\gamma_{j,t}(u_{t+j-1} - u^{\max}_{t+j-2|t-1})^2\right] \quad (14)$$

where (the new variables are)

- $\gamma_{j,t}$ is a new weighting sequence,
- $u^{\max}_{t+j-2|t-1}$ is the largest control determined by all of the GPC subcontrollers at time $t-1$, see Fig. 1 in **[B4]**.

The cost function expressed in Eq. (14) differs from the GPC cost function Eq. (12) in the last term, and it can be minimized in a similar way. This, however, will not be pursued any further here.

5 Optimal Operation of Heat Storage – A Stochastic Approach (Summary of Papers [C1] and [C2])

5.1 Introduction

The relatively low maximum supply temperature (120 °C and normal supply temperature below 100 °C) in the Danish district heating systems allows installation of pressureless steel heat storage (or accumulation) tanks. The advantages of such installation are obvious when the heat storage is combined with CHP plants, viz., it can smooth the heat production during the day making it possible to produce the electricity needed, since the consumption of electricity and heat varies during the day. Furthermore, the heat storage can serve as a back-up reserve for short term faults, avoid use of expensive oil-fired peak load stations, and smooth the variations in heat-only (oil-fired) stations. The heat storage is also useful for maintaining pressure in the district heating network and compensates for water losses on leakages.

The operation of the heat storage can be considered as an optimization problem, particularly in connection with CHP plants. Thus, optimal operation of heat storages has been discussed in, e.g., Mosbech (1982), Nielsen (1991), Ravn (1987) and Ravn and Rygaard (1992) where the problem is formulated and solved as a deterministic optimization problem.

The study presented in [C2] discusses this problem, i.e., the heat storage combined with CHP extraction plant (see description of extraction plant in Section 2.2.1, Horlock (1987) or [C2]). Hence, the problem is to find how the heat storage can be optimally loaded and unloaded, assuming that the power production is subject to some uncertainty and that accurate predictions of the future heat consumption are available.

The objective is to minimize the expected operating cost within the planning horizon considered. The fuel consumption is approximated by a second order polynomial in the heat production, where the coefficients in the polynomial are functions of the power production. Furthermore, given the power output, the heat output can be chosen independently within certain limits, see a sketch of an operating region in Fig. 3 in [C2]. I.e., the figure shows how the maximum heat output depends on the power output.

Hence, the stochastic behavior of the power production influences the optimization problem in two different ways, it makes the maximum heat production uncertain, and the coefficients in the objective function uncertain.

The method used to solve the stochastic problem in [C2] is the new Progressive Hedging Algorithm (PHA), first presented in Rockafellar and Wets (1991). This method is based on scenario analysis. Thus, the stochastic behavior of the future power production is assumed represented by a finite number of scenarios, where each scenario is assumed to have a finite, known probability of occurrence. The scenario analysis and the PHA are described in [C1] where the method is illustrated by small numerical examples. The relevant parts of the method are also described in [C2].

In the following a very brief description of the scenario analysis and the PHA is given. This is followed by a summary of the main results obtained in [C1] and [C2].

5.2 The Scenario Analysis and the Progressive Hedging Algorithm

The PHA is based on scenario analysis, as previously mentioned. The information structure in the scenario analysis can be represented in terms of a tree, as illustrated in Fig. 2, where the stochastic variable takes two values each time. Each scenario is assumed to have a fixed, known probability of occurrence. (These probabilities may either be subjective or objective.) The scenarios are aggregated into scenario bundles, where the scenarios in each bundle share a common history, until that moment of time. The sixteen scenarios in Fig. 2 are aggregated into fifteen scenario bundles, where the first seven are numbered in Roman.

The central point in this procedure is as follows: *at the time the optimization is done the optimizer does not know which scenario will occur. Therefore, scenarios sharing a common history up to any moment of time, viz., scenarios in the same scenario bundle, must also share a common decision up to that moment.* This requirement is called (by Rockafellar and Wets (1991)) the implementability constraint.

The purpose of the PHA is then to find an optimal decision or policy, that is both implementable and admissible, hence feasible. Admissible means that the ordinary constraints in the optimization problem must not be violated.

The PHA finds such a decision in an iterative way, by solving deterministic scenario subproblems, for each scenario in each iteration, and introducing a Lagrange multiplier relative to the implementability constraint and adding a penalty term that ensures the convergence of the algorithm. The algorithm is described in details in both [C1] and [C2].

The deterministic scenario subproblems are solved, in [C2], by a Forward Maximum Principle, see Ravn (1987). This method is based on some assumptions, which the subproblems addressed in [C2] fulfill.

A new feature in [C2] is that the scenario analysis and PHA are used in a receding horizon manner, i.e., the scenario tree is updated each time the stochastic outcome is realized. Hence, this procedure can be used in on-line optimization, this is shown by a flow chart in Fig. 7 in [C2].

5.3 The Main Results

Two simple stochastic optimization problems are considered in [C1] in order to illustrate how the algorithm works. Intermediate results from the iterations are given. The solutions obtained by PHA are compared to solutions obtained by the classical optimization methods (Dynamic Programming and Mathematical Programming). The solutions compare nicely, furthermore, the results indicate that the penalty parameter should be chosen with care, to ensure a fast and accurate convergence.

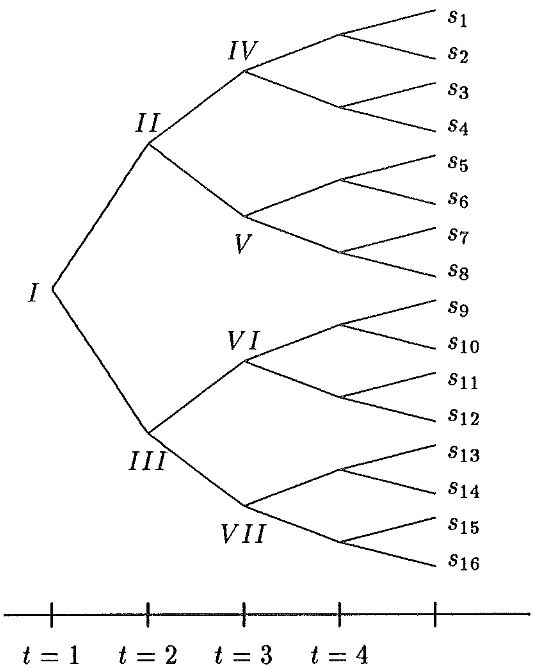

Fig. 2 A scenario tree with sixteen scenarios, [C2].

In [C2] the PHA is applied in order to solve the heat storage problem previously described. In the simulation study four cases of choosing the next scenario path are considered:

- the scenario path is chosen by a random number generator,
- the most probable scenario path is chosen each time,
- the least probable scenario is chosen each time,
- the deterministic equivalent of the problem is considered.

The four cases demonstrate the feasibility of incorporating stochastic elements in the modeling of heat storage operation. This permits more realistic modeling.

The heat production characteristics are different in the four cases presented. In particular this is seen with respect to the upper production limit. However, the variations in the heat storage content are much less outspoken.

5.4 Discussion

It is concluded in [C1] that the PHA may be an attractive alternative to the classical methods.

Traditionally, in stochastic Dynamic Programming all decision variables are discretized and an enumeration algorithm is used to compute an optimal solution at each time stage in a dynamic programming recursion. Hence, computation time grows rapidly with the level of discretization.

One of the advantages of the scenario analysis is that the only thing needed to be discretized is the stochastic distribution. The growth in problem size for PHA is linear in the number of scenarios considered, but the number of scenarios grows exponentially with the number of time steps and the number of stochastic outcome each time. Since the rate of convergence is the weak part of the PHA, the number of scenarios should be kept reasonably small.

The results in [C2] illustrate one of the rationalities for having a heat storage, viz., the possibility of smoothing production over time and - as demonstrated in [C2] - the stochastic variations.

In the study in [C2] the stochastic power production is assumed to take two values each time, and the decision horizon applied is 6 hours, hence, there are 64 deterministic scenario subproblems to be solved in each iteration. The results indicate that this choice of horizon is too short. However, it should be noted, that although it is conceptually straightforward to increase the horizon, it is computationally demanding. This is because the number of individual deterministic scenario problems grows exponentially. Thus, while $2^6 = 64$ scenarios are solved in [C2] in each iteration, a doubling of the horizon to 12 periods calls for solution of $2^{12} = 4096$ scenarios. This is the main difficulty in optimization of stochastic systems.

6 Conclusions

The present thesis considers different aspects of stochastic modeling, control and optimization in connection with district heating systems. The modeling part is concerned with dynamic modeling and parameter estimation of heat exchangers. The control part deals with stochastic predictive controllers, suitable for controlling the supply temperature in district heating systems, and the optimization part introduces stochasticity in the management of a heat storage.

The stochastic approach is advantageous in various respects. In the modeling part the different assumptions made and the uncertainty in the measurements are handled by introducing process and observation noise in the models. This also results in an estimate of the uncertainty in the estimated parameters. In the control part the stochastic predictive controllers are based on transfer function models, obtained from relevant measurements, thus, they do not depend on a huge amount of physical data. The uncertainty in the output prediction is used in the calculation of the reference temperatures, i.e., the temperatures that the controller should aim at. Finally, in the optimization part the uncertainty in the future power production is considered, this implies uncertain parameters in the objective function and uncertain maximum limits on the heat production. Although the future power production may be known with good accuracy, emergency situations may occur and these have consequences for the acceptable ranges of heat production.

The conclusions drawn are listed, for the respective parts in the previous sections. The main conclusions of the thesis can be summarized as follows:

- Physical interpretable low order state space models, where the mass-flow and/or temperature dependence of the heat convection is explicitly incorporated, give a satisfactory description of the dynamics of heat exchangers over a wide range of operating conditions.
- Model-based predictive controllers, especially the modified Generalized Predictive Controller, are well suited for controlling the supply

temperature in district heating systems. The modified GPC controller is able to cope with time-varying dynamics, contrary to the ordinary GPC. The modified GPC controller, based on a time-varying model representation (due to diurnal variations), keeps the supply temperature in the network close to, but, with a given probability, over an ambient air dependent minimum, and penalizes large variations in the supply temperature from the plant.

- The stochastic variation in the power production may advantageously be taken into account in the operational optimization of a heat storage. Furthermore, it is concluded that it is both convenient and natural to represent the uncertainty by scenarios, and the solution method applied is found to be an attractive alternative to the classical stochastic optimization tools.

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Modeling and Parameter Estimation of Heat Exchangers - A Statistical Approach

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Abstract

A continuous in time dynamic model of counterflow and parallelflow heat exchangers is presented. The model is on state space form and it is shown how to use statistical methods, Kalman filtering and least squares, to estimate the model parameters. Empirical relations for convection heat transfer are incorporated in the parameters in order to improve the model performance. The parameters in these empirical formulas may be estimated with the same methods, thereby adjusting the relations to the specific heat exchanger. As a case-study, a small counterflow plate heat exchanger is modeled and some results from the parameter estimation are given.

Key Words: Heat Exchangers; Statistical Parameter Estimation; Kalman Filtering; Empirical Relations.

1 Introduction

Heat exchangers are examples of distributed systems in which the dynamics in principle may be described by physical laws concerning mass, energy and momentum. One important property of the heat exchangers is that the dynamic response depends upon the operating points of the massflows and temperatures. For instance, transport lags and time constants depend on massflow, and the heat transfer coefficient is a function of both temperature and massflow. This has to be accounted for in the model if it is to be valid over a wide range of operating conditions, hereby yielding either a non-linear model or a linear model with different parameter sets for various operating intervals.

Heat exchangers have been modeled in many ways as the literature in this field clearly demonstrates, see e.g., Jonsson and Holst (1989), Palsson (1989), Bittanti *et al.* (1982), Humo and Popovic (1982) and Masada and Wormley (1982) for three basically different modeling approaches.

Basically, there are two possibilities to obtain the parameters for heat exchangers. First, it is possible to use the geometry of the heat exchanger along with empirical relations for the heat transfer coefficients to determine the parameter values, see e.g., Gummerus (1988). There is a vast literature on empirical relations for heat exchangers, e.g., showing the dependence of the heat transfer coefficient upon temperature and massflow, Kakac *et al.* (1983), Holman (1974) and Kays and London (1964). If the heat exchanger model is of lumped nature the accuracy of this approach is strongly dependent upon the number of sections used and the approximations made in the modeling. The second possibility is to use some suitable statistical identification method, e.g., maximum likelihood or least squares estimation, to determine the parameters.

However, it seems intuitively attractive to combine these two identification procedures. By combining empirical relations with parameters, which are identified by statistical methods, it is possible to derive an accurate model which is valid over a wide range of operating points for the actual heat exchanger. Such a model is well suited for simulation focusing, e.g., on control strategies.

This is in fact the approach taken in this paper. It can be described as follows: based on energy consideration a state space compartmental model is established (lumped parameter modeling). The model parameters are then determined by minimizing a least squares based score function measuring the sum of the squared one step ahead prediction residuals for both outlet temperatures of the heat exchanger.

2 The Heat Exchanger Model

The heat exchanger model is based on what may be called "Direct Lumping of the Process". This approach has been used by many authors, see e.g., Jonsson and Holst (1989), Palsson (1989), Gummerus (1988), Shoureshi and Paynter (1983) and Steiner (1989). The heat exchanger is divided into sections where the temperature in each section is a model state. The main assumptions used and justified by many authors, e.g., Masada and Wormley (1982) and Ito and Masubuchi (1978), are

- the heat transfer to the surroundings is negligible
- there is no heat conduction in the direction of the flow in the metal between the fluids nor in the fluids themselves
- there is a uniform temperature in each section.

The Basic Model

The aim of the modeling is to derive a state space model capable of describing the outlet temperatures subject to a given course of variation. Based on energy balance and, to begin with, neglecting the effect of the metal between the two media, then for the hot section number i and cold section number j one has

$$M_h c_h \frac{dT_{h,i}(t)}{dt} = \dot{m}_h(t)c_h[T_{h,i-1}(t) - T_{h,i}(t)] - A_h U(t)\Delta T_{ij}(t) \quad (1)$$

$$M_c c_c \frac{dT_{c,j}(t)}{dt} = \dot{m}_c(t)c_c[T_{c,j-1}(t) - T_{c,j}(t)] + A_c U(t)\Delta T_{ij}(t) \quad (2)$$

where i and j ranges from 1 to ns and $T_{h,0} = T_{h,\text{in}}$ and $T_{c,0} = T_{c,\text{in}}$. U is written as time dependent to indicate that it is a function of the massflows and even temperature. The coupling term $AU(t)\Delta T_{ij}(t)$ describes the energy transfer between the two sections. There are several ways to represent $\Delta T_{ij}(t)$, and Palsson (1989) and Steiner (1989) compared three possibilities

1. $\Delta T_{ij}(t) = T_{h,i}(t) - T_{c,j}(t)$
2. $\Delta T_{ij}(t) = [T_{h,i-1}(t) + T_{h,i}(t)]/2 - [T_{c,j-1}(t) + T_{c,j}(t)]/2$
3. $\Delta T_{ij}(t) = \frac{[T_{h,i-1}(t) - T_{c,j}(t)] - [T_{h,i}(t) - T_{c,j-1}(t)]}{\ln([T_{h,i-1}(t) - T_{c,j}(t)]/[T_{h,i}(t) - T_{c,j-1}(t)])}$

It should be noted that Eqs. (1) and (2) apply to both parallel- and counterflow heat exchangers. In the parallelflow case, $i = j$ in ΔT_{ij}, and $j = ns - i + 1$ when modeling a counterflow heat exchanger.

It turns out, see e.g., Jonsson and Holst (1989) or Palsson (1989), that alternative 1 yields too small a temperature difference between the two sections meaning that if the parameters are determined from the geometry and empirical relations for $U(t)$, the model may have considerable steady state error which magnitude mostly depends on the number of sections used in the model. This bias problem is avoided if the parameters are determined by statistical methods. It is, however, desirable that the statistically obtained values coincide with what can be estimated from the relations $\dot{m}_h c_h[T_{h,\text{in}} - T_{h,\text{out}}] = AU\Delta T_{\text{LMTD}} = \dot{m}_c c_c[T_{c,\text{out}} - T_{c,\text{in}}]$, say. This is indeed the case when alternative 2 or 3 is applied instead of alternative 1. Using alternative 3 introduces extra non-linearities in the model and thus it is easier to use alternative 2 than 3 in the model.

With only one section on each side ($ns = 1$), by introducing

$$\alpha(t) = \frac{A_h U(t)}{\dot{m}_h(t) c_h}, \quad \beta(t) = \frac{A_c U(t)}{\dot{m}_c(t) c_c}, \quad \tau_h(t) = \frac{M_h}{\dot{m}_h(t)}, \quad \tau_c(t) = \frac{M_c}{\dot{m}_c(t)} \quad (3)$$

and using alternative 2, Eqs. (1) and (2) can be written on state space form (for simplicity not indicating the time dependence)

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{c,1} \end{bmatrix} &= \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_{h,1} \\ T_{c,1} \end{bmatrix} \\ &+ \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & (1-\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \end{aligned} \quad (4)$$

Including the Effect of the Metal

If the heat capacity of the metal between the two media is taken into account, two additional parameters and one extra state is needed for each pair of sections, i.e., the additional state is the temperature in the metal between the hot and the cold sections where the heat transfer occurs.

The differential equation is given by

$$M_m c_m \frac{dT_{m,i,j}}{dt} = A_h h_h(t) \Delta T_{ij}^h - A_c h_c(t) \Delta T_{ij}^c \quad (5)$$

where ΔT_{ij} now becomes, for the hot and the cold side respectively (using alternative 2 as before)

$$\begin{aligned} \Delta T_{ij}^h &= [T_{h,i-1}(t) + T_{h,i}(t)]/2 - T_{m,i,j}(t) \\ \Delta T_{ij}^c &= T_{m,i,j}(t) - [T_{c,j-1}(t) + T_{c,j}(t)]/2 \end{aligned}$$

The relations between the indices i and j are as before.

The equation analogous to Eq. (4) becomes

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{m,1,1} \\ T_{c,1} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha_m}{2})/\tau_h & \frac{\alpha_m}{\tau_h} & 0 \\ \frac{\alpha_m}{2\tau_{mh}} & -(\frac{\alpha_m}{\tau_{mh}}+\frac{\beta_m}{\tau_{mc}}) & \frac{\beta_m}{2\tau_{mc}} \\ 0 & \frac{\beta_m}{\tau_c} & -(1+\frac{\beta_m}{2})/\tau_c \end{bmatrix}\begin{bmatrix} T_{h,1} \\ T_{m,1,1} \\ T_{c,1} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha_m}{2})/\tau_h & 0 \\ \frac{\alpha_m}{2\tau_{mh}} & \frac{\beta_m}{2\tau_{mc}} \\ 0 & (1-\frac{\beta_m}{2})/\tau_c \end{bmatrix}\begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{6}$$

where the new parameters are given by ($\tau_h(t)$ and $\tau_c(t)$ remain unchanged)

$$\alpha_m(t) = \frac{A_h h_h(t)}{\dot{m}_h(t)c_h}, \beta_m(t) = \frac{A_c h_c(t)}{\dot{m}_c(t)c_c}, \tau_{mh}(t) = \frac{M_m c_m}{\dot{m}_h(t)c_h}, \tau_{mc}(t) = \frac{M_m c_m}{\dot{m}_c(t)c_c} \tag{7}$$

Discussion

The models given by Eqs. (4) and (6) apply to parallel- as well as counterflow heat exchangers since $ns = 1$. If ns is greater than 1 the models differ due to the different indexing in ΔT_{ij} as previously pointed out, but the modeling principles are the same.

Note also that when the model is given by Eq. (6), $U(t)$ in α and β is replaced by the heat transfer coefficients from the hot side to the metal and the metal to the cold side, noted as h_h and h_c respectively.

Generally, Eqs. (4) and (6) can be written as

$$\frac{d}{dt}\underline{T} = \boldsymbol{A}(\underline{\dot{m}})\underline{T} + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}} \tag{8}$$

Equation (8) is easily extended to include any number of sections. With ns sections the model is either of order $n = 3ns$ or $n = 2ns$ depending on whether the effect of the metal is modeled or not. The model yields absolute temperatures. It is linear in the temperatures but non-linear in the massflows. The state matrices in Eq. (8) are shown as functions of the massflows. They are also functions of the temperatures via $U(t)$ but this dependence is not as strong as of the massflows. Letting $\boldsymbol{A}$ and $\boldsymbol{B}$ be temperature dependent makes the model non-linear in the states. This increased complexity should be weighted against the additional model accuracy that is obtained.

3 Including Empirical Relations in the Model Parameters

The main reason for the non-linearity in heat exchangers is the strong massflow dependence and even temperature dependence of the heat transfer coefficients. In order to account for this, empirical relations of the heat transfer coefficients can be used and incorporated in the model parameters.

In the present case

$$\mathrm{Nu} = C\mathrm{Pr}^x\mathrm{Re}^y \tag{9}$$

is used which is applicable to both sides of the heat exchanger, see e.g., Kakac *et al.* (1983), Holman (1974) and Kays and London (1964). Inserting the expressions for Nu, Pr and Re gives

$$\frac{hd}{k} = C(\frac{c_p\mu}{k})^x(\frac{ud\rho}{\mu})^y \tag{10}$$

With $u = \frac{\dot{m}}{\rho A_{\mathrm{cross}}}$ and by rearranging, Eq. (10) becomes

$$h = \underbrace{C\frac{c_p^x d^{(y-1)}}{A_{\mathrm{cross}}^y}}_{i}\underbrace{\mu^{(x-y)}k^{(1-x)}}_{ii}\underbrace{\dot{m}^y}_{iii} \tag{11}$$

The term (i) is constant, i.e., independent of the massflows and temperatures. (ii) is temperature dependent, and (iii) is massflow dependent. This means that Eq. (11) can be written as

$$h = K\{\mu^{(x-y)}k^{(1-x)}\}(\dot{m}^y) \tag{12}$$

and if the temperature dependence is neglected, this simplifies to

$$h = C'\dot{m}^y \tag{13}$$

Including the temperature dependence of h in the parameters means that the model from Eq. (8) becomes non-linear in the states, which means that it is not correct to use standard Kalman filtering for the state estimation. Instead, the extended Kalman filter can be used. This, however, will not be pursued any further here, but will be the subject of a later publication.

The model parameters are given by Eq. (3), and also Eq. (7) if the effect of metal is included in the model. It is observed that the massflows enter the denominator in all the parameters. To account for variations in the massflows it is practical to rewrite them as

$$\begin{aligned}
\alpha_m(\dot{m}_h(t)) &= \frac{h_{h*}A_h}{\dot{m}_{h*}c_h} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} = \alpha_{m*} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} \\
\tau_h(\dot{m}_h(t)) &= \frac{M_h}{\dot{m}_{h*}} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} = \tau_{h*} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} \\
\beta_m(\dot{m}_c(t)) &= \frac{h_{c*}A_c}{\dot{m}_{c*}c_c} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)} = \beta_{m*} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)} \\
\tau_c(\dot{m}_c(t)) &= \frac{M_c}{\dot{m}_{c*}} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)} = \tau_{c*} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)} \\
\tau_{mh}(\dot{m}_h(t)) &= \frac{M_m c_m}{\dot{m}_{h*}c_h} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} = \tau_{mh*} \times \frac{\dot{m}_{h*}}{\dot{m}_h(t)} \\
\tau_{mc}(\dot{m}_c(t)) &= \frac{M_m c_m}{\dot{m}_{c*}c_c} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)} = \tau_{mc*} \times \frac{\dot{m}_{c*}}{\dot{m}_c(t)}
\end{aligned} \tag{14}$$

i.e., the actual massflow $\dot{m}(t)$ is scaled by some reference massflow $\dot{m}_*$. By doing this, the model parameters are related to the reference conditions. The reference values should be chosen from some typical operating conditions of the heat exchanger. Note that in Eq. (14), the heat transfer coefficients are assumed constant and given the index $*$ to relate to the reference conditions.

By using Eq. (13), the massflow dependence in h_h and h_c, α_m and β_m from Eq. (14) become (other parameters are unchanged)

$$\alpha_m(\dot{m}_h(t), y) = \frac{h_{h*}A_h}{\dot{m}_{h*}c_h}\frac{\dot{m}_{h*}}{\dot{m}_h(t)}\frac{h_h(t)}{h_{h*}}$$

$$
\begin{aligned}
&= \alpha_{m*} \frac{\dot{m}_{h*}}{\dot{m}_h(t)} \frac{C' \dot{m}_h^y(t)}{C' \dot{m}_{h*}^y} = \alpha_{m*} \frac{\dot{m}_{h*}^{1-y}}{\dot{m}_h^{1-y}(t)} \\
\beta_m(\dot{m}_c(t), y) &= \frac{h_{c*} A_c}{\dot{m}_{c*} c_c} \frac{\dot{m}_{c*}}{\dot{m}_c(t)} \frac{h_c(t)}{h_{c*}} \\
&= \beta_{m*} \frac{\dot{m}_{c*}}{\dot{m}_c(t)} \frac{C' \dot{m}_c^y(t)}{C' \dot{m}_{c*}^y} = \beta_{m*} \frac{\dot{m}_{c*}^{1-y}}{\dot{m}_c^{1-y}(t)}
\end{aligned} \tag{15}
$$

where it has been assumed that the same relations apply for the heat transfer coefficients on both sides. This is a good approximation in the present case. One may assume different constants, C' and even y, which increases the number of parameters to be estimated, and can introduce problems in the parameter estimation process. Note that α_m and β_m in Eq. (15) are written as functions of the exponent y in order to differ from the notation in Eq. (14).

When the metal effect is not included in the model, one can use the relation $\frac{1}{U} = \frac{1}{h_h} + \frac{1}{h_c}$, i.e., neglect the thermal resistance in the metal between the two media and assume that $A_h = A_c$. $\frac{h(t)}{h_*}$ in Eq. (15) is then replaced by

$$
\frac{U(t)}{U_*} = \frac{(\dot{m}_h(t)\dot{m}_c(t))^y}{(\dot{m}_h(t) + \dot{m}_c(t))} / \frac{(\dot{m}_{h*}\dot{m}_{c*})^y}{(\dot{m}_{h*} + \dot{m}_{c*})} \tag{16}
$$

4 Parameter Estimation

Preliminaries

The heat exchanger model as given by Eq. (8) is continuous in time and until now deterministic. To allow for variations between the model and the true temperatures, a noise term is added in Eq. (8). The model is then described by a stochastic differential equation and written as

$$
d\underline{T}(t) = \boldsymbol{A}(\underline{\dot{m}})\underline{T}(t)dt + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}}(t)dt + d\underline{w}_c(t) \tag{17}
$$

where $\underline{w}_c(t)$ is assumed to be a Wiener process with the incremental covariance $Q'(t)dt$. Stochastic differential equation models are described, e.g., in Åström (1970), Gard (1988) and Gardiner (1983).

By assuming that $\underline{T}_{\text{in}}$ and the model parameters are constant between sampling points, i.e., the inlet temperatures and massflows are constant between sampling instants, the model can be written in discrete time (see e.g., Åström (1970))

$$\underline{T}(t+\Delta t) = \mathbf{\Phi}(\underline{\dot{m}}, \Delta t)\underline{T}(t) + \mathbf{\Gamma}(\underline{\dot{m}}, \Delta t)\underline{T}_{\text{in}}(t) + \underline{w}(t+\Delta t) \tag{18}$$

where

$$\mathbf{\Phi}(\underline{\dot{m}}, \Delta t) = e^{\boldsymbol{A}(\underline{\dot{m}})} = \boldsymbol{I} + \boldsymbol{A}(\underline{\dot{m}}) + \frac{\boldsymbol{A}(\underline{\dot{m}})^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{\boldsymbol{A}(\underline{\dot{m}})^j}{j!} \tag{19}$$

and

$$\mathbf{\Gamma}(\underline{\dot{m}}, \Delta t) = [\int_0^{\Delta t} e^{\boldsymbol{A}(\underline{\dot{m}})s} ds]\boldsymbol{B}(\underline{\dot{m}}) \tag{20}$$

$\{\underline{w}(t)\}$ is a Gaussian white noise vector sequence with zero mean and covariance $\boldsymbol{Q}(t)$. Indeed, the covariance should be a function of the sampling interval and the massflow, but the sampling interval is constant in present case and for simplicity the dependence on massflow is neglected.

The measurement model is assumed to be

$$\underline{T}_{\text{out}}(t) = \boldsymbol{C}\underline{T}(t) + \underline{v}(t) \tag{21}$$

with $\{\underline{v}(t)\}$ a white noise sequence with zero mean and covariance $\boldsymbol{R}(t)$, $\underline{v}(t) \in N(\underline{0}, \boldsymbol{R}(t))$. $\boldsymbol{C}$ is the constant measurement matrix. It describes the linear combinations of the state variables, which comprise $\underline{T}_{\text{out}}$ in the absence of noise. Furthermore, it is assumed, that $\underline{w}(t)$ and $\underline{v}(t)$ are uncorrelated, i.e., $E[\underline{w}(t)\underline{v}^T(s)] = 0$ for all t, s.

Using Eqs. (18), (19) and (20), the model is discretized and propagated between the sampling points. $\mathbf{\Phi}(\underline{\dot{m}}, \Delta t)$ and $\mathbf{\Gamma}(\underline{\dot{m}}, \Delta t)$ need to be calculated each time the massflow or the sampling interval changes, see e.g., Moler and Loan (1978) for a detailed discussion of how to efficiently calculate

$\mathbf{\Phi}(\underline{\dot{m}}, \Delta t)$. $\mathbf{\Gamma}(\underline{\dot{m}}, \Delta t)$ is readily obtained from the relation $\mathbf{\Gamma} = \boldsymbol{A}^{-1}(\mathbf{\Phi} - \boldsymbol{I})\boldsymbol{B}$, see Åström and Wittenmark (1984), when the inverse of $\boldsymbol{A}$ exists, which it does in the present case.

The Kalman Filter

The aim of the parameter estimation is to determine the parameter set which minimizes a score function based on the residual sequences, i.e., the difference between the measurements and the model outputs. In the present case there are two residual sequences, the residuals on the hot and cold side of the heat exchanger.

Let $\hat{\underline{T}}^-(t+\Delta t)$ denote the estimated value of the model states at time $t+\Delta t$ given all measurements up to time t. The optimal estimate, $\hat{\underline{T}}^-(t+\Delta t)$, is the conditional mean $E[\underline{T}(t+\Delta t)|\underline{T}(t), \underline{T}(t-\Delta t), \ldots, \underline{T}(1)]$. Similarly, $\hat{\underline{T}}^-_{\text{out}}(t+\Delta t)$ is the output estimate. The residuals at the time $t+\Delta t$ are then given by

$$\underline{\varepsilon}(t+\Delta t) = \underline{T}_{\text{out}}(t+\Delta t) - \boldsymbol{C}\hat{\underline{T}}^-(t+\Delta t) = \underline{T}_{\text{out}}(t+\Delta t) - \hat{\underline{T}}^-_{\text{out}}(t+\Delta t) \quad (22)$$

$\hat{\underline{T}}^-(t+\Delta t)$ is easily computed recursively using a standard Kalman filter, see e.g., Åström and Wittenmark (1984), Gelb (1974), and Åström (1970). Adapting the notation from Gelb (1974) the algorithm is ($\mathbf{\Phi} = \mathbf{\Phi}(\underline{\dot{m}}, \Delta t), \mathbf{\Gamma} = \mathbf{\Gamma}(\underline{\dot{m}}, \Delta t)$):

System model

$$\underline{T}(t+\Delta t) = \mathbf{\Phi}\underline{T}(t) + \mathbf{\Gamma}\underline{T}_{\text{in}}(t) + \underline{w}(t+\Delta t) \quad (23)$$

Measurement model

$$\underline{T}_{\text{out}}(t+\Delta t) = \boldsymbol{C}\underline{T}(t+\Delta t) + \underline{v}(t+\Delta t) \quad (24)$$

State estimate extrapolation

$$\hat{\underline{T}}^-(t+\Delta t) = \mathbf{\Phi}\hat{\underline{T}}^+(t) + \mathbf{\Gamma}\underline{T}_{\text{in}}(t) \quad (25)$$

Error covariance extrapolation

$$P^{-}(t+\Delta t) = \Phi P^{+}(t)\Phi^{T} + Q(t+\Delta t) \tag{26}$$

State estimate update

$$\hat{\underline{T}}^{+}(t+\Delta t) = \hat{\underline{T}}^{-}(t+\Delta t) + K(t+\Delta t)[\underline{T}_{\text{out}}(t+\Delta t) - C\hat{\underline{T}}^{-}(t+\Delta t)] \tag{27}$$

Error covariance update

$$P^{+}(t+\Delta t) = [I - K(t+\Delta t)C]P^{-}(t+\Delta t) \tag{28}$$

Kalman gain matrix

$$K(t+\Delta t) = P^{-}(t+\Delta t)C^{T}[CP^{-}(t+\Delta t)C^{T} + R(t+\Delta t)]^{-1} \tag{29}$$

where $\underline{w}(t) \in N(\underline{0}, Q(t))$ and $\underline{v}(t) \in N(\underline{0}, R(t))$. The index "+" denotes values of estimates and covariances of the state estimation errors at time $t+\Delta t$ after measurement and "–" indicates the corresponding values after propagation from t to $t+\Delta t$. To start the Kalman filter, initial values of the state vector, $\underline{T}_0$, and the covariance matrix, P_0, are needed. The noise matrices $Q(t)$ and $R(t)$ are either assumed known or they could also be estimated along with the model parameters.

The above formulation of the Kalman filter assumes that the model parameters are known. This is, however, not the case here. There are several possibilities to circumvent this problem. One is to estimate the model parameters simultaneously with the state estimation, leading to an extended Kalman filter, see e.g., Gelb (1974). This approach results in non-linear estimation of time varying parameters. Here, all parameters are assumed to be constant (apart from the deterministic dependence in flow and temperature), and estimated from the whole dataset. Equation (22) is used to calculate the residuals, $\underline{\varepsilon}(1), \ldots, \underline{\varepsilon}(N)$, which are used in the score function which is then minimized w.r.t. the parameters by using a minimization routine.

The Score Function

As mentioned earlier, results from the parameter estimation, when the score function is based on least squares techniques, will be shown. It is given by

$$V(\underline{\theta}) = \sum_{t=t_1}^{N} \underline{\varepsilon}(t)^T \underline{\varepsilon}(t) \tag{30}$$

where $\underline{\theta}$ is a vector containing the parameters included in the model. The reason for summing from $t = t_1$ is to allow for the transient effects to vanish, when the Kalman filter is tuning the model states. Other forms of $V(\underline{\theta})$ are quite possible. For instance, $V(\underline{\theta})$ can be based on multistep prediction residuals instead of one step only. It is also possible to scale the residual sequences as is done in weighted least squares, or to base the score function on the maximum likelihood principle instead of least squares.

Referring to the discussion in the previous chapter, $\underline{\theta}$ may contain up to seven parameters when the noise covariance parameters are supposed to be known.

$$\underline{\theta} = \{\alpha_{()*}, \beta_{()*}, \tau_{h*}, \tau_{c*}, \tau_{mh*}, \tau_{mc*}, y\} \tag{31}$$

The number of parameters is independent of the number of sections used in the model and depends only on the complexity of the chosen model, cf., Eqs. (4) and (6).

To obtain an estimate of the uncertainty in the parameters, the information available from the Hessian matrix, $\boldsymbol{H}$, of the score function, $V(\underline{\theta})$, is used (see e.g., Bard (1974))

$$\mathbf{Cov}[\hat{\underline{\theta}}] \simeq 2\hat{\sigma}^2 \boldsymbol{H}^{-1} = \frac{2V(\hat{\underline{\theta}})}{N-(t_1+l)} \boldsymbol{H}^{-1} \tag{32}$$

and the variance of the i'th parameter is accordingly

$$\mathrm{Var}[\hat{\theta}_i] = 2\hat{\sigma}^2 \{\boldsymbol{H}_{ii}^{-1}\} \tag{33}$$

where $\{\boldsymbol{H}_{ii}^{-1}\}$ denotes diagonal element no. i in the inverse Hessian matrix.

The Algorithm

Summarizing, the basic parameter estimation procedure may be divided into the following steps

1. Supply an estimate of $\underline{\theta}$
2. Use the whole dataset and compute for each measurement point:
   - Discretize the heat exchanger model using Eqs. (19) and (20) if necessary.
   - Calculate the residual vector, $\underline{\varepsilon}(t)$.
   - Update the score function, $V(\underline{\theta})$.
   - Use Kalman filtering to predict the model output temperatures, $\hat{\underline{T}}_{\text{out}}^{-}(t+\Delta t)$, the covariance matrix, $\boldsymbol{P}^{-}(t+\Delta t)$ etc.
3. Iterate until the minimum of $V(\underline{\theta})$ is found.

For the minimization of $V(\underline{\theta})$, routines from the NAG and IMSL libraries which only require function values have been used.

5 Results

The Data

The heat exchanger used in the case study is a small water to water counterflow plate heat exchanger from Alfa-Laval, type CB25, with dimensions $304 \times 103 \times 56$ mm and 14 plates. It is a part of a heat exchanger setup located at the Department of Heat and Power Engineering, Lund Institute of Technology, University of Lund, Sweden.

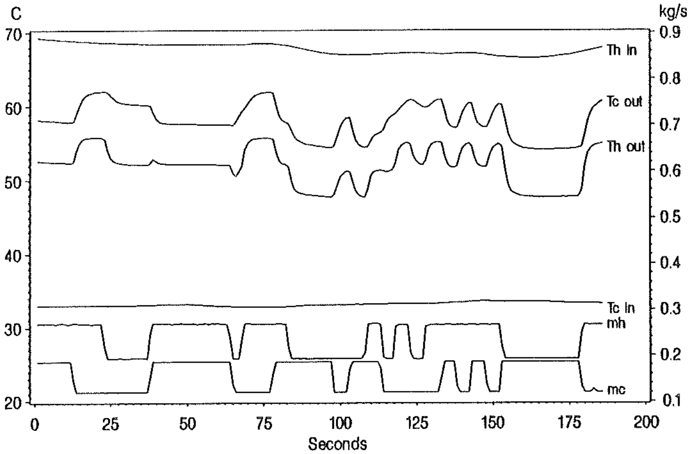


Fig. 1 The experiment used in the parameter estimation.

Figure 1 shows the experiment that was used. The heat exchanger was excited by changing both of the massflows. The temperature measurements are in degrees Centigrade and the accuracy is in the first decimal. The flow measurements are in kg/s and the accuracy is in the third decimal.

The sampling interval was one second and constant throughout the experiment.

The Results

There are at least three different ways to improve the performance of the heat exchanger model given by Eq. (4)

1. Include the temperature of the metal between the two media as a model state, i.e., use models like the one given by Eq. (6).

2. Increase the number of sections.

3. Let the heat transfer coefficients be functions of the massflows.

The first alternative means one new state for each pair of sections and two new parameters, τ_{mh*} and τ_{mc*}, in the model. The second alternative means two or three new states, when ns is increased by one, and the third alternative means one new parameter in the model, i.e., the exponent y, see Eq. (11). This should be kept in mind, when determining what type of model to choose.

Table 1 contains the resulting minimum values of the score function, $V(\hat{\underline{\theta}})$, for different versions of the model. Results when the metal is modeled separately or not, along with different number of sections and two representations of the heat transfer coefficients are shown. As starting values for the Kalman filter $\boldsymbol{P}_0 = 10 \times \boldsymbol{I}$ were used, and the covariance matrices were fixed at $\boldsymbol{R} = 0.006 \times \boldsymbol{I}$ and $\boldsymbol{Q} = 0.006 \times \boldsymbol{I}$. The value of the measurement covariance matrix, $\boldsymbol{R}$, is found from the accuracy of the temperature measurements (± 0.05 degrees Centigrade $\simeq 2\sigma \Rightarrow \sigma^2 \simeq 0.006$). Experiments with different values of the matrix $\boldsymbol{Q}$ showed that the ratio between $\boldsymbol{Q}$ and $\boldsymbol{R}$ was of more importance than the actual values. From the results of these experiments $\boldsymbol{Q}/\boldsymbol{R} = 1$ was chosen. Experiments also indicate that different values of the covariance matrices have little influence on the parameter estimates, but on the other hand they influence the value of the score function and the variances of the parameter estimates. Different initial values of the parameter vector, $\underline{\theta}_0$, were tried in order to verify, that the obtained minimum was global. The starting values of $\underline{T}_0$ are not critical, since the states are quickly tuned in by the Kalman filter. The reference massflows were chosen as $\dot{m}_{h*} = \dot{m}_{c*} = 0.15$ kg/s.

When the metal was included as a model state and the heat transfer coefficient U was constant some identification problems were encountered. These cases are marked by $*$ in Table 1.

The reason for these problems was that the values of τ_{mh*} and τ_{mc*}, which minimize the score function, turned out to be so small (< 0.0002 [s]) that the score function was insensitive to the exact value of the parameters.

Table 1 Value of the score function, $V(\hat{\underline{\theta}})$, for different model types. $*$ and $**$ means that identification problems were encountered.

Metal	ns	n	U const	U $\dot{m}$ dep.
no	1	2	125.8	51.26
	2	4	101.7	38.10
	3	6	101.1	34.70
yes	1	3	125.8*	35.54**
	2	6	101.7*	34.27
	3	9	101.1*	32.66

The problem in finding the optimal parameters are, of course, of numerical character, but they are the result of the fact that it is not possible to estimate the size of the heat capacity in an intermediate metal section using this particular observation set and model.

This conclusion can be derived by examining the time constants. For instance, two of the time constants in the estimate of the third order model with metal effect and constant U are the very same as those found in the second order model without metal effect (5.6 [s] and 1.9 [s]), and the third time constant is effectively zero (0.00002 [s]). Thus, the dynamics of the estimated third order model are the same as that of the corresponding model without metal effect. Moreover, in all three cases the minimum value of the score function is the same as for the corresponding model without metal effect. Therefore, the conclusion must be that, when the heat transfer is modeled as being independent of massflow, modeling the heat capacity in the metal separation explicitly is not improving the model at all.

Also in the case marked by $**$ in Table 1 problems in the numerical minimization occurred. None of the parameters, however, converged to extreme values, but the explanation might be that a model with one section on each side is not sufficient in giving a description that matches the actual dynamical relations and transport lags in the exchanger. This is indicated by the results for models without metal effect. The reason that the score function is so close to the result with more sections might be that the intermediate state takes over the role, from additional sections in the water, of modeling the dynamics and transport lags.

Based on the score function values in Table 1, it may be concluded, that a suitable choice is a model with at least two sections on each side, i.e., $ns = 2$, and the heat transfer coefficient massflow dependent (Eq. (16)). Since there are two states and two parameters more, when the metal is included, the conclusion is that very little is gained by doing so in the present application.

Consider the model with $ns = 2$ and U massflow dependent. The following parameter values and their standard deviations were obtained for that model

$$\begin{aligned} \alpha_* &= 0.79 \ (0.01), & \tau_{h*} &= 1.5 \ (0.16)\ [\mathrm{s}] \\ \beta_* &= 0.81 \ (0.01), & \tau_{c*} &= 1.9 \ (0.15)\ [\mathrm{s}] \\ y &= 0.60 \ (0.05) \end{aligned}$$

Perhaps y is the most interesting parameter. The literature recommends $y \in [0.65, 0.85]$ for a plate heat exchanger, see e.g., Kakac *et al.* (1983). The statistically obtained value is just below this interval.

From α_* or β_*, it is possible to estimate the overall heat transfer coefficient in the heat exchanger. As an example

$$U_* = \frac{\alpha_* \dot{m}_{h*} c_h}{A/2} = \frac{2 \times 0.790 \times 0.15 \times 4180}{12 \times 0.0255} \simeq 3.24\ \mathrm{kW/m^2\,^\circ C} \qquad (34)$$

U can also be computed from the steady state relations $\dot{m}_h c_h (T_{h,\mathrm{in}} - T_{h,\mathrm{out}}) = AU\Delta T_{\mathrm{LMTD}}$. The starting values of the dataset give (see Fig. 1)

$$U = \frac{\dot{m}_h c_h (T_{h,\mathrm{in}} - T_{h,\mathrm{out}})}{A \Delta T_{\mathrm{LMTD}}} = \frac{0.27 \times 4180 \times (69.3 - 52.6)}{12 \times 0.0255 \times 14.95} = 4.12\ \mathrm{kW/m^2\,^\circ C}$$

Equation (16) gives $\frac{U}{U_*} = 1.27$ and therefore

$$U_* = 4120/1.27 \simeq 3.24\ \mathrm{kW/m^2\,^\circ C}$$

which compares nicely with the value from Eq. (34). Analogous results are obtained when β_* is used.

It may be asked whether there is anything gained by using statistical methods in the parameter estimation since the values above coincide with what can be calculated in other and more traditional ways. There are at least two reasons. First, the values of e.g., the exponent y that are found in the literature are guidelines and the "correct" value may differ considerably from one heat exchanger to another although they are basically of the same structure. Secondly, the parameters τ_{h*} and τ_{c*} should, according to the specifications of the heat exchanger used here, be approximately 1.0 and 1.2 seconds respectively. Compared to the values above, i.e., 1.5 and 1.9 seconds respectively the difference is about 50%. This difference is, at least partly, explained by acknowledging that τ_{h*} and τ_{c*} are parameters that are tuned in order to make the transient response of the model match the measurements. Furthermore, τ_{h*} and τ_{c*} represent factors not accounted for explicitly in the model, like the effect of the metal between the two media. This would be difficult to accomplish satisfactorily without using statistical methods.

Figure 2 shows the model output and the measurements, when the optimum parameters were used. The agreement is quite satisfactory. Note that the model output in the figure is obtained by simulating the model from time t_1, i.e., after the Kalman filter has tuned in the states, it is removed and the model simply integrated or propagated between the sampling points. This was not the case in the parameter estimation, where the Kalman filter was on all the time.

Finally, Fig. 3 shows the model response when the same optimum parameters were used, but applied to data from another experiment. The model was propagated in the same manner as described above. The agreement is not as profound as in Fig. 2. This is because the model parameters were determined from another series where the situation was not exactly the same. For instance, the inlet temperatures are at different levels and the heat transfer coefficient is not assumed temperature dependent which introduces some error. Nevertheless, the agreement between the measured and predicted outlet temperatures in Fig. 3 is quite acceptable.

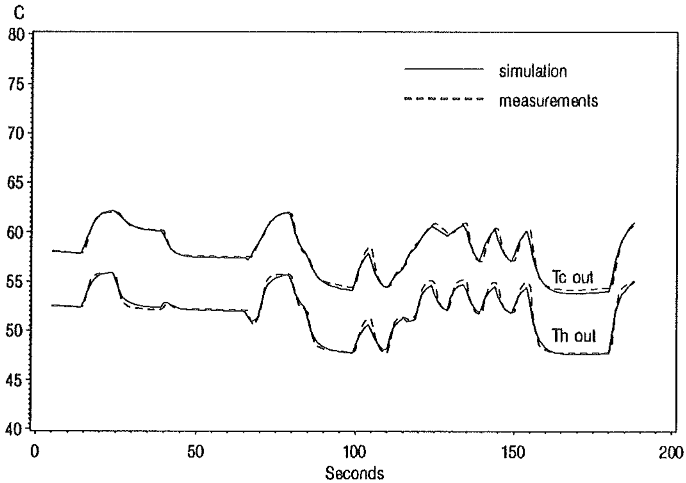


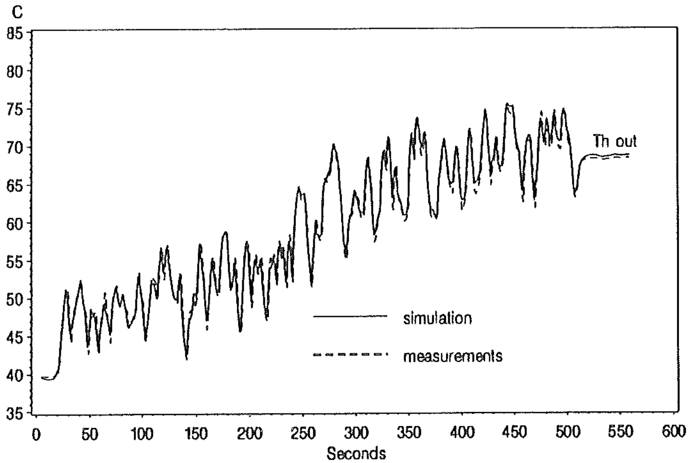
Fig. 2 The measurements and the model output. The same dataset is used for estimation and simulation. T_h and T_c are the hot and the cold outlet temperature respectively.

6 Summary and Conclusions

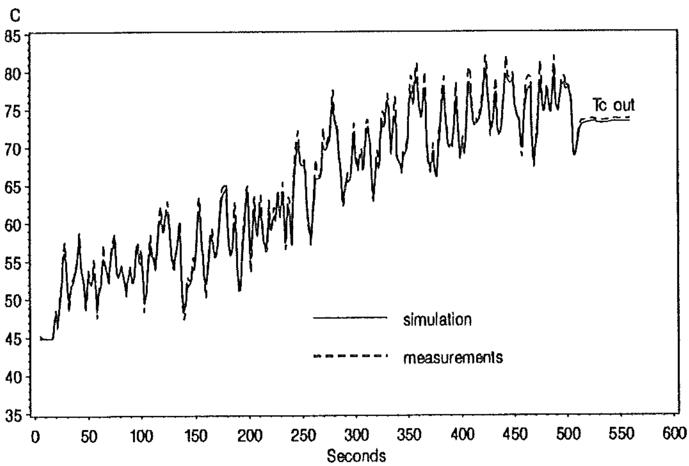
The aim of the paper is twofold. First, to obtain a model that is capable of describing the dynamics of a heat exchanger over a wide range of operating conditions and secondly to show, how the parameters in the model can be estimated by means of statistical methods given a series of measurements.

For that purpose, a time continuous model of a heat exchanger is presented. The model is on state space form and based on lumping of the process. The temperatures in each section are the model states. The model is linear in the states but non-linear in the massflows as they enter the model through the state matrices. Two basic versions of the model are shown, i.e., with and without including the temperature of the metal between the two media as a model state.

The problem of identifying the model parameters is addressed in detail.



(a)



(b)

Fig. 3 The measurements and the model output. The model parameters were estimated from a different dataset. T_h, (a), and T_c, (b), are the hot and the cold outlet temperature respectively.

First of all, the parameters can be determined from the geometry of the heat exchanger and by using empirical relations. This approach is strongly dependent upon the assumptions done in the modeling and can easily give parameter values, which are far away from the correct values for the actual heat exchanger. However, by using data obtained from experiments on the heat exchanger in question and using statistical methods to determine the model parameters, such parameter values will be obtained that yield a good model response, as long as the model itself is capable of providing the dynamics, that are needed. If the approximations made in the modeling are many and crude, the statistically determined values will probably differ severely from what might be expected and even calculated from steady state considerations. If the modeling is adequate, the statistically determined parameters should be in good coherence with the expected values. In any case, using statistical methods to identify the model parameters ensure parameter values that result in correct model response and any moderate discrepancies in the modeling approach are compensated for in the parameter values. Last but not least, by using statistical methods one obtains a measure of the uncertainty in the parameter values.

An important feature of the present approach is that given one experiment which preferably scans the whole operating area of the heat exchanger (both massflows and temperatures), one can include the coefficients in empirical formulas of the heat transfer coefficients, say, in the parameter estimation and simultaneously estimate them with the rest of the model parameters.

Based on the data used in the present study, it was found that a model with two sections on each side, and not explicitly including the effects of the metal between the two media, gave a very satisfactory performance. The values of some of the model parameters can be verified from the steady state conditions of the heat exchanger and, furthermore, from values in empirical formulas of the heat transfer coefficients, given as guidelines in the literature. The comparison shows that the statistically determined values compare favorably with those values which, accompanied by the good dynamic description of the heat exchanger, indicates a good modeling of the process.

Acknowledgements

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Nomenclature

A	=	heat transfer area
$\boldsymbol{A}$	=	$(n \times n)$ system matrix
$\boldsymbol{B}$	=	$(n \times m)$ input matrix
c_p	=	specific heat
C, C'	=	constants
$\boldsymbol{C}$	=	$(m \times n)$ measurement matrix
Cov	=	covariance matrix
d	=	diameter
h	=	heat transfer coefficient
$\boldsymbol{H}$	=	$(l \times l)$ Hessian matrix
$\boldsymbol{I}$	=	identity matrix
k	=	thermal conductivity
K	=	constant
$\boldsymbol{K}$	=	$(n \times m)$ Kalman filter gain matrix
l	=	number of parameters
M	=	mass in each section
m	=	number of inputs and outputs
$\underline{\dot{m}}$	=	m dimensional massflow vector
$\dot{m}$	=	massflow
n	=	number of states
N	=	number of time points

ns	=	number of sections
$\boldsymbol{P}$	=	$(n \times n)$ error covariance matrix
$\boldsymbol{Q}$	=	$(n \times n)$ system noise covariance matrix
$\boldsymbol{Q}'(t)dt$	=	incremental covariance matrix
$\boldsymbol{R}$	=	$(m \times m)$ measurement noise covariance matrix
t	=	time
T	=	temperature
$\underline{T}$	=	n dimensional state vector
U	=	overall heat transfer coefficient
$\underline{v}$	=	m dimensional white noise vector, $\underline{v} \in N(\underline{0}, \boldsymbol{R})$
V	=	score function
$\underline{w}$	=	n dimensional white noise vector, $\underline{w} \in N(\underline{0}, \boldsymbol{Q})$
$w_c(t)$	=	Wiener process
x, y	=	exponents
Δt	=	time step (sampling interval)
ΔT	=	temperature difference
ΔT_{LMTD}	=	log-mean-temperature-difference

Greek Letters

α	=	model parameter
β	=	model parameter
$\boldsymbol{\Gamma}$	=	$(n \times m)$ discrete input matrix
$\underline{\varepsilon}$	=	m dimensional residual vector
$\underline{\theta}$	=	l dimensional parameter vector
μ	=	dynamic viscosity
ρ	=	density
σ	=	standard deviations
τ	=	model parameter
$\boldsymbol{\Phi}$	=	$(n \times n)$ discrete system matrix

Dimensionless Groups

Nu	=	Nusselt number (hd/k)
Pr	=	Prandtl number ($c_p\mu/k$)
Re	=	Reynolds number ($\rho ud/\mu$)

Superscripts and Subscripts

c	=	cold
cross	=	cross sectional (area)
h	=	hot
in	=	input
m	=	metal
mc	=	the cold side of the metal
mh	=	the hot side of the metal
m, i, j	=	metal between hot no. i and cold no. j
out	=	output

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Use of Empirical Relations in the Parameters of Heat Exchanger Models

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Abstract

This paper shows how empirical relations of the heat transfer coefficients can be included in the parameters of heat exchanger models, thereby making them valid over a wide range of operating conditions. Some of the parameters in these relations are estimated along with the remaining parameters of the models by using statistical parameter estimation methods. It is possible to use simple models with this approach, as any moderate discrepancies in the modeling are successfully compensated for in the model parameters since they are determined by matching the model response to actual measurements.

Furthermore, the difference in the model performance when using statistically estimated parameters instead of parameters derived solely from the geometry of the heat exchanger and steady state considerations is demonstrated. The comparison is in favor of the statistical approach.

Key Words: Heat Exchangers; Physical State Space Model; Parameter Estimation; Multistep Prediction; Empirical Relations.

1 Introduction

The aim of this paper is to show how empirical relations of the heat transfer coefficient can be incorporated in the parameters in a heat exchanger model. Some of the parameters in these relations are estimated by statistical methods along with the parameters of the model. This means that the empirical formulas are adjusted to the specified heat exchanger, resulting in a considerably more accurate description of the dynamics of the heat exchanger compared to what is obtained that if the empirical formulas were used directly or adjusted in some "manual" way.

Heat exchangers constitute distributed parameter systems. They are nonlinear, mainly because the heat transfer is a function of the massflows and the temperatures. This must therefore be accounted for, if the model is to be able to describe the dynamics accurately. This is most easily done by using empirical formulas for the heat transfer coefficients. Those are given in abundance in the literature for various types of heat exchangers, see e.g., Kakac *et al.* (1983), Holman (1976) and Kays and London (1964). However, if they are applied without being adjusted to the specified heat exchanger, it is very likely that the heat exchanger model will give an unnecessarily poor description of the heat exchanger dynamics.

There are several ways to adjust the formulas. They can be adjusted manually but that is difficult if there are more than just a few parameters to tune. Another possibility is to do many experiments with the heat exchanger and by nonlinear regression analysis based on steady state calculations of the energy transfer, determine the parameters in the empirical formulas, see e.g., Gummérus (1988). There are at least two drawbacks with this approach. First, it demands many steady state values, i.e., many experiments. Secondly, and more seriously, this verification does not take into account any modeling approximations. This means that if the model is based on a lumping of the process, which is very common, one is inevitably forced to use many sections in the modeling in order to reduce the lumping effect. This is in contrast with one of the general aims of the modeling procedure, i.e., to use as low order models as possible.

The third possibility, which is the one that this paper discusses, is to incorporate the empirical formulas in the model and via parameter estimation in the model, adapt the coefficients in the formulas to the specified heat exchanger. Through one experiment, which scans the operating area of the heat exchanger, in which the model is to be valid, the coefficients along with the remaining model parameters are estimated by statistical methods. It is possible to use models of a very low order with this approach. If the modeling is poor, it is expected that the obtained parameters differ from what can be expected and vice versa if the models are good. In any case, the resulting parameter values ensure that the model describes the dynamics of the heat exchanger accurately, as long as the model structure is capable of supplying the dynamics that are needed.

The paper starts with a brief description of the heat exchanger model, followed by a discussion of how the empirical relations are included therein. The parameter estimation procedure is then briefly described and finally some results are given.

The measurements used were obtained from a heat exchanger rig located at the Laboratory of Heat and Power Engineering, Lund Institute of Technology, Sweden. The heat exchanger is a small water to water counterflow plate heat exchanger, (Alfa-Laval CB25) which for instance is very common in subcentrals in district heat networks in Sweden.

2 The Heat Exchanger Model

The model is based on direct lumping of the process. This approach is also applied in Jonsson and Holst (1989), Steiner (1989) and Palsson (1989). Figure 1 shows how the heat exchanger is divided into ns sections on each side and the arrows shown indicate the energy flow. Using the following assumptions

- the heat transfer to the surroundings is negligible

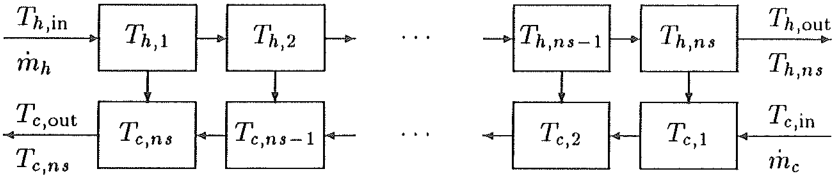

Fig. 1 Schematic diagram of the heat exchanger model.

- there is no heat conduction in the direction of the flow in the metal between the fluids nor in the fluids themselves
- there is a uniform temperature in each section
- the specific heat capacities are constant

for hot section number i and cold section number j the following differential equations are obtained:
Hot section:

$$M_h c_h \frac{dT_{h,i}(t)}{dt} = \dot{m}_h(t) c_h [T_{h,i-1}(t) - T_{h,i}(t)] - A_h U(t) \Delta T_{ij}(t) \quad (1)$$

Cold section:

$$M_c c_c \frac{dT_{c,j}(t)}{dt} = \dot{m}_c(t) c_c [T_{c,j-1}(t) - T_{c,j}(t)] + A_c U(t) \Delta T_{ij}(t) \quad (2)$$

i and j range from 1 to ns and $T_{h,0} = T_{h,\text{in}}$, $T_{h,ns} = T_{h,\text{out}}$, $T_{c,0} = T_{c,\text{in}}$ and $T_{c,ns} = T_{c,\text{out}}$. Equations (1) and (2) apply to both parallel- and counterflow heat exchangers. In the parallelflow case, $i = j$ in ΔT_{ij} and $j = ns - i + 1$ when modeling a counterflow heat exchanger.

The term, $A_{()} U(t) \Delta T_{ij}(t)$, denotes the energy transfer between the hot and the cold sections. There are several ways to represent ΔT_{ij}, see e.g., Steiner

(1989) and Palsson (1989), but here

$$\Delta T_{ij}(t) = [T_{h,i-1}(t) + T_{h,i}(t)]/2 - [T_{c,j-1}(t) + T_{c,j}(t)]/2 \tag{3}$$

has been used. Introducing

$$\alpha(t) = \frac{A_h U(t)}{\dot{m}_h(t) c_h}, \quad \beta(t) = \frac{A_c U(t)}{\dot{m}_c(t) c_c}, \quad \tau_h(t) = \frac{M_h}{\dot{m}_h(t)}, \quad \tau_c(t) = \frac{M_c}{\dot{m}_c(t)} \tag{4}$$

and using Eq. (3) along with $ns = 1$, the heat exchanger model on state space form becomes (for simplicity not indicating the time dependence)

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{c,1} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_{h,1} \\ T_{c,1} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & (1-\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{5}$$

and with $ns = 2$

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & 0 & \frac{\alpha}{2\tau_h} & \frac{\alpha}{2\tau_h} \\ (1-\frac{\alpha}{2})/\tau_h & -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} & 0 \\ \frac{\beta}{2\tau_c} & \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c & 0 \\ \frac{\beta}{2\tau_c} & 0 & (1-\frac{\beta}{2})/\tau_c & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix} \times \begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & 0 \\ 0 & \frac{\alpha}{2\tau_h} \\ 0 & (1-\frac{\beta}{2})/\tau_c \\ \frac{\beta}{2\tau_c} & 0 \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{6}$$

or generally

$$\frac{d}{dt}\underline{T} = \boldsymbol{A}(\underline{\dot{m}})\underline{T} + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}} \tag{7}$$

α and β are dimensionless and τ_h and τ_c are in the units of time. It is observed that the state matrices, $\boldsymbol{A}$ and $\boldsymbol{B}$, are shown as function of the massflows, $\underline{\dot{m}}$. They are, in fact, also functions of the temperatures due to the temperature dependence of U and the specific heat capacities, c_h and c_c. That will, however, make the model nonlinear in the states and therefore the parameter identification procedure described later on is not applicable without some modifications. U will therefore only be assumed to be massflow dependent in the following. Parameter estimation results with a temperature dependent U will be the subject of a later publication. In the present case the temperature dependence of c_h and c_c is quite moderate and therefore it is neglected.

The model given by Eq. (5) is the same whether a counterflow or a parallelflow heat exchanger is modeled. However, when ns is greater than 1, the models will differ due to the coupling term $A_{()}U\Delta T_{ij}$. Thus Eq. (6) applies to a counterflow heat exchanger. Whatever the case, it is straightforward to write down the model for any number of sections, see e.g., Jonsson and Holst (1989) and Palsson (1989) for details. Furthermore, it is noted that A_h, A_c, M_h and M_c are the heat transfer area and the amount of water in each section. Hence, they depend on the number of sections used in the modeling, i.e., compared to the case when $ns = 1$, they are halved when $ns = 2$, etc.

Including the Effect of the Metal

If the heat capacity of the metal between the two media is taken into account, two additional parameters and one extra state, $T_{m,i,j}$, is needed for each pair of sections, i.e., the additional state is the temperature in the metal between the hot and the cold sections where the heat transfer occurs.

The differential equation is given by

$$M_m c_m \frac{dT_{m,i,j}}{dt} = A_h h_h(t)\Delta T_{ij}^h - A_c h_c(t)\Delta T_{ij}^c \tag{8}$$

where ΔT_{ij} now becomes, for the hot and the cold side respectively

$$\begin{aligned} \Delta T_{ij}^h &= [T_{h,i-1}(t) + T_{h,i}(t)]/2 - T_{m,i,j}(t) \\ \Delta T_{ij}^c &= T_{m,i,j}(t) - [T_{c,j-1}(t) + T_{c,j}(t)]/2 \end{aligned}$$

The relations between the indexes i and j are as before.

The equation analogous to Eq. (5) becomes

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{m,1,1} \\ T_{c,1} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha_m}{2})/\tau_h & \frac{\alpha_m}{\tau_h} & 0 \\ \frac{\alpha_m}{2\tau_{mh}} & -(\frac{\alpha_m}{\tau_{mh}}+\frac{\beta_m}{\tau_{mc}}) & \frac{\beta_m}{2\tau_{mc}} \\ 0 & \frac{\beta_m}{\tau_c} & -(1+\frac{\beta_m}{2})/\tau_c \end{bmatrix}\begin{bmatrix} T_{h,1} \\ T_{m,1,1} \\ T_{c,1} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha_m}{2})/\tau_h & 0 \\ \frac{\alpha_m}{2\tau_{mh}} & \frac{\beta_m}{2\tau_{mc}} \\ 0 & (1-\frac{\beta_m}{2})/\tau_c \end{bmatrix}\begin{bmatrix} T_{h,in} \\ T_{c,in} \end{bmatrix} \tag{9}$$

where the new parameters are given by ($\tau_h(t)$ and $\tau_c(t)$ remain unchanged)

$$\alpha_m(t) = \frac{A_h h_h(t)}{\dot{m}_h(t)c_h}, \beta_m(t) = \frac{A_c h_c(t)}{\dot{m}_c(t)c_c}, \tau_{mh}(t) = \frac{M_m c_m}{\dot{m}_h(t)c_h}, \tau_{mc}(t) = \frac{M_m c_m}{\dot{m}_c(t)c_c} \tag{10}$$

Models where the intervening metal is modeled explicitly will not be considered any further in the paper.

3 Empirical Relations in the Model Parameters

To make the model valid over a wide range of operating conditions, U needs to be massflow (and possibly temperature) dependent. In the present case the empirical relation

$$\mathrm{Nu} = \frac{hd}{k} = C\mathrm{Pr}^x\mathrm{Re}^y = C(\frac{c\mu}{k})^x(\frac{ud\rho}{\mu})^y \tag{11}$$

has been used (see e.g., Kakac *et al.* (1983) or Holman (1976)). Equation (11) is applicable to both sides of the heat exchanger in the present case.

With $u = \dot{m}/\rho A_{\text{cross}}$ and after rearranging, Eq. (11) becomes

$$h = \underbrace{K}_{i}\underbrace{\mu^{(x-y)}k^{(1-x)}}_{ii}\underbrace{\dot{m}^{y}}_{iii} \tag{12}$$

The term (i) is constant, i.e., independent of the massflows and temperatures, (ii) is temperature dependent and (iii) is massflow dependent. Neglecting the temperature dependence, Eq. (12) simplifies to

$$h = C'\dot{m}^{y} \tag{13}$$

By assuming that Eq. (13) applies to both of the heat transfer coefficients, h_h and h_c, the overall heat transfer coefficient, U, is given by

$$U(t) = \frac{h_h(t)h_c(t)}{h_h(t) + h_c(t)} = \frac{C'(\dot{m}_h(t)\dot{m}_c(t))^y}{(\dot{m}_h^y(t) + \dot{m}_c^y(t))} \tag{14}$$

where the thermal resistance in the metal is neglected. It is not necessary to assume that C' and y are the same for h_h and h_c. This is, however, quite reasonable for the heat exchanger used later on in the case study. Furthermore, it is desirable to keep the number of parameters to be estimated as low as possible to prevent potential identification problems.

It is observed from Eq. (4) that the massflows enter the denominator in all of the parameters. For practical reasons it is convenient to normalize them and U in Eq. (14) by some reference flows, $\dot{m}_{h*}$ and $\dot{m}_{c*}$. Thus

$$\frac{U(t)}{U_*} = \frac{(\dot{m}_h(t)\dot{m}_c(t))^y}{(\dot{m}_h^y(t) + \dot{m}_c^y(t))} \Big/ \frac{(\dot{m}_{h*}\dot{m}_{c*})^y}{(\dot{m}_{h*}^y + \dot{m}_{c*}^y)} \tag{15}$$

and

$$\begin{aligned}
\alpha(t) &= U_*A_h/\dot{m}_{h*}c_h &\times\ & \dot{m}_{h*}/\dot{m}_h(t) &= \alpha_*\dot{m}_{h*}/\dot{m}_h(t) \\
\tau_h(t) &= M_h/\dot{m}_{h*} &\times\ & \dot{m}_{h*}/\dot{m}_h(t) &= \tau_{h*}\dot{m}_{h*}/\dot{m}_h(t) \\
\beta(t) &= U_*A_c/\dot{m}_{c*}c_c &\times\ & \dot{m}_{c*}/\dot{m}_c(t) &= \beta_*\dot{m}_{c*}/\dot{m}_c(t) \\
\tau_c(t) &= M_c/\dot{m}_{c*} &\times\ & \dot{m}_{c*}/\dot{m}_c(t) &= \tau_{c*}\dot{m}_{c*}/\dot{m}_c(t)
\end{aligned} \tag{16}$$

The parameters α_*, τ_{h*}, β_* and τ_{c*} are estimated, $\alpha(t)$, $\tau_h(t)$, $\beta(t)$ and $\tau_c(t)$ are then obtained by multiplying them by $\dot{m}_{h*}/\dot{m}_h(t)$ or $\dot{m}_{c*}/\dot{m}_c(t)$.

If the massflow dependence in U is taken into account, only α and β from Eq. (16) are affected. They are then written as (given the index $\dot{m}$ to differ from Eq. (16))

$$\begin{aligned} \alpha_{\dot{m}}(t) &= \alpha_* \frac{\dot{m}_{h*}}{\dot{m}_h(t)} \frac{U(t)}{U_*} \\ \beta_{\dot{m}}(t) &= \beta_* \frac{\dot{m}_{c*}}{\dot{m}_c(t)} \frac{U(t)}{U_*} \end{aligned} \tag{17}$$

The parameters in Eq. (16) form the basic parameter set in the heat exchanger model. Using them only, will in the following be referred to as Case 1 whereas when U is assumed to be massflow dependent will be referred to as Case 2. In Case 2 one additional parameter is estimated, i.e., the exponent y. It should also be noted that the number of parameters is independent of the number of sections used in the modeling.

4 Parameter Estimation

Obtaining the Model Output at Discrete Time Instants

The heat exchanger model as given by Eq. (7) is time continuous and until now deterministic. To allow for variations between the model and the true temperatures, it is appropriate to add a noise term in Eq. (7). The model is then described by a stochastic differential equation and written as

$$d\underline{T}(t) = \boldsymbol{A}(\underline{\dot{m}})\underline{T}(t)dt + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}}(t)dt + d\underline{w}_c(t) \tag{18}$$

where $\underline{w}_c(t)$ is assumed to be a Wiener process with the incremental covariance $\boldsymbol{Q}'(t)dt$.

Assuming that $\underline{T}_{\text{in}}$ and the model parameters are constant between sampling points, i.e., the inlet temperatures and massflows are constant between sampling instants, the model can be discretized (see e.g., Åström (1970))

$$\underline{T}(t + \Delta t) = \boldsymbol{\Phi}(\underline{\dot{m}}, \Delta t)\underline{T}(t) + \boldsymbol{\Gamma}(\underline{\dot{m}}, \Delta t)\underline{T}_{\text{in}}(t) + w(t + \Delta t) \tag{19}$$

where

$$\boldsymbol{\Phi}(\underline{\dot{m}}, \Delta t) = e^{\boldsymbol{A}(\underline{\dot{m}})\Delta t} = \boldsymbol{I} + A(\underline{\dot{m}})\Delta t + \frac{A(\underline{\dot{m}})^2 \Delta t^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{A(\underline{\dot{m}})^j \Delta t^j}{j!} \tag{20}$$

and

$$\boldsymbol{\Gamma}(\underline{\dot{m}}, \Delta t) = [\int_0^{\Delta t} e^{\boldsymbol{A}(\underline{\dot{m}})s} ds]\boldsymbol{B}(\underline{\dot{m}}) \tag{21}$$

$\{\underline{w}(t)\}$ is a white noise vector sequence with zero means and covariance $\boldsymbol{Q}$.

The model for the measurements is assumed to be

$$\underline{T}_{\text{out}}(t) = \boldsymbol{C}\underline{T}(t) + \underline{v}(t) \tag{22}$$

with $\{\underline{v}(t)\}$ as $\{\underline{w}(t)\}$, but with covariance $\boldsymbol{R}$, $\underline{v}(t) \in N(\underline{0}, \boldsymbol{R})$. The matrix $\boldsymbol{C}$ is constant in the present case (all entries are 0 or 1). Furthermore, it is assumed, that $\underline{w}(t)$ and $\underline{v}(t)$ are uncorrelated, i.e., $E[\underline{w}(t)\underline{v}^T(s)] = 0$ for all t, s.

Using Eqs. (20) and (21), the continuous model is discretized and is then propagated between the sampling points. $\boldsymbol{\Phi}(\underline{\dot{m}}, \Delta t)$ and $\boldsymbol{\Gamma}(\underline{\dot{m}}, \Delta t)$ need to be calculated each time the massflows or the sampling interval changes, see e.g., Moler and van Loan (1978) for a detailed discussion of how to efficiently calculate $\boldsymbol{\Phi}(\underline{\dot{m}}, \Delta t)$. Instead of using Eq. (21) for calculating $\boldsymbol{\Gamma}(\underline{\dot{m}}, \Delta t)$, the relationship $\boldsymbol{\Gamma} = \boldsymbol{A}^{-1}(\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{B}$ (see Åström and Wittenmark (1984)) can be used when the inverse of $\boldsymbol{A}$ exists, as it does in the present case.

An alternative way to obtain the model output at the sampling instants is simply to integrate Eq. (18) by some suitable integration method.

The Score Function

The aim of the parameter estimation is to determine the parameter set which minimizes some score function. There are many possibilities to form the score function. Jonsson and Holst (1989) and Palsson (1989) use score functions based on the squared one step ahead prediction residuals, i.e., the difference between the measurements and the model output for parameter estimation in heat exchanger models. The score functions are either based on least squares or maximum likelihood principles. To obtain the one step prediction residuals, a Kalman filter is used for the state estimation.

Here, another variant of the score function is applied. Instead of using one step predictions only, the score function is based on multistep prediction errors, i.e.

$$V(\underline{\theta}) = \sum_{t=t_1+1}^{N} \underline{\varepsilon}^T(t|t_1)\underline{\varepsilon}(t|t_1) \tag{23}$$

$\underline{\varepsilon}(t|t_1)$ is the prediction error vector at time t based on the output measurements at time t and the predicted output of the model at time t done at time t_1, i.e.

$$\underline{\varepsilon}(t|t_1) = \underline{T}_{\text{out}}(t) - \boldsymbol{C}\hat{\underline{T}}(t|t_1) = \underline{T}_{\text{out}}(t) - \hat{\underline{T}}_{\text{out}}(t|t_1) \tag{24}$$

The multistep prediction values, $\hat{\underline{T}}(t|t_1)$, are calculated recursively according to (with $\Delta t = 1$)

$$\begin{aligned}
\hat{\underline{T}}(t_1+1|t_1) &= \boldsymbol{\Phi}(\underline{\dot{m}}_{t_1})\underline{T}(t_1) + \boldsymbol{\Gamma}(\underline{\dot{m}}_{t_1})\underline{T}_{\text{in}}(t_1) \\
\hat{\underline{T}}(t_1+2|t_1) &= \boldsymbol{\Phi}(\underline{\dot{m}}_{t_1+1})\hat{\underline{T}}(t_1+1|t_1) + \boldsymbol{\Gamma}(\underline{\dot{m}}_{t_1+1})\underline{T}_{\text{in}}(t_1+1) \\
&\vdots \\
\hat{\underline{T}}(N|t_1) &= \boldsymbol{\Phi}(\underline{\dot{m}}_{N-1})\hat{\underline{T}}(N-1|t_1) + \boldsymbol{\Gamma}(\underline{\dot{m}}_{N-1})\underline{T}_{\text{in}}(N-1)
\end{aligned} \tag{25}$$

The argumentation for using the kind of score function as given by Eq. (23) is that by using multistep predictions the model is in fact being simulated or propagated from one time point to another without any correction of the model states as is the case when a Kalman filter is applied, cf. Jonsson and Holst (1989) or Palsson (1989) for a discussion of how a Kalman filter can be applied in this context.

When using multistep predictions as above, it is important that at time t_1 the model states are tuned in. Otherwise there will be a constant error when the model is propagated according to Eq. (25) since there is no feedback or updating from new measurements of the outlet temperatures. A Kalman filter was therefore used to tune the model states to their correct values and from time t_1 it was removed and the model propagated as described.

Referring to the discussion in the previous chapter the parameter vector, $\underline{\theta}$, contains four or five parameters. In Case 1

$$\underline{\theta} = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}\} \tag{26}$$

and in Case 2

$$\underline{\theta} = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}, y\} \tag{27}$$

The inverse Hessian matrix of the score function is used as an estimate of the uncertainty in the parameter values. An unbiased estimate of the covariance matrix is (see e.g., Bard (1974))

$$\mathbf{Cov}[\hat{\underline{\theta}}] \simeq 2\hat{\sigma}^2 \boldsymbol{H}^{-1} = \frac{2\sum \underline{\varepsilon}^T \underline{\varepsilon}}{N - (t_1 + l)} \boldsymbol{H}^{-1} \tag{28}$$

i.e. the variance of parameter number i is given by

$$\mathrm{Var}[\hat{\theta}_i] = 2\hat{\sigma}^2 \{\boldsymbol{H}_{ii}^{-1}\} \tag{29}$$

where $\{\boldsymbol{H}_{ii}^{-1}\}$ is the diagonal element number i in the inverse Hessian matrix.

The Algorithm

The parameter sets given by Eq. (26) and Eq. (27) are determined by minimizing the score function given by Eq. (23). Summarizing, the parameter estimation procedure may be divided into the following steps

1. Supply an estimate of $\underline{\theta}$

2. Use the whole dataset and compute for each measurement point:

   - if needed, the discrete time heat exchanger model using Eqs. (20) and (21)
   - the model output temperatures, $\hat{\underline{T}}_{\text{out}}(t|t_1)$ from Eq. (25)
   - the residual vector, $\underline{\varepsilon}(t|t_1)$
   - the updated score function, $V(\underline{\theta})$

3. Iterate Step 2 until the minimum of $V(\underline{\theta})$ is found.

Routines from the NAG and IMSL libraries have been used for the minimization of $V(\underline{\theta})$. They only require function values.

5 A Case Study

The Data

The heat exchanger used in the case study is a small water to water counterflow plate heat exchanger from Alfa-Laval, type CB25, with dimensions $304 \times 103 \times 56$ mm and 14 plates. It is a part of a heat exchanger setup located at the Department of Heat and Power Engineering, Lund Institute of Technology, University of Lund, Sweden.

Figure 2 shows the experiment that was used in the parameter estimation. The major excitation is in the massflow on the hot side of the heat exchanger. Furthermore, the cold inlet temperature is increased slowly. The experiment is called dataset A in the following.

The sampling interval was one second and constant during the experiment.

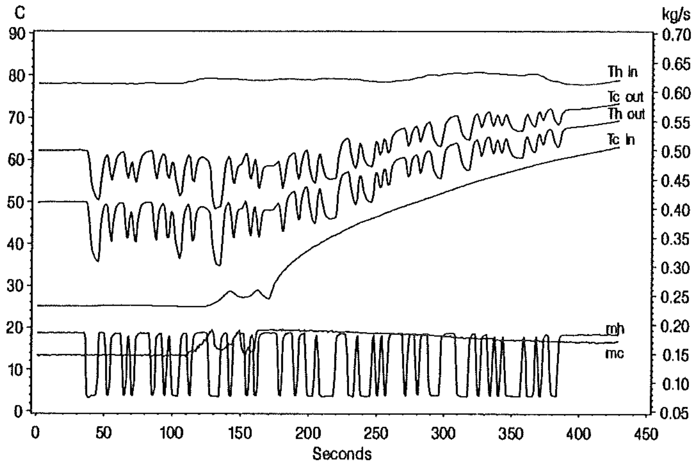

Fig. 2 The experiment used for the parameter estimation (dataset A).

The Results

Table 1 shows the results from the parameter estimation for $ns = 1, 2, 3$ and U constant or massflow dependent. The values of the score function, V, clearly demonstrate the positive effect of letting the overall heat transfer coefficient be a function of the massflows. Increasing ns further revealed that there is nothing gained by such an increase. It may therefore be concluded that a model with two sections on each side and U massflow dependent gives a good description of the process, but even with $ns = 1$ the results are quite acceptable.

This is also evident from Fig. 3 which shows the model output and the measurements when $ns = 2$ and the optimum parameter values were used. In order to verify the model further, the same model and the same parameters were used on another data set in order to see how accurately the model would simulate another experiment, called dataset B, from the same heat exchanger. Figure 4 shows the results. The agreement is quite satisfactory and not far away from what was observed in Fig. 3.

Table 1 Results from the parameter estimation. The standard deviation of the parameters are given in parenthesis. The reference flows were $\dot{m}_{h*} = \dot{m}_{c*} = 0.15$ kg/s (dataset A).

	1 section on each side		2 sections on each side		3 sections on each side	
	U const.	U $\dot{m}$ dep.	U const.	U $\dot{m}$ dep.	U const.	U $\dot{m}$ dep.
α_*	1.16 (.02)	1.59 (.01)	.803 (.006)	.804 (.005)	.539 (.004)	.538 (.003)
τ_{h*}	4.38 (.22)	3.35 (.17)	2.28 (.09)	1.78 (.08)	1.52 (.05)	1.21 (.05)
β_*	1.67 (.02)	1.68 (.01)	.847 (.006)	.849 (.006)	.567 (.004)	.568 (.004)
τ_{c*}	2.78 (.38)	3.40 (.22)	0.59 (.09)	1.59 (.11)	0.46 (.06)	1.11 (.08)
y		0.69 (.04)		.56 (.04)		.54 (.04)
V	1803	1034	1294	991	1214	936

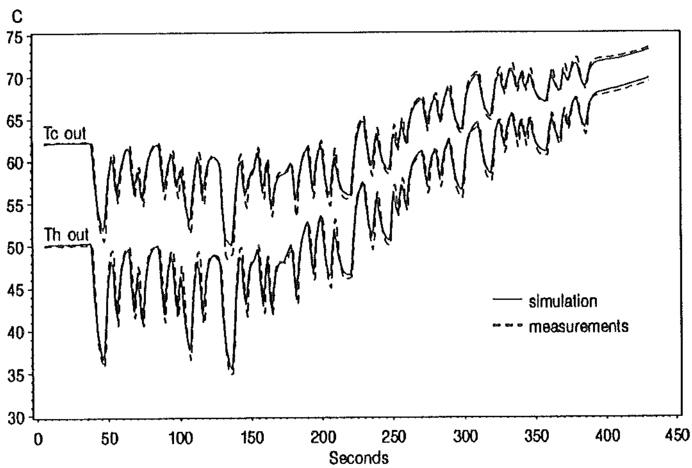

Fig. 3 Model and measured output from the heat exchanger. The optimum parameter values are used, $ns = 2$ and U is massflow dependent (dataset A).

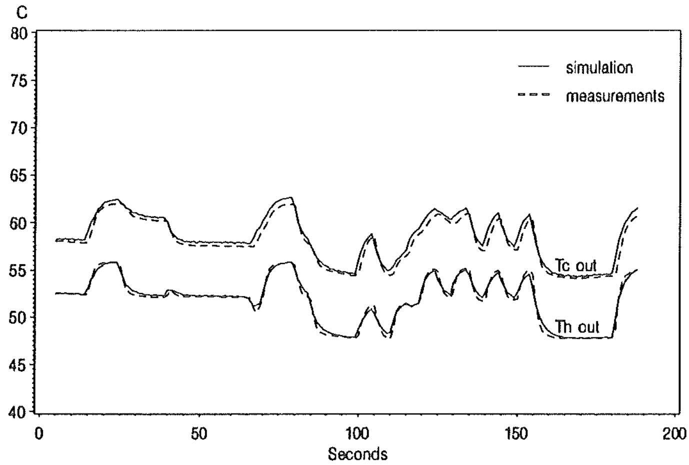

Fig. 4 Model and measured output from the heat exchanger (dataset B). The parameter values were determined from dataset A, $ns = 2$ and U is massflow dependent.

Statistical vs. Physical Parameter Estimation

How do the results above compare to the case when the model parameters were determined from steady state values of the heat exchanger and from its geometry. Consider the case when the heat transfer coefficient is massflow dependent. From Eq. (16)

$$\alpha_* = \frac{U_* A_h}{\dot{m}_{h*} c_h}$$

Furthermore, by using the temperature and massflow values in the beginning of dataset A, shown in Fig. 2, then

$$\begin{aligned} U &= \frac{\dot{m}_h c_h (T_{h,\text{in}} - T_{h,\text{out}})}{A_h \Delta T_{\text{LMTD}}} \\ &= \frac{0.185 \times 4180 \times (78.05 - 49.94)}{12 \times 0.0255 \times 20.02} = 3549 \quad \text{W/m}^2\,^\circ\text{C} \end{aligned}$$

The literature (see e.g., Kakac *et al.* (1983)) recommends $y \in [0.65, 0.85]$ for the heat exchanger used here (i.e., a water to water plate heat exchanger).

Then from Eq. (15)

$$\begin{aligned} U_* &= U(0) \times \frac{(\dot{m}_h^y(0) + \dot{m}_c^y(0))}{(\dot{m}_h(0)\dot{m}_c(0))^y} \times \frac{(\dot{m}_{h*}\dot{m}_{c*})^y}{(\dot{m}_{h*}^y + \dot{m}_{c*}^y)} \\ &= 3549 \times \frac{0.185^{0.7} + 0.147^{0.7}}{(0.185 \times 0.147)^{0.7}} \times \frac{(0.15 \times 0.15)^{0.7}}{0.15^{0.7} + 0.15^{0.7}} = 3300 \quad \mathrm{W/m^2{}^\circ C} \end{aligned}$$

where the value of $y = 0.70$ has been used. Then

$$\alpha_* = \frac{3330 \times 12 \times 0.0255}{0.15 \times 4180} = 1.62$$

and similarly for β_* (with $U_* = \frac{\dot{m}_c c_c (T_{c,\mathrm{out}} - T_{c,\mathrm{in}})}{A_c \Delta T_{\mathrm{LMTD}}} / 1.065 = 3540$)

$$\beta_* = \frac{3540 \times 12 \times 0.0255}{0.15 \times 4180} = 1.70$$

Note that there is about 5% difference between α_* and β_* when in fact they should be equal. The reason for this is that occasionally problems were encountered during the experiment, such that the energy balance not always was correct. Instead of scaling the series in order to compensate for this effect it was decided to let the model parameters take care of this unbalance. Furthermore, it turns out that mainly α_* and β_*, which determine the steady state behavior of the model, see Jonsson and Holst (1989) and Palsson (1989), are affected.

Furthermore, the time constants for the heat exchanger, determined from its geometry, are

$$\begin{aligned} \tau_{h*} &= \frac{0.3}{0.15} = 2.0 \text{ s} \\ \tau_{c*} &= \frac{0.3 \times \frac{7}{6}}{0.15} = 2.32 \text{ s} \end{aligned}$$

Thus, the parameter set obtained in this way is given by

$$\underline{\hat{\theta}}_{\mathrm{physic}} = \{1.62, 2.0, 1.70, 2.32, 0.70\} \tag{30}$$

This parameter set applies to the case when $ns = 1$. When ns is greater than one, the parameters except the exponent y must be scaled accordingly,

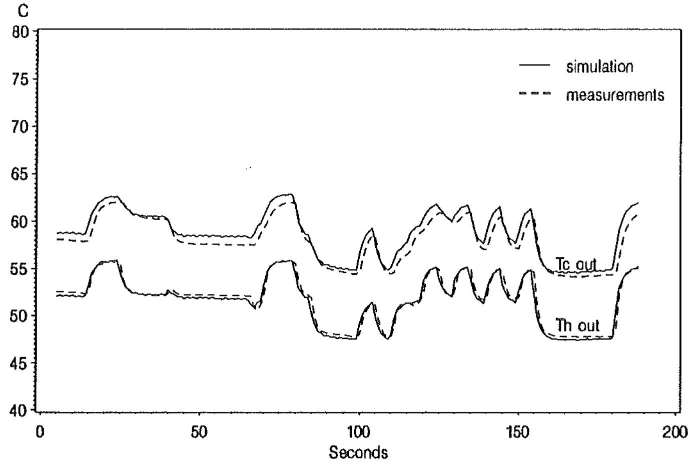

Fig. 5 Model and measured output from the heat exchanger, $ns = 2$ and U is massflow dependent and the parameter set from Eq. (30) is used (dataset B).

i.e., halved when $ns = 2$ etc. Comparing these parameter values with the corresponding values given in Table 1 ($ns = 1, U$ massflow dependent) shows that the values of α_* and β_* are quite similar and y is not very different. However, the values of τ_{h*} and τ_{c*} differ considerably. This difference is maintained even when $ns = 2$ or 3. τ_{h*} and τ_{c*} from Eq. (30) and their scaled values when $ns = 2$ or 3 are about $55 - 60\%$ and 70% of the values determined from the experimental data and given in Table 1. However, as ns increases the value of y in Table 1 decreases, the greatest decrease occurring when going from $ns = 1$ to $ns = 2$.

To demonstrate what effect this difference in the parameter values has on the model performance, the model with $ns = 2$ and the parameter set from Eq. (30) was used to simulate the two datasets used in Figures 3 and 4, i.e., datasets A and B. The resulting values of the score function, $V(\underline{\theta})$, were 1877 and 196 for the datasets A and B shown in Figs. 3 and 4 respectively. Compared to 991 from Table 1 (dataset A) and 89 (dataset B) when the optimum parameters were used, the difference is quite significant.

Finally, Fig. 5 shows the model output and the measurements when $ns = 2$ and the parameter set from Eq. (30) was used on dataset B. Compared with Fig. 4 the difference is noticeable and the model performance is higher when the estimated parameters are used instead of the computed parameters given by Eq. (30).

6 Summary and Discussion

This paper focuses on how information from empirical relations of heat transfer coefficients can be incorporated in the parameters of heat exchanger models, which then are estimated along with some of the parameters in the empirical relations from experiments by using statistical methods.

The modeling approach is based on a direct lumping of the process. Based upon energy considerations, a state space model is formed. The temperature of each section is a model state. The model is linear in the temperatures but nonlinear in the massflows.

The parameters are determined by minimizing a score function which is based on multistep prediction residuals, i.e., the difference between the measured outlet temperatures and the estimated values.

Some of the key issues in using the approach discussed in the paper are as follows:

- The nonlinearities in the process are coped with by using empirical relations of the heat transfer coefficient showing the massflow and possibly temperature dependence of the heat transfer.
- Some of the coefficients in the empirical relations are estimated along with the remaining model parameters, thereby adjusting the formulas to the specified heat exchanger. This is done in one experiment which therefore should be representative for the whole operating range of the heat exchanger with respect to the massflows and the temperatures.

- Estimating the parameters statistically provides an estimate of their uncertainty.
- If the modeling is sufficient, the resulting parameter values will at least be in the neighborhood of what is expected from physical considerations.
- If the modeling is poor, the resulting parameters will differ from what is expected.
- In any case, using statistical methods to estimate parameters in physically derived models gives a good indication of the quality of the modeling. Furthermore, parameter values are obtained that will ensure good model performance as long as the model structure is capable of providing the dynamics needed.

Based on the data used in this study, it is shown that using two sections on each side ($ns = 2$) and letting the overall heat transfer coefficient be massflow dependent gives a very satisfactory description of the dynamics of the heat exchanger.

Furthermore, comparison of the model performance when using statistically obtained parameter values versus physically obtained parameter values shows that a more accurate description of the dynamics is achieved by using statistical methods.

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Nomenclature

A	= heat transfer area
$\boldsymbol{A}$	= $(n \times n)$ system matrix
$\boldsymbol{B}$	= $(n \times m)$ input matrix
c	= specific heat
C, C', K	= constant
$\boldsymbol{C}$	= $(m \times n)$ measurement matrix
$\mathbf{Cov}$	= covariance matrix
d	= diameter
h	= heat transfer coefficient
$\boldsymbol{H}$	= $(l \times l)$ Hessian matrix
$\boldsymbol{I}$	= identity matrix
k	= thermal conductivity
l	= number of parameters
M	= mass in each section
m	= number of inputs and outputs
$\underline{\dot{m}}$	= m dimensional massflow vector
$\dot{m}$	= massflow
n	= number of states
N	= number of time points
ns	= number of sections
$\boldsymbol{Q}$	= $(n \times n)$ system noise covariance matrix
$\boldsymbol{Q}'(t)dt$	= incremental covariance matrix
$\boldsymbol{R}$	= $(m \times m)$ measurement noise covariance matrix
t	= time
T	= temperature
$\underline{T}$	= n dimensional state vector
u	= velocity
U	= overall heat transfer coefficient
$\underline{v}$	= m dimensional white noise vector $\in N(\underline{0}, \boldsymbol{R})$
V	= score function

$\underline{w}$	=	n dimensional white noise vector $\in N(\underline{0}, \boldsymbol{Q})$
$w_c(t)$	=	Wiener process
x, y	=	exponent
Δt	=	time step (sampling interval)
ΔT	=	temperature difference
ΔT_{LMTD}	=	log-mean-temperature-difference

Greek Letters

α	=	model parameter
β	=	model parameter
$\boldsymbol{\Gamma}$	=	$(n \times m)$ discrete input matrix
$\underline{\varepsilon}$	=	m dimensional residual vector
$\underline{\theta}$	=	l dimensional parameter vector
μ	=	dynamic viscosity
ρ	=	density
σ	=	standard deviations
τ	=	model parameter
$\boldsymbol{\Phi}$	=	$(n \times n)$ discrete system matrix

Dimensionless Groups

Nu	=	Nusselt number (hd/k)
Pr	=	Prandtl number $(c_p\mu/k)$
Re	=	Reynolds number $(\rho ud/\mu)$

Subscripts and Superscripts

c	=	cold
cross	=	cross sectional (area)
h	=	hot

in	=	input
m	=	metal
out	=	output
*	=	reference conditions

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An Application of Extended Kalman Filtering to Heat Exchanger Models

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Abstract

This paper shows how an extended Kalman filter can be applied to the parameter estimation in continuous time heat exchanger models. The model is based on lumping of the heat exchanger. It is on state space form where the temperature in each section is a model state. By letting the model parameters be functions of the massflows and the temperatures one obtains a model that is capable of accurately describing the dynamics of the heat exchanger for all relevant working conditions. Since the parameters are functions of temperature, the model is non-linear in the states and an extended Kalman filter is applied to the state estimation. Empirical relations of the heat transfer coefficients are incorporated in the model parameters in order to cope with the massflow and temperature dependence. Some of the parameters in the empirical relations are also estimated, thereby adjusting the formulas to the specific heat exchanger.

Key Words: Heat Exchangers; Physical State Space Models; Parameter Estimation; Empirical Relations; Extended Kalman Filter.

1 Introduction

Modeling heat exchangers has been the subject of numerous publications. There are many modeling approaches depending on, e.g., the intended use of the model. Some of the approaches involve a linearization of the process which reduces the potential use of the resulting models, see e.g., Franck and Rake (1985), Bittanti *et al.* (1982), Masada and Wormley (1982) and Ito and Masubuchi (1978) for a discussion of those types of models. In Jonsson (1990) it was found that models based on lumping the process directly (no partial differential equations) can be made very general and with the correct parametrization they can be used to describe the dynamic behavior of the whole heat exchanger working area.

The parametrization in the models is very important since the non-linearity of the heat transfer process is mainly handled by the parameters. The main reason for the non-linearity is the heat transfer dependence upon massflows and temperatures. For this reason a number of empirical relations showing this dependence have been developed for the heat transfer coefficients, see e.g., Palen (1986), Kakac *et al.* (1983) and Kays and London (1964). Different relations apply to different types of heat exchangers. For a given type of heat exchanger, however, one may expect that the empirical relations will only give a rough indication of the true heat transfer coefficients. A certain adjustment of the relations is therefore required for an accurate model of the specific heat exchanger.

This can be accomplished in several ways. One possibility is to do a lot of experiments and on the basis of steady state considerations, calculate the heat transfer coefficients for different massflow and temperature values. By means of regression analysis the coefficients of the empirical formulas can then be evaluated. This has been done in, e.g., Gummérus (1988). The second possibility is to try to adjust the coefficients manually. This is obviously not feasible, especially if there are more than two parameters to tune in, say.

The third possibility is to include the structure of the relations in the model parameters and use the model to adapt the coefficients in the relations to the specified heat exchanger. The coefficients, together with the remaining model parameters, are estimated through one experiment which activates the heat exchanger over the entire intended working area. This approach, given that the model structure is sufficient to provide the dynamics needed, yields parameter values that ensure good model performance. Furthermore, by estimating the coefficients this way one compensates for any discrepancies that possibly may be found in the modeling procedure. This enables, among other things, the use of very low order models.

This approach has been discussed in detail in Jonsson and Palsson (1991), Jonsson *et al.* (1992), Jonsson and Holst (1989) and Palsson (1989). There it is shown how the structure of the empirical relations is included in the parameters of the heat exchanger models, based on lumping the process directly. The models are time continuous and the temperature of each section is a model state. The massflows enter the model through the state matrices. Applications to both shell and tube and plate heat exchangers are given and different ways of estimating the parameters statistically are discussed.

In the above references, the major emphasis is on results when the heat transfer coefficients are constant or massflow dependent. In that case a standard Kalman filter is applied for the state estimation in the model.

This paper takes the discussion one step further. It deals with the case when the heat transfer coefficients are also assumed temperature dependent, making the model non-linear in the model states. This means that instead of the standard Kalman filter, an extended Kalman filter is applied to the state estimation. Since the models are discussed in detail in the previously mentioned references they will only be stated here for the sake of completeness. The same applies to the parameter estimation procedure.

The paper starts with a short description of the heat exchanger model and how the empirical relations are incorporated in the model parameters. This is followed by a discussion of the extended Kalman filter. Then it is shown how the models are prepared for the parameter estimation and finally some results are given.

The measurements used in the case study are from a heat exchanger rig located at the Laboratory of the Heat and Power Engineering, Lund Institute of Technology, Sweden. The heat exchanger is a small water to water counterflow plate heat exchanger.

2 The Heat Exchanger Model

2.1 The Model

The model is based on direct lumping of the process. Assume that the heat transfer to the surroundings is negligible, there is no heat conduction in the direction of the flow in the metal between the fluids nor in the fluids themselves, there is a uniform temperature in each section, and the specific heat capacities are constant. Furthermore, it was found, see e.g., Jonsson *et al.* (1992), that for the present heat exchanger nothing is gained by modeling the metal between the two media as a separate state variable. For hot section number i and cold section number j the following differential equations are then obtained:
Hot section:

$$M_h c_h \frac{dT_{h,i}(t)}{dt} = \dot{m}_h(t) c_h [T_{h,i-1}(t) - T_{h,i}(t)] - A_h U(t) \Delta T_{ij}(t) \quad (1)$$

Cold section:

$$M_c c_c \frac{dT_{c,j}(t)}{dt} = \dot{m}_c(t) c_c [T_{c,j-1}(t) - T_{c,j}(t)] + A_c U(t) \Delta T_{ij}(t) \quad (2)$$

where

$$\Delta T_{ij}(t) = [T_{h,i-1}(t) + T_{h,i}(t)]/2 - [T_{c,j-1}(t) + T_{c,j}(t)]/2 \quad (3)$$

i and j range from 1 to ns and $T_{h,0} = T_{h,\text{in}}$, $T_{h,ns} = T_{h,\text{out}}$, $T_{c,0} = T_{c,\text{in}}$ and $T_{c,ns} = T_{c,\text{out}}$. Equations (1) and (2) apply to both parallel- and counterflow heat exchangers. In the parallelflow case, $i = j$ in ΔT_{ij} and $j = ns - i + 1$ when modeling a counterflow heat exchanger. The general form of the model on state space form is thus given by

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{h,2} \\ \vdots \\ T_{h,ns} \\ T_{c,1} \\ T_{c,2} \\ \vdots \\ T_{c,ns} \end{bmatrix} =$$

$$\begin{bmatrix}
-\frac{(1+\frac{\alpha}{2})}{\tau_h} & 0 & \cdots & 0 & 0 & 0 & \frac{\alpha}{2\tau_h} & \frac{\alpha}{2\tau_h} \\
\frac{(1-\frac{\alpha}{2})}{\tau_h} & -\frac{(1+\frac{\alpha}{2})}{\tau_h} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \frac{\alpha}{2\tau_h} & \vdots & \cdots & \vdots \\
0 & \cdots & \frac{(1-\frac{\alpha}{2})}{\tau_h} & -\frac{(1+\frac{\alpha}{2})}{\tau_h} & \frac{\alpha}{2\tau_h} & 0 & \cdots & 0 \\
0 & 0 & \frac{\beta}{2\tau_c} & \frac{\beta}{2\tau_c} & -\frac{(1+\frac{\beta}{2})}{\tau_c} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \frac{(1-\frac{\beta}{2})}{\tau_c} & -\frac{(1+\frac{\beta}{2})}{\tau_c} & \cdots & 0 \\
\frac{\beta}{2\tau_c} & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\beta}{2\tau_c} & 0 & \cdots & 0 & 0 & \cdots & \frac{(1-\frac{\beta}{2})}{\tau_c} & -\frac{(1+\frac{\beta}{2})}{\tau_c}
\end{bmatrix}$$

$$\times \begin{bmatrix} T_{h,1} \\ T_{h,2} \\ \vdots \\ T_{h,ns} \\ T_{c,1} \\ T_{c,2} \\ \vdots \\ T_{c,ns} \end{bmatrix} + \begin{bmatrix} \frac{(1-\frac{\alpha}{2})}{\tau_h} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \frac{\alpha}{2\tau_h} \\ 0 & \frac{(1-\frac{\beta}{2})}{\tau_c} \\ 0 & 0 \\ \vdots & \vdots \\ \frac{\beta}{2\tau_c} & 0 \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{4}$$

or

$$\frac{d}{dt}\underline{T} = \boldsymbol{A}\underline{T} + \boldsymbol{B}\underline{T_{\text{in}}} \tag{5}$$

Where the elements in the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are functions of massflows and temperatures.

In the following, the interest is focused on the case when $ns = 1$ or $ns = 2$. With $ns = 1$ the model becomes

$$\frac{d}{dt}\begin{bmatrix} T_h \\ T_c \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_h \\ T_c \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} \\ \frac{\beta}{2\tau_c} & (1-\frac{\beta}{2})/\tau_c \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \tag{6}$$

and with $ns = 2$

$$\frac{d}{dt}\begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} = \begin{bmatrix} -(1+\frac{\alpha}{2})/\tau_h & 0 & \frac{\alpha}{2\tau_h} & \frac{\alpha}{2\tau_h} \\ (1-\frac{\alpha}{2})/\tau_h & -(1+\frac{\alpha}{2})/\tau_h & \frac{\alpha}{2\tau_h} & 0 \\ \frac{\beta}{2\tau_c} & \frac{\beta}{2\tau_c} & -(1+\frac{\beta}{2})/\tau_c & 0 \\ \frac{\beta}{2\tau_c} & 0 & (1-\frac{\beta}{2})/\tau_c & -(1+\frac{\beta}{2})/\tau_c \end{bmatrix}$$

$$\times \begin{bmatrix} T_{h,1} \\ T_{h,2} \\ T_{c,1} \\ T_{c,2} \end{bmatrix} + \begin{bmatrix} (1-\frac{\alpha}{2})/\tau_h & 0 \\ 0 & \frac{\alpha}{2\tau_h} \\ 0 & (1-\frac{\beta}{2})/\tau_c \\ \frac{\beta}{2\tau_c} & 0 \end{bmatrix} \begin{bmatrix} T_{h,\text{in}} \\ T_{c,\text{in}} \end{bmatrix} \quad (7)$$

2.2 The Parametrization

CASE 1: Constant Heat Transfer

The models given by Eqs. (4) - (7) contain four basic parameters which are given by

$$\alpha(t) = \frac{A_h U}{\dot{m}_h(t) c_h}, \quad \tau_h(t) = \frac{M_h}{\dot{m}_h(t)}, \quad \beta(t) = \frac{A_c U}{\dot{m}_c(t) c_c}, \quad \tau_c(t) = \frac{M_c}{\dot{m}_c(t)} \quad (8)$$

Note that the overall heat transfer coefficient, U, is assumed constant in Eq. (8). Hence the parameters are only massflow dependent. In order to relate to some reference conditions the parameters are scaled. Thus

$$
\begin{aligned}
\alpha(t) &= U_* A/\dot{m}_{h*} c_h &\times \quad \dot{m}_{h*}/\dot{m}_h(t) &= \alpha_* &\times \quad \dot{m}_{h*}/\dot{m}_h(t) \\
\tau_h(t) &= M_h/\dot{m}_{h*} &\times \quad \dot{m}_{h*}/\dot{m}_h(t) &= \tau_{h*} &\times \quad \dot{m}_{h*}/\dot{m}_h(t) \\
\beta(t) &= U_* A/\dot{m}_{c*} c_c &\times \quad \dot{m}_{c*}/\dot{m}_c(t) &= \beta_* &\times \quad \dot{m}_{c*}/\dot{m}_c(t) \\
\tau_c(t) &= M_c/\dot{m}_{c*} &\times \quad \dot{m}_{c*}/\dot{m}_c(t) &= \tau_{c*} &\times \quad \dot{m}_{c*}/\dot{m}_c(t)
\end{aligned}
\tag{9}
$$

where the reference flows can be chosen arbitrarily. The parameter set

$$
\underline{\theta}_1 = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}\} \tag{10}
$$

is estimated by statistical methods. The model with the parameters defined by Eq. (9) will be referred to as Case 1.

CASE 2: Massflow Dependent Heat Transfer

Assuming that the heat transfer coefficient, h, is a function of the massflows, i.e., (see also Case 3)

$$
h_h(t) = C' \dot{m}_h^y(t), \qquad h_c(t) = C' \dot{m}_c^y(t) \tag{11}
$$

and with $U^{-1} \simeq (\frac{1}{h_h} + \frac{1}{h_c})$ (neglecting the thermal resistance in the metal and assuming that $A_h = A_c$) this leads to

$$
U(t) = \frac{h_h(t) h_c(t)}{h_h(t) + h_c(t)} = \frac{C'(\dot{m}_h(t)\dot{m}_c(t))^y}{(\dot{m}_h^y(t) + \dot{m}_c^y(t))} \tag{12}
$$

The overall heat transfer coefficient at the reference massflows, $\dot{m}_{h*}$ and $\dot{m}_{c*}$, is similarly

$$
U_* = \frac{h_{h*} h_{c*}}{h_{h*} + h_{c*}} = \frac{C'(\dot{m}_{h*}\dot{m}_{c*})^y}{(\dot{m}_{h*}^y + \dot{m}_{c*}^y)} \tag{13}
$$

The parameter set from Eq. (9) now becomes

$$
\begin{aligned}
\alpha(t) &= \alpha_* \times \dot{m}_{h*}/\dot{m}_h(t) \times U(t)/U_* \\
\tau_h(t) &= \tau_{h*} \times \dot{m}_{h*}/\dot{m}_h(t) \\
\beta(t) &= \beta_* \times \dot{m}_{c*}/\dot{m}_c(t) \times U(t)/U_* \\
\tau_c(t) &= \tau_{c*} \times \dot{m}_{c*}/\dot{m}_c(t)
\end{aligned} \tag{14}
$$

where it is observed that only α and β are effected by the massflow dependence in U. Furthermore, there is one additional parameter to be estimated, the exponent y giving

$$\underline{\theta}_2 = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}, y\} \tag{15}$$

This will be referred to as Case 2. It is of course possible to work with different values of the constants and the exponents on the hot and cold sides of the heat exchanger. This is however not necessary for the heat exchanger used in the case study later on.

CASE 3: Massflow and Temperature Dependent Heat Transfer

The following empirical relation of the heat transfer coefficients applies to both sides of the heat exchanger used here, see e.g., Palen (1986) or Kakac *et al.* (1983)

$$\mathrm{Nu} = h\frac{d}{k} = C\mathrm{Pr}^x\mathrm{Re}^y = C(\frac{c_p\mu}{k})^x(\frac{ud\rho}{\mu})^y \tag{16}$$

Inserting for the Reynolds number ($\mathrm{Re} = ud\rho/\mu = \dot{m}d/A_{\mathrm{cross}}\mu$) and the Prandtl number ($\mathrm{Pr} = c_p\mu/k$), Eq. (16) becomes

$$h = \underbrace{C\frac{c_p^x d^{y-1}}{A_{\mathrm{cross}}^y}}_{1}\underbrace{\mu^{(x-y)}k^{(1-x)}}_{2}\underbrace{\dot{m}^y}_{3} \tag{17}$$

Term 1 is constant, term 2 is temperature dependent and term 3 is massflow dependent. Equation (17) can thus be written as

$$h = C''\{\mu^{(x-y)}k^{(1-x)}\}\dot{m}^y = C''g(T)\dot{m}^y \tag{18}$$

where it is noticed that with $g(T)$ = constant Eq. (11) is obtained. The overall heat transfer coefficient, scaled by its reference value becomes

$$\frac{U}{U_*} = \frac{g(T_h)\dot{m}_h^y(t)g(T_c)\dot{m}_c(t)^y}{(g(T_h)\dot{m}_h^y(t) + g(T_c)\dot{m}_c^y(t))} / \frac{g(T_{h*})\dot{m}_{h*}^y g(T_{c*})\dot{m}_{c*}^y}{(g(T_{h*})\dot{m}_{h*}^y + g(T_{c*})\dot{m}_{c*}^y)} \tag{19}$$

The parameter set is the same as given by Eq. (14) but with one additional parameter, i.e.

$$\underline{\theta}_3 = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}, y, x\} \tag{20}$$

Observe that the constant C'' in Eq. (18) disappears when $U(t)$ is scaled with U_* and needs not to be estimated.

The following polynomial approximations for μ and k have been used to represent $g(T)$

$$\begin{aligned} \mu(T) &= 1.7227 \times 10^{-3} - 4.3197 \times 10^{-5}T + 5.2608 \times 10^{-7}T^2 \\ &- 3.1990 \times 10^{-9}T^3 + 9.2675 \times 10^{-12}T^4 - 1.0157 \times 10^{-14}T^5 \\ k(T) &= 0.566 + 2.00 \times 10^{-3}T - 7.64 \times 10^{-6}T^2 - 2.67 \times 10^{-8}T^3 \\ &+ 2.22 \times 10^{-10}T^4 \end{aligned} \tag{21}$$

they were obtained by polynomial fitting using data from Holman (1976). T is in degrees Centigrade.

CASE 4: Massflow Dependent Heat Transfer and Linearly Dependent upon Temperature

It is shown in Jonsson and Holst (1989) and Palsson (1989) that the temperature dependence of h from Eqs. (16) or (17) is approximately linear in the temperature interval $[0, 100]$ degrees Centigrade. In other words, it can be assumed that

$$g(T) \simeq C'''(1 + bT) \tag{22}$$

Since U is scaled by U_* as before, only b is estimated. The parameter vector $\underline{\theta}$ is thus given by

$$\underline{\theta}_4 = \{\alpha_*, \tau_{h*}, \beta_*, \tau_{c*}, y, b\} \tag{23}$$

where $\alpha_*, \tau_{h*}, \ldots$ are defined as before. This will be referred to as Case 4.

3 Parameter Estimation

The overall heat transfer coefficient, $U(t)$, is not assumed temperature dependent in Cases 1 and 2. An ordinary Kalman filter can therefore be applied to the state estimation. Kalman filtering in this context is discussed in detail in, e.g., Jonsson *et al.* (1992) and for the sake of completeness a part of the discussion is included here in appendix. In Cases 3 and 4, when U is also assumed temperature dependent, the model is non-linear in the states and therefore an extended Kalman filter has been used for the state estimation. Although the main emphasis here is on results obtained when applying the extended Kalman filter, results from Cases 1 and 2 will also be shown in order to demonstrate the difference in the model performance when using different representations of the heat transfer coefficient.

3.1 The Extended Kalman Filter, EKF

The EKF is based on linearization around the current state estimates, $\underline{\hat{T}}$, see e.g., Gelb (1974) for details. The continuous-discrete version of the EKF is quite similar to the standard discrete Kalman filter. The difference is that the state vector and the covariance matrix can not be propagated in the same manner due to the non-linearity in the model. The system model is now written as (replaces Eq. (46))

$$d\underline{\hat{T}}(t) = f(\underline{T}(t), t)dt + dw_c(t) \tag{24}$$

where

$$f(\underline{T}(t), t) = \boldsymbol{A}(\underline{\dot{m}}, \underline{T}(t))\underline{T}(t) + \boldsymbol{B}(\underline{\dot{m}}, \underline{T}(t))\underline{T}_{\text{in}}(t) \tag{25}$$

Instead of Eqs. (48) and (49) (see appendix) the following update between sampling instants is used

State estimate propagation

$$\frac{d}{dt}\underline{\hat{T}}(t) = f(\underline{\hat{T}}(t), t) \tag{26}$$

Error covariance propagation

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{F}(\hat{\underline{T}}(t),t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{F}^T(\hat{\underline{T}}(t),t) + \boldsymbol{Q}(t) \tag{27}$$

where

$$\boldsymbol{F}(\hat{\underline{T}}(t),t) = \frac{\partial f(\underline{T}(t),t)}{\partial \underline{T}(t)}\Big|_{\underline{T}(t)=\hat{\underline{T}}(t)} \tag{28}$$

Calculation of $\boldsymbol{F}(\hat{\underline{T}}(t),t)$

One Section on Each Side ($ns = 1$)

In this case $\boldsymbol{F}(\hat{\underline{T}}(t),t)$ contains four elements

$$F(,) = \begin{bmatrix} \partial f_1/\partial T_h & \partial f_1/\partial T_c \\ \partial f_2/\partial T_h & \partial f_2/\partial T_c \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \tag{29}$$

Based on the model from the Eq. (6) the elements are

$$\begin{aligned} f_{11} &= -(1+\frac{\alpha}{2})/\tau_h + [T_c + T_{c,\text{in}} - T_h - T_{h,\text{in}}]\frac{1}{2\tau_h}\frac{\partial\alpha}{\partial T_h} \\ f_{12} &= \frac{\alpha}{2\tau_h} + [T_c + T_{c,\text{in}} - T_h - T_{h,\text{in}}]\frac{1}{2\tau_h}\frac{\partial\alpha}{\partial T_c} \\ f_{21} &= \frac{\beta}{2\tau_c} + [T_h + T_{h,\text{in}} - T_c - T_{c,\text{in}}]\frac{1}{2\tau_c}\frac{\partial\beta}{\partial T_h} \\ f_{22} &= -(1+\frac{\beta}{2})/\tau_c + [T_h + T_{h,\text{in}} - T_c - T_{c,\text{in}}]\frac{1}{2\tau_c}\frac{\partial\beta}{\partial T_c} \end{aligned} \tag{30}$$

From Eq. (14) one has that

$$\frac{\partial\alpha}{\partial T_{()}} = \alpha_* \frac{\dot{m}_{h*}}{\dot{m}_h}\frac{1}{U_*}\frac{\partial U}{\partial T_{()}} \tag{31}$$

and furthermore

$$\frac{\partial U}{\partial T_h} = \frac{\partial}{\partial T_h}\left[\frac{C''g(T_h)\dot{m}_h^y C''g(T_c)\dot{m}_c^y}{C''g(T_h)\dot{m}_h^y + C''g(T_c)\dot{m}_c^y}\right] \tag{32}$$

Using $\partial U/\partial T_h = -U^2 \partial U^{-1}/\partial T_h$ and that

$$\frac{\partial U^{-1}}{\partial T_h} = \frac{\partial}{\partial T_h}\left[\frac{1}{C''g(T_h)\dot{m}_h^y} + \frac{1}{C''g(T_c)\dot{m}_c^y}\right] = -\frac{d}{dT_h}g(T_h)/C''\dot{m}_h^y g^2(T_h) \tag{33}$$

this leads to

$$\frac{\partial U}{\partial T_h} = \frac{C''\dot{m}_h^y g^2(T_c)\dot{m}_c^{2y}}{[g(T_h)\dot{m}_h^y + g(T_c)\dot{m}_c^y]^2}\frac{d}{dT_h}g(T_h) \tag{34}$$

It is observed that C'' cancels out in Eq. (31) due to the reference factor $1/U_*$.

In the same manner one needs for $\partial\alpha/\partial T_c$

$$\frac{\partial U}{\partial T_c} = \frac{C''\dot{m}_c^y g^2(T_h)\dot{m}_h^{2y}}{[g(T_h)\dot{m}_h^y + g(T_c)\dot{m}_c^y]^2}\frac{d}{dT_c}g(T_c) \tag{35}$$

In Case 3 one obtains (with $T = T_h$ or $T = T_c$)

$$\begin{aligned}\frac{d}{dT}g(T) &= (x-y)\mu^{(x-y-1)}(T)k^{(1-x)}(T)\frac{d}{dT}\mu(T) \\ &+ (1-x)\mu^{(x-y)}(T)k^{-x}(T)\frac{d}{dT}k(T)\end{aligned} \tag{36}$$

where $\frac{d}{dT}\mu(T)$, $\frac{d}{dT}k(T)$ are easily calculated when μ and k are given by Eq. (21).

In Case 4 one simply has that

$$\frac{d}{dT}g(T) = b \tag{37}$$

The procedure for $\partial\beta/\partial T_c$ and $\partial\beta/\partial T_h$ is completely analogous.

Two Sections on Each Side ($ns = 2$)

In this case $\boldsymbol{F}(\hat{\underline{T}}(t), t)$ contains 16 elements. When the number of sections is two or more the possibility arises of using different α and β for each pair of sections. Thus, by introducing the notation that α_{ij} is a function of the temperature in hot section no i and cold section no j and similarly for β_{ij} the elements in $\boldsymbol{F}$ are

$$\begin{aligned}
f_{11} = \frac{\partial f_1}{\partial T_{h,1}} &= -(1+\frac{\alpha_{12}}{2})/\tau_h + [T_{c,1}+T_{c,2}-T_{h,1}-T_{h,\text{in}}]\frac{1}{2\tau_h}\frac{\partial \alpha_{12}}{\partial T_{h,1}} \\
f_{12} = \frac{\partial f_1}{\partial T_{h,2}} &= 0 \\
f_{13} = \frac{\partial f_1}{\partial T_{c,1}} &= \frac{\alpha_{12}}{2\tau_h} \\
f_{14} = \frac{\partial f_1}{\partial T_{c,2}} &= \frac{\alpha_{12}}{2\tau_h} + [T_{c,1}+T_{c,2}-T_{h,1}-T_{h,\text{in}}]\frac{1}{2\tau_h}\frac{\partial \alpha_{12}}{\partial T_{c,2}} \\
f_{21} = \frac{\partial f_2}{\partial T_{h,1}} &= (1-\frac{\alpha_{21}}{2})/\tau_h \\
f_{22} = \frac{\partial f_2}{\partial T_{h,2}} &= -(1+\frac{\alpha_{21}}{2})/\tau_h + [T_{c,1}+T_{c,\text{in}}-T_{h,1}-T_{h,2}]\frac{1}{2\tau_h}\frac{\partial \alpha_{21}}{\partial T_{h,2}} \\
f_{23} = \frac{\partial f_2}{\partial T_{c,1}} &= \frac{\alpha_{21}}{2\tau_h} + [T_{c,1}+T_{c,\text{in}}-T_{h,1}-T_{h,2}]\frac{1}{2\tau_h}\frac{\partial \alpha_{21}}{\partial T_{c,1}} \\
f_{24} = \frac{\partial f_2}{\partial T_{c,2}} &= 0 \qquad (38) \\
f_{31} = \frac{\partial f_3}{\partial T_{h,1}} &= \frac{\beta_{21}}{2\tau_c} \\
f_{32} = \frac{\partial f_3}{\partial T_{h,2}} &= \frac{\beta_{21}}{2\tau_c} + [T_{h,1}+T_{h,2}-T_{c,1}-T_{c,\text{in}}]\frac{1}{2\tau_c}\frac{\partial \beta_{21}}{\partial T_{h,2}} \\
f_{33} = \frac{\partial f_3}{\partial T_{c,1}} &= -(1+\frac{\beta_{21}}{2})/\tau_c + [T_{h,1}+T_{h,2}-T_{c,1}-T_{c,\text{in}}]\frac{1}{2\tau_c}\frac{\partial \beta_{21}}{\partial T_{c,1}} \\
f_{34} = \frac{\partial f_3}{\partial T_{c,2}} &= 0 \\
f_{41} = \frac{\partial f_4}{\partial T_{h,1}} &= \frac{\beta_{12}}{2\tau_c} + [T_{h,1}+T_{h,\text{in}}-T_{c,1}-T_{c,2}]\frac{1}{2\tau_c}\frac{\partial \beta_{12}}{\partial T_{h,1}} \\
f_{42} = \frac{\partial f_4}{\partial T_{h,2}} &= 0
\end{aligned}$$

$$f_{43} = \frac{\partial f_4}{\partial T_{c,1}} = (1 - \frac{\beta_{12}}{2})\tau_c$$
$$f_{44} = \frac{\partial f_4}{\partial T_{c,2}} = -(1 + \frac{\beta_{12}}{2})/\tau_c + [T_{h,1} + T_{h,\text{in}} - T_{c,1} - T_{c,2}]\frac{1}{2\tau_c}\frac{\partial \beta_{12}}{\partial T_{c,2}}$$

The procedure for calculating $\partial\alpha_{12}/\partial T_{h,1}$, $\partial\alpha_{12}/\partial T_{c,2}$, ... is the same as described previously in the case when $ns = 1$.

Proceeding from two to three or even more sections is straightforward, especially by recognizing the structure of $\boldsymbol{F}$. For instance there are always two elements in each row in $\boldsymbol{F}$ that contain non-zero derivatives of α or β and the temperatures for the section pair numbered ij are $T_{h,i}$, $T_{h,i-1}$ and $T_{c,j}$, $T_{c,j-1}$ with $T_{h,0} = T_{h,\text{in}}$, $T_{c,0} = T_{c,\text{in}}$.

3.2 The Parameter Estimation Algorithm

The score function used in the parameter estimation is based on the squared one step ahead prediction errors of both of the outlet temperatures

$$V(\underline{\theta}) = \sum_{t=t_1}^{N} \underline{\varepsilon}^T(t|t-\Delta t)\underline{\varepsilon}(t|t-\Delta t) \tag{39}$$

where the one step ahead prediction residuals are computed by

$$\underline{\varepsilon}(t|t-\Delta t) = \underline{T}_{\text{out}}(t) - \boldsymbol{C}\underline{\hat{T}}(t|t-\Delta t) = \underline{T}_{\text{out}}(t) - \underline{\hat{T}}_{\text{out}}(t|t-\Delta t) \tag{40}$$

$\underline{\hat{T}}(t|t-\Delta t)$ is easily computed recursively using a standard or an extended Kalman filter, see appendix.

The reason for summing from t_1 is to allow the transients to vanish when the Kalman filter is tuning in the states.

To obtain an estimate of the uncertainty in the parameters, the information available from the Hessian matrix, $\boldsymbol{H}$, of the score function, $V(\underline{\theta})$, is used (see e.g., Bard (1974))

$$\mathbf{Cov}[\underline{\hat{\theta}}] \simeq 2\hat{\sigma}^2\boldsymbol{H}^{-1} = \frac{2V(\underline{\hat{\theta}})}{N-(t_1+l)}\boldsymbol{H}^{-1} \tag{41}$$

and the variance of the i'th parameter is accordingly

$$\mathrm{Var}[\hat{\theta}_i] = 2\sigma^2\{\boldsymbol{H}_{ii}^{-1}\} \tag{42}$$

where $\{\boldsymbol{H}_{ii}^{-1}\}$ denotes the diagonal element no. i in the inverse Hessian matrix. For a more thorough description of the parameter estimation algorithm, see Jonsson *et al.* (1992).

For the minimization of the score function and the integration of Eqs. (26) and (27), routines from the NAG and IMSL libraries have been used. The minimization routines use a quasi-Newton method and a finite-difference gradient, see IMSL (1987) or Dennis and Schnabel (1983). Hence, the routines only require function values.

4 Results from the Parameter Estimation

4.1 The Data

The heat exchanger used in the case study is a small water to water counterflow plate heat exchanger from Alfa-Laval, type CB25, with dimensions 304 × 103 × 56 mm and 14 plates. It is a part of a heat exchanger setup located at the Department of Heat and Power Engineering, Lund Institute of Technology.

Figure 1 shows data from an experiment that was performed. Both of the massflows and the inlet temperatures, particularly on the cold side are excited. The excitation of the massflows is based on the Pseudo Random Binary Signal (PRBS) techniques, see e.g., Godfrey (1980). It was not possible to activate the hot inlet temperature more than shown in Fig. 1 whereas the cold inlet temperature could be changed rapidly, although a PRBS form was impossible.

The sampling interval was 1 second and constant.

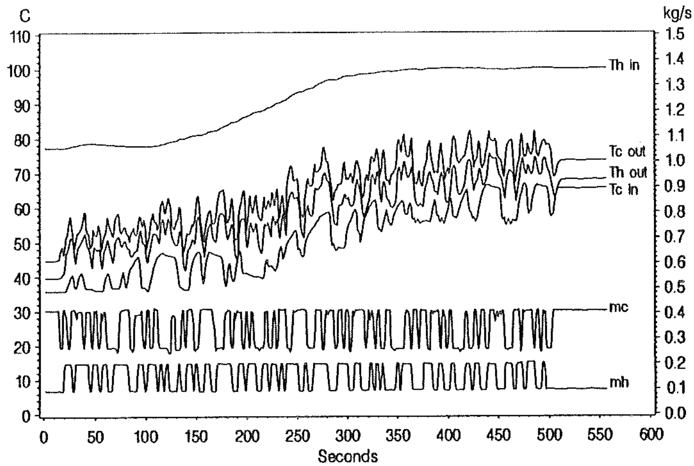

Fig. 1 The experiment used in the parameter estimation.

The temperature measurements, both the inlet and the outlet temperatures, enter explicitly (directly) in the Kalman filters. The massflow measurements, on the other hand, enter the Kalman filters implicitly (indirectly) through the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ because of the massflow scaling in Eqs. (9) and (14).

4.2 Starting Values for the Kalman Filters

In all four cases the starting values $\boldsymbol{P}_0 = 10 \times \boldsymbol{I}$, $\boldsymbol{R} = 0.1 \times \boldsymbol{I}$ and $\boldsymbol{Q} = 0.1 \times \boldsymbol{I}$ were used. No major effort was made to optimize those values nor were they included in the parameter estimation although that is quite possible. The initial values of the state vector, $\underline{T}(0)$, were based on the initial values of the temperatures in the experiment. t_1 in Eq. (39) was in all instances 25, i.e., the first 25 residuals were excluded from the score function $V(\underline{\hat{\theta}})$. The reference massflows were chosen as $\dot{m}_{h*} = \dot{m}_{c*} = 0.15$ kg/s and the reference temperatures as $T_{h*} = 61.0\,^{\circ}\mathrm{C}$ and $T_{c*} = 45.0\,^{\circ}\mathrm{C}$.

Table 1 Parameter estimation results. $ns = 1$. The standard deviations of the parameter estimates are given in parenthesis.

$ns = 1$	Case 1	Case 2	Case 3	Case 4
α_*	1.83 (0.02)	1.41 (0.01)	1.37 (0.02)	1.37 (0.02)
τ_{h*}	6.3 (0.4)	3.8 (0.2)	3.9 (0.2)	3.9 (0.2)
β_*	1.93 (0.03)	1.49 (0.02)	1.45 (0.02)	1.45 (0.02)
τ_{c*}	4.2 (0.4)	3.5 (0.2)	3.4 (0.2)	3.4 (0.2)
y		0.83 (0.03)	0.84 (0.02)	0.84 (0.03)
x			0.68 (0.05)	
b				0.0033 (0.001)
$V(\underline{\theta})$	2436	1175	1144	1144

4.3 The Results

Tables 1 and 2 contain results from the parameter estimation when $ns = 1$ and $ns = 2$ respectively. The results clearly indicate the effect of letting U be at least massflow dependent instead of only a constant. The effect of including the temperature dependence of U does not seem to be of great importance considering the reduction in the score function which is moderate in both instances, i.e., when $ns = 1$ and $ns = 2$. This was also anticipated since the massflow dependence of U is much stronger than the temperature dependence. However, the reduction in $V(\underline{\theta})$ between Cases 2 and 3 or 2 and 4 is greater when $ns = 2$. This is probably due to the fact that effectively, the model in Cases 3 and 4 works with two α and β (see Section 3.1), i.e., one for each pair of the sections (hot and cold) instead of only one when $ns = 1$. The temperature effect is therefore better represented with larger number of sections.

It also turns out that the improvement in the model performance in Cases 3

Table 2 Parameter estimation results. $ns = 2$. The standard deviations of the parameter estimates are given in parenthesis.

$ns = 2$	Case 1	Case 2	Case 3	Case 4
α_*	1.03 (0.01)	0.81 (0.01)	0.77 (0.01)	0.77 (0.01)
τ_{h*}	2.2 (0.1)	1.92 (0.07)	1.94 (0.07)	1.94 (0.07)
β_*	1.07 (0.01)	0.85 (0.01)	0.81 (0.01)	0.81 (0.01)
τ_{c*}	1.6 (0.1)	1.73 (0.09)	1.71 (0.08)	1.71 (0.08)
y		0.62 (0.03)	0.63 (0.03)	0.63 (0.03)
x			0.39 (0.05)	
b				0.0058 (0.001)
$V(\underline{\theta}_0)$	1710	1083	1011	1009

and 4 compared to Case 2 mainly is due to the improved steady state description of the model. This is also observed by comparing Figs. 2, 3 and 4 which show the measured outlet temperatures along with the model outputs for Cases 1, 2 and 3 respectively when $ns = 2$ and the optimum parameter values (Table 2) were used. Case 4 is omitted since it essentially yields the same results as Case 3.

The main difference between Fig. 3 and Fig. 4 is that the steady state error is reduced to practically zero in Case 3. This is most evident at the beginning and at the end of the dataset. Furthermore, no attempt is made to scale the residuals in the score function, Eq. (39). This means that the steady state residuals have little influence since they are very small compared to the residuals that may occur when, e.g., the massflows are changed suddenly. As an example, with 500 data points, say, a reduction of 0.2 degrees Centigrade in steady state error at each measurement point will reduce the score function by $500 \times 0.2^2 = 20$ whereas the same effect is obtained from a single residual of about 4.5 degrees Centigrade.

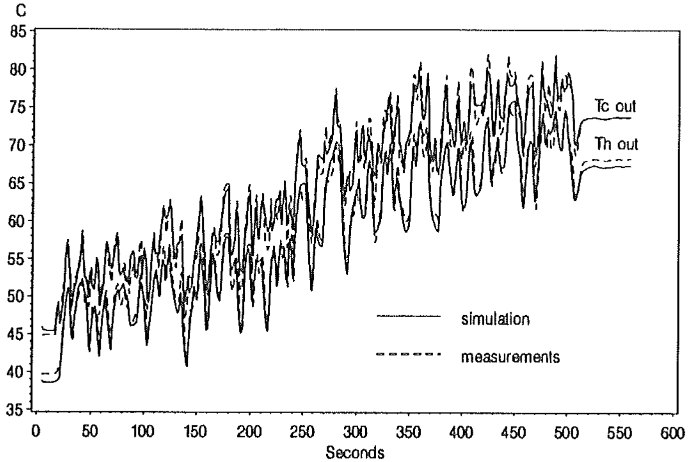

Fig. 2 Measured and model output temperatures in Case 1. Optimum parameters, $ns = 2$.

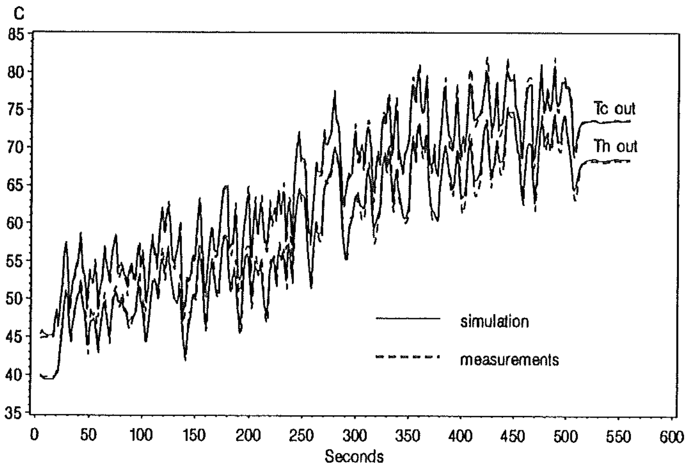

Fig. 3 Measured and model output temperatures in Case 2. Optimum parameters, $ns = 2$.

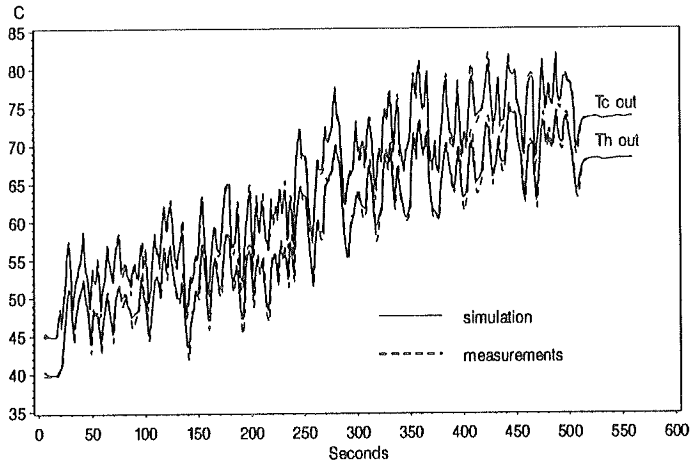

Fig. 4 Measured and model output temperatures in Case 3. Optimum parameters, $ns = 2$.

Finally, consider the parameter values that were obtained. It has been demonstrated (see Jonsson and Palsson (1991)) that the statistically obtained parameters yield better model performance for the same heat exchanger as is used in the present case study compared to parameter values which have been determined by considering the geometry and the steady state values of the heat exchanger. This is also the case when the two differently obtained parameter sets were tried on another experiment than the one they were estimated from. This will thus not be discussed any further here. It is interesting, however, to note the difference in the exponents, x and y, in Case 3 and b and y in Case 4 when $ns = 1$ and $ns = 2$. The literature recommends values in the neighborhood of $y = 2/3$ and $x = 1/3$ for the type of heat exchanger used here, see e.g., Kakac *et al.* (1983). From Tables 1 and 2 it is observed that much better coherence with those values is accomplished when $ns = 2$ compared to $ns = 1$. The same applies to the remaining parameter values, i.e., α_*, τ_{h*} etc. This indicates that the physics of the process are better modeled using $ns = 2$. The difference in the score function values, however, is not that great, i.e., 1144 and 1011 for $ns = 1$ and $ns = 2$ respectively for Case 3. This demonstrates clearly what can be accomplished by estimating the parameters statistically. Although the dynamics available from a model with $ns = 1$ are not sufficient for describing the process accurately, the statistical procedure provides parameter values that still yield good model performance. The price is that at least some of the resulting parameter values differ considerably from what is expected from physical considerations. In other words, using $ns = 1$ and not estimating the parameters statistically is likely to result in a poor model performance.

The value of b in Case 4 differs also considerably between $ns = 1$ and $ns = 2$. Comparing these values is very difficult since U is a function of the temperature on both sides of the heat exchanger and furthermore the temperature difference between the two sides is not the same when $ns = 1$ and $ns = 2$. Nevertheless, it may be concluded that the temperature dependence of h (see e.g., Eq. (17)) indeed is linear since the model parameters are practically the same in Cases 3 and 4, i.e., those that are comparable.

5 Summary and Discussion

The paper demonstrates how non-linearities in heat exchangers can be handled by letting the model parameters be functions of massflows and temperatures. The model is based on lumping the process directly where the temperature of each section is a model state. The massflows enter the model via the state matrices.

Several versions of the parametrization are considered and compared. The main reason for the non-linearities in the heat exchanger is the heat transfer dependence on massflows and temperatures. The parametrization is thus based on different approximations of the heat transfer coefficients. Four cases are discussed

- constant heat transfer (Case 1)
- massflow dependent heat transfer (Case 2)

- massflow and temperature dependent heat transfer (Case 3)
- massflow dependent heat transfer and also linearly dependent upon temperature (Case 4)

A standard Kalman filter can be applied in the state estimation in the first two cases whereas an extended Kalman filter is applied in the last two due to the non-linearity in the model states that arises when U also depends on the temperatures.

The model parameters are determined by using statistical methods. The score function is based on the one step ahead prediction errors for both of the outlet temperatures. These prediction errors are obtained from the Kalman filters.

The experiment used in the parameter estimation was designed so that the resulting model would be capable of describing accurately the dynamics of the heat exchanger over its entire working area.

The following conclusions can be drawn based on the results presented.

- The model performance is drastically improved when the heat transfer is assumed massflow dependent instead of being constant.
- The steady state behavior is improved when heat transfer also is assumed temperature dependent.
- Analogous results are obtained (i.e., in Cases 3 and 4) when the heat transfer is assumed linearly dependent upon temperature.
- The parameter values are much closer to what can be evaluated from physical considerations, i.e., the geometry and the steady state conditions in the heat exchanger when $ns = 2$ compared to when $ns = 1$.
- However the reduction in the score function, is not that large going from $ns = 1$ to $ns = 2$, except in Case 1. This means that although the physics of the process are much poorer described with $ns = 1$ the

statistical parameter estimation still yields parameter values that ensure good model performance. In other words, if $ns = 1$ and the model parameters are not determined statistically, that will most probably result in insufficient model performance.

The question of how many sections to use in the modeling depends of course on the heat exchanger in question. In the present case, an excellent model performance is obtained both with $ns = 1$ and $ns = 2$ and the correct parametrization. Since the parameter values in the latter case coincide much better with what can be evaluated from physical considerations, then $ns = 2$ is a suitable choice. On the other hand this also shows what can be accomplished by using statistical methods in the parameter estimation. Even though a model with $ns = 1$ does not contain the appropriate dynamics for describing the heat exchanger, a good model performance is accomplish by using "biased" parameters, i.e., values that can not be evaluated from physical considerations.

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Nomenclature

A = heat transfer area

$\boldsymbol{A}$	=	$(n \times n)$ system matrix
b	=	model parameter
$\boldsymbol{B}$	=	$(n \times m)$ input matrix
c_p	=	specific heat
$C,^{\prime},^{\prime\prime},^{\prime\prime\prime}$	=	constants
$\boldsymbol{C}$	=	$(m \times n)$ measurement matrix
Cov	=	covariance matrix
d	=	diameter
$\underline{f}$	=	model vector
$\boldsymbol{F}$	=	model matrix
g	=	function
h	=	heat transfer coefficient
$\boldsymbol{H}$	=	Hessian matrix of the score function
$\boldsymbol{I}$	=	identity matrix
k	=	thermal conductivity
$\boldsymbol{K}$	=	Kalman filter gain
l	=	number of parameters
M	=	mass in each section
m	=	number of inputs and outputs
$\underline{\dot{m}}$	=	m dimensional massflow vector
$\dot{m}$	=	massflow
n	=	number of states
N	=	number of time points
ns	=	number of sections
$\boldsymbol{P}$	=	$(n \times n)$ state covariance matrix
$\boldsymbol{Q}$	=	$(n \times n)$ system noise covariance matrix
$\boldsymbol{R}$	=	$(m \times m)$ measurement noise covariance matrix
t	=	time
T	=	temperature
$\underline{T}$	=	n dimensional state vector
u	=	velocity
U	=	overall heat transfer coefficient
$\underline{v}$	=	m dimensional white noise vector $\in N(\underline{0}, \boldsymbol{R})$
V	=	score function
$\underline{w}$	=	n dimensional white noise vector $\in N(\underline{0}, \boldsymbol{Q})$
x, y	=	exponents
Δt	=	time step (sampling interval)

Greek Letters

α	=	model parameter
β	=	model parameter
$\mathbf{\Gamma}$	=	$(n \times m)$ discrete input matrix
$\underline{\varepsilon}$	=	m dimensional residual vector
$\underline{\theta}$	=	l dimensional parameter vector
μ	=	dynamic viscosity
ρ	=	density
σ	=	standard deviation
τ	=	model parameter
$\mathbf{\Phi}$	=	$(n \times n)$ discrete system matrix

Dimensionless Groups

Nu	=	Nusselt number (hd/k)
Pr	=	Prandtl number $(c_p\mu/k)$
Re	=	Reynolds number $(\rho ud/\mu)$

Subscripts

c	=	cold
cross	=	cross sectional (area)
h	=	hot
in	=	input
out	=	output
$*$	=	reference conditions

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APPENDIX: Kalman Filtering

In the following it is shown how a standard Kalman filter can be applied for the state estimation when the model is linear in the states.

To allow for variations between the model and the true temperatures, the time continuous heat exchanger model (see Eqs. (4), (6) or (7)) is written as a stochastic differential equation

$$d\underline{T}(t) = \boldsymbol{A}(\underline{\dot{m}})\underline{T}(t)dt + \boldsymbol{B}(\underline{\dot{m}})\underline{T}_{\text{in}}(t)dt + d\underline{w}_c(t) \tag{43}$$

i.e., a noise term has been added to the model where $\underline{w}_c(t)$ is assumed to be a Wiener process with the incremental covariance $\boldsymbol{Q}'(t)dt$. Stochastic differential equation models are described, e.g., in Åström (1970), Gard (1988) and Gardiner (1983).

To obtain the model output at discrete time instants the model from Eq. (43) can either be discretized and propagated from one time point to another (see e.g., Jonsson *et al.* (1992)) or integrated by some suitable integration routine.

The model for the measurements is assumed to be

$$\underline{T}_{\text{out}}(t) = \boldsymbol{C}\underline{T}(t) + \underline{v}(t) \tag{44}$$

with $\underline{v}(t) \in N(\underline{0}, \boldsymbol{R})$. $\boldsymbol{C}$ is the measurement matrix and constant in the present case (all entries 0 or 1).

The Kalman Filter

Let $\hat{\underline{T}}^{-}(t+\Delta t)$ denote the estimated value of the model states at time $t+\Delta t$ given all measurements up to time t. The optimal estimate, $\hat{\underline{T}}^{-}(t+\Delta t)$, is the conditional mean, i.e., $E[\underline{T}(t+\Delta t)|\underline{T}(t),\underline{T}(t-\Delta t),\dots,\underline{T}(1)]$. Similarly, $\hat{\underline{T}}^{-}_{\text{out}}(t+\Delta t)$, is the output estimate. The one step ahead prediction errors at the time $t+\Delta t$ are then given by

$$\underline{\varepsilon}(t+\Delta t) = \underline{T}_{\text{out}}(t+\Delta t) - \boldsymbol{C}\hat{\underline{T}}^{-}(t+\Delta t) = \underline{T}_{\text{out}}(t+\Delta t) - \hat{\underline{T}}^{-}_{\text{out}}(t+\Delta t) \quad (45)$$

$\hat{\underline{T}}^{-}(t+\Delta t)$ is easily computed recursively using a standard Kalman filter. Adapting the notation from Gelb (1974) and assuming that the model from Eq. (43) has been discretized, i.e., $\boldsymbol{\Phi}(\underline{\dot{m}}) = e^{\boldsymbol{A}(\underline{\dot{m}})} = \boldsymbol{I} + \boldsymbol{A}(\underline{\dot{m}}) + \frac{\boldsymbol{A}(\underline{\dot{m}})^2}{2!} + \dots = \sum_{j=0}^{\infty} \frac{\boldsymbol{A}(\underline{\dot{m}})^j}{j!}$ and $\boldsymbol{\Gamma}(\underline{\dot{m}}) = [\int_0^{\Delta T} e^{\boldsymbol{A}(\underline{\dot{m}})s} ds]\boldsymbol{B}$ the Kalman filter equations are

System model

$$\underline{T}(t+\Delta t) = \boldsymbol{\Phi}\underline{T}(t) + \boldsymbol{\Gamma}\underline{T}_{\text{in}}(t) + \underline{w}(t+\Delta t) \quad (46)$$

Measurement model

$$\underline{T}_{out}(t+\Delta t) = \boldsymbol{C}\underline{T}(t+\Delta t) + \underline{v}(t+\Delta t) \quad (47)$$

State estimate extrapolation

$$\hat{\underline{T}}^{-}(t+\Delta t) = \boldsymbol{\Phi}\hat{\underline{T}}^{+}(t) + \boldsymbol{\Gamma}\underline{T}_{\text{in}}(t) \quad (48)$$

Error covariance extrapolation

$$\boldsymbol{P}^{-}(t+\Delta t) = \boldsymbol{\Phi}\boldsymbol{P}^{+}(t)\boldsymbol{\Phi}^T + \boldsymbol{Q}(t+\Delta t) \quad (49)$$

State estimate update

$$\hat{\underline{T}}^{+}(t+\Delta t) = \hat{\underline{T}}^{-}(t+\Delta t) + \boldsymbol{K}(t+\Delta t)[\underline{T}_{\text{out}}(t+\Delta t) - \boldsymbol{C}\hat{\underline{T}}^{-}(t+\Delta t)] \quad (50)$$

Error covariance update

$$P^{+}(t+\Delta t) = [I - K(t+\Delta t)C]P^{-}(t+\Delta t) \tag{51}$$

Kalman gain matrix

$$K(t+\Delta t) = P^{-}(t+\Delta t)C^{T}[CP^{-}(t+\Delta t)C^{T} + R(t+\Delta t)]^{-1} \tag{52}$$

where $\underline{w}(t) \in N(\underline{0}, Q)$ and $\underline{v}(t) \in N(\underline{0}, R)$. The index "+" denotes values of estimates and covariances of the state estimation errors at time $t+\Delta t$ after measurement and "−" indicates the corresponding values after propagation from t to $t+\Delta t$. To start the Kalman filter initial values of the state vector, $\underline{T}(0)$, and the covariance matrix, $P(0)$, are needed and the noise matrices Q and R. Another possibility is to estimate the elements in Q and R along with the parameters.

The above formulation of the Kalman filter assumes that the model parameters are known. This is however not the case here. There are several possibilities to circumvent this problem. One is to estimate the model parameters simultaneously with the state estimation, leading to an extended Kalman filter, see e.g., Wiberg and DeWolf (1991) and Gelb (1974). This results in non-linear estimation of time varying parameters and will not be pursued any further here. Here, all parameters are assumed to be constant (apart from the deterministic dependence in flow and temperature), and estimated from the whole dataset. Eq. (45) is used to calculate the residuals, $\underline{\varepsilon}(1), \ldots, \underline{\varepsilon}(N)$, and the parameters are determined in an iterative manner on the basis of some score function, using a search algorithm.

Predictor-based Optimal Control of Supply Temperature in District Heating Systems

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[B1]

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Abstract

A modified version of the linear quadratic Gaussian controller is presented, which is build upon the prediction form of the model. This implies that the controller is more capable of handling non-stationarities, like time-varying model parameters, than the classical type of linear quadratic Gaussian controller, which usually is based on the solution of the Diophantine or the Riccati Equation (in the state-space case). The linear quadratic Gaussian controller is used in a simulation study together with a transfer function model, with time-varying parameters, that describes the relations between supply temperature of the water from a district heating plant and the supply temperature at specific locations in the distribution network. The simulation results show that the variance of the differenced control signal can be reduced drastically without affecting the performance of the controller significantly.

Key Words: Predictive Control; Linear Quadratic Gaussian Control; Minimum Variance Control; Time-varying Systems; District Heating; Temperature Control.

1 Introduction

The minimum variance controller has been frequently used since it was first presented in Åström (1970), and it has been discussed in abundance in the literature, see e.g., Davis and Vinter (1984) and Lewis (1986). One of its advantages is that it is extremely easy to compute on-line, since the current control value is a linear combination of a finite number of past control values and outputs. The minimum variance controller is therefore popular in self-tuning control, where the control parameters can be estimated directly or indirectly (the model parameters are estimated adaptively using, e.g., least-square techniques and the control parameters are then computed).

The minimum variance controller minimizes the variance of the controlled signal (which can be shown to be a moving average process, see Åström (1970)). However, it is evident, that the consequence is that the minimum variance controller uses inadequately amount of control effort. In particular this is the case if the system is non-minimum phase. In such cases some form of control costing must be incorporated. This type of controller was first presented in Clarke and Gawthrop (1975) (generalized minimum variance controller) where the self-tuning principle is used to minimize a cost function which incorporates fluctuations of inputs, outputs and set points, while the system parameters are assumed constant but unknown. Another version of practically the same controller is described in Lewis (1986). There it is called polynomial linear quadratic Gaussian regulator and the problem is solved using the Diophantine Equations. The linear quadratic Gaussian problem can also be solved using the state-space representation of the model, leading to the solution of the Riccati Equation, see e.g., Bertsekas (1987) and Davis and Vinter (1984).

This paper presents a linear quadratic Gaussian controller which uses the predictive form of the model successively. Unlike the above mentioned controllers which use the Diophantine or the Riccati Equations, built on the assumption of time-invariance, this controller is capable of handling time-varying model parameters. The idea is adapted from Søgaard (1988) where it is used in relation with a traditional minimum variance controller. As an introduction, a general discussion of predictor based type of linear quadratic

Gaussian controllers is given. This is then applied to special models which are used in a simulation study. Finally, it is shown how the proposed controller can be used to control supply temperature of water from a district heating plant. The model used in the simulation study is taken from Søgaard (1988) and describes the relations between the supply temperature of water from the district heating plant and supply temperature at specified locations in a distribution network.

2 Linear Quadratic Gaussian Control

This section starts with a brief description of the transfer function model used. Then the cost function is discussed. This is followed by a description of the controller and the optimal predictions used in the controller formulation. Finally, an example is given.

2.1 The Model

The transfer function model considered belongs to the single-input single-output ARMAX(n, m, r) (Auto-Regressive-Moving-Average-eXtraneous) structure

$$A_t(q^{-1})y_t = B_t(q^{-1})q^{-k}u_t + C_t(q^{-1})\varepsilon_t \tag{1}$$

where $\{y_t\}$ and $\{u_t\}$ are the output and the input (control) signal, respectively. $\{\varepsilon_t\}$ is a Gaussian white noise process with variance, σ_ε^2. The integer, $k \geq 1$, denotes a delay (dead time) from input to output, and q^{-1} is the back shift operator defined by: $q^{-1}y_t = y_{t-1}$. $A_t(q^{-1})$, $B_t(q^{-1})$ and $C_t(q^{-1})$ are polynomials in q^{-1}:

$$\begin{aligned} A_t(q^{-1}) &= 1 + a_{1,t}q^{-1} + \cdots + a_{n,t}q^{-n} \\ B_t(q^{-1}) &= b_{0,t} + b_{1,t}q^{-1} + \cdots + b_{m,t}q^{-m} \\ C_t(q^{-1}) &= 1 + c_{1,t}q^{-1} + \cdots + c_{r,t}q^{-r} \end{aligned} \tag{2}$$

where $a_{i,t}; i \in \{1, \ldots, n\}, b_{j,t}; j \in \{0, \ldots, m\}, (b_{0,t} \neq 0)$ and $c_{l,t}; l \in \{1, \ldots, r\}$ are the model parameters. As indicated by the index t the model parameters may be time-varying. The time-delay, k, is assumed known. The (exact) time-delay being unknown is one of the major problems in control theory. In Søgaard and Madsen (1991) methods for tracking time-varying time-delays are described.

2.2 The Cost Function

Depending on the application considered, it is possible to form various types of cost functions. A general quadratic cost function is

$$J_t = (P(q^{-1})y_{t+k} - Q(q^{-1})y_{t+k}^{\text{ref}})^2 + (R(q^{-1})u_t)^2 \tag{3}$$

with weighting polynomials

$$\begin{aligned} P(q^{-1}) &= 1 + p_1 q^{-1} + \cdots + p_{n_P} q^{-n_P} \\ Q(q^{-1}) &= q_0 + q_1 q^{-1} + \cdots + q_{n_Q} q^{-n_Q} \\ R(q^{-1}) &= r_0 + r_1 q^{-1} + \cdots + r_{n_R} q^{-n_R} \end{aligned} \tag{4}$$

This formulation is also found in Lewis (1986) and Clarke and Gawthrop (1975).

In the present case the cost function considered is on the form

$$J_t = (y_{t+k} - y_{t+k}^{\text{ref}})^2 + \lambda(u_t - u_{t-1})^2 \tag{5}$$

i.e., the purpose is to keep the output close to some predetermined reference output and at the same time to keep the changes in the control signal small ($P(q^{-1}) = 1$, $Q(q^{-1}) = 1$, and $R(q^{-1}) = \sqrt{\lambda}(1 - q^{-1})$ in Eq. (3)). The cost parameter λ can be adjusted to put more or less penalty on changes in the control signal.

The cost function in Eq. (5) has been chosen with special attention to the case-study. However, the principles of the technique demonstrated for this cost function may as well be used for other cost functions of the type shown in Eq. (3), e.g.

$$J_t = (y_{t+k} - y_{t+k}^{\text{ref}})^2 + \lambda u_t^2 \tag{6}$$

or even

$$J_t = (y_{t+k} - y_{t+k}^{\text{ref}})^2 + \lambda_1(u_t - u_{t-1})^2 + \lambda_2(u_t - u_t^{\text{ref}})^2 \tag{7}$$

The following section shows how these types of cost functions are minimized.

2.3 The Controller

The linear quadratic Gaussian problem is to determine a control sequence that minimizes the expected cost

$$j_t = E[J_t] = E[(y_{t+k} - y_{t+k}^{\text{ref}})^2 + \lambda(u_t - u_{t-1})^2] \tag{8}$$

For $\lambda = 0$ the cost on the control is zero and by minimizing j_t with respect to u_t the minimum variance controller is obtained.

The output y_{t+k} can be decomposed into two parts

$$y_{t+k} = \hat{y}_{t+k|t} + \tilde{y}_{t+k|t} \tag{9}$$

where $\hat{y}_{t+k|t}$ is the k-step prediction and $\tilde{y}_{t+k|t}$ is the k-step prediction error. These parts are orthogonal (uncorrelated) since the prediction depends only on values of the noise at time t and earlier $(\varepsilon_t, \varepsilon_{t-1}, \ldots)$, and the error depends only on noise values subsequent to time t $(\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots, \varepsilon_{t+k})$, see e.g., Lewis (1986). Hence, Eq. (8) can be written as

$$j_t = [(\hat{y}_{t+k|t} - y_{t+k}^{\text{ref}})^2 + \lambda(u_t - u_{t-1})^2] + E[\tilde{y}_{t+k|t}^2] \tag{10}$$

The prediction $\hat{y}_{t+k|t}$ in Eq. (10) can be written as a linear function of u_t (this will be shown later on)

$$\hat{y}_{t+k|t} = \alpha_t u_t + \beta_t \tag{11}$$

where β_t depends on observations known at time t $(u_{t-1}, u_{t-2}, \ldots, y_t, y_{t-1}, \ldots)$ and the model parameters. α_t depends only on the model parameters. Inserting Eq. (11) in Eq. (10) gives

$$j_t = [(\alpha_t u_t + \beta_t - y_{t+k}^{\text{ref}})^2 + \lambda(u_t - u_{t-1})^2] + E[\tilde{y}_{t+k|t}^2] \tag{12}$$

which takes its minimum with respect to u_t when

$$\frac{\partial j_t}{\partial u_t} = 0 \tag{13}$$

i.e.,

$$2\alpha_t(\alpha_t u_t + \beta_t - y^{\text{ref}}_{t+k}) + 2\lambda(u_t - u_{t-1}) = 0 \tag{14}$$

or for $\alpha_t \neq 0$

$$\left(\frac{\lambda}{\alpha_t} + \alpha_t\right) u_t - \frac{\lambda}{\alpha_t} u_{t-1} + \beta_t - y^{\text{ref}}_{t+k} = 0 \tag{15}$$

Then the controller is obtained by isolating u_t

$$u_t = \frac{-1}{\left(\frac{\lambda}{\alpha_t} + \alpha_t\right)} \left[\frac{-\lambda}{\alpha_t} u_{t-1} + \beta_t - y^{\text{ref}}_{t+k}\right] \tag{16}$$

2.4 Optimal Prediction

In Lewis (1986) and other references the optimal prediction for an ARMAX model is computed by some polynomial manipulation of the Diophantine Equation or polynomial division. One drawback of this method is that the model parameters must be time-invariant.

Another way to compute the optimal prediction is to use the conditional expectation, as described, e.g., in Abraham and Ledolter (1983) and Madsen (1989). This leads to a method that is capable of handling time-varying systems. A brief discussion of this method is given in the following.

The k-step prediction can be written as

$$\hat{y}_{t+k|t} = E[y_{t+k}|y_t, y_{t-1}, \ldots; u_t, u_{t-1}, \ldots] = E[y_{t+k}|\cdot] \tag{17}$$

where $E[y_{t+k}|\cdot]$ is the conditional expectation. This representation is convenient, since the prediction can be calculated from the difference equation form of the model. The difference equation form of the model in Eq. (1) is

$$\begin{aligned} y_t = & - a_{1,t}y_{t-1} - a_{2,t}y_{t-2} - \cdots - a_{n,t}y_{t-n} \\ & + b_{0,t}u_{t-k} + b_{1,t}u_{t-1-k} + \cdots + b_{m,t}u_{t-m-k} \\ & + \varepsilon_t + c_{1,t}\varepsilon_{t-1} + \cdots + c_{r,t}\varepsilon_{t-r} \end{aligned} \tag{18}$$

The k-step prediction is then

$$\begin{aligned} E[y_{t+k}|\cdot] = & - a_{1,t+k}E[y_{t+k-1}|\cdot] - \cdots - a_{n,t+k}E[y_{t+k-n}|\cdot] \\ & + b_{0,t+k}E[u_t|\cdot] + b_{1,t+k}E[u_{t-1}|\cdot] + \cdots \\ & + b_{m,t+k}E[u_{t-m}|\cdot] + E[\varepsilon_{t+k}|\cdot] \\ & + c_{1,t+k}E[\varepsilon_{t+k-1}|\cdot] + \cdots + c_{r,t+k}E[\varepsilon_{t+k-r}|\cdot] \end{aligned} \tag{19}$$

or simply

$$\begin{aligned} \hat{y}_{t+k|t} = & - a_{1,t+k}\bar{y}_{t+k-1} - \cdots - a_{n,t+k}\bar{y}_{t-n+k} \\ & + b_{0,t+k}u_t + b_{1,t+k}u_{t-1} + \cdots + b_{m,t+k}u_{t-m} \\ & + c_{k,t+k}\varepsilon_t + \cdots + c_{r,t+k}\varepsilon_{t-r+k} \end{aligned} \tag{20}$$

where the following identities have been used, see Abraham and Ledolter (1983)

$$E[\varepsilon_{t+j}|\cdot] = 0 \quad \text{if} \quad j > 0 \tag{21}$$

$$E[\varepsilon_{t+j}|\cdot] = \varepsilon_{t+j} \quad \text{if} \quad j \leq 0 \tag{22}$$

$$E[y_{t+j}|\cdot] = \bar{y}_{t+j} = \hat{y}_{t+j|t} \quad \text{if} \quad j > 0 \tag{23}$$

$$E[y_{t+j}|\cdot] = \bar{y}_{t+j} = y_{t+j} \quad \text{if} \quad j \leq 0 \tag{24}$$

$$E[u_{t+j}|\cdot] = u_{t+j} \quad \text{if} \quad j \leq 0 \tag{25}$$

The following example illustrates the above described optimal prediction and controller. It is clearly seen from Eq. (20) that the prediction can be written as stated in Eq. (11).

2.5 Example

Consider an ARMAX(1, 3, 0) model

$$y_t = -a_t y_{t-1} + b_{0,t} u_{t-k} + b_{1,t} u_{t-k-1} + b_{2,t} u_{t-k-2} + \varepsilon_t \tag{26}$$

with time-varying a_t and $b_{j,t}$ parameters, and a fixed time-delay k. The 1-step prediction is then given by

$$\hat{y}_{t+1|t} = -a_{t+1} y_t + b_{0,t+1} u_{t-k+1} + b_{1,t+1} u_{t-k} + b_{2,t+1} u_{t-k-1} \tag{27}$$

and the k-step prediction is

$$\hat{y}_{t+k|t} = -a_{t+k} \hat{y}_{t+k-1|t} + b_{0,t+k} u_t + b_{1,t+k} u_{t-1} + b_{2,t+k} u_{t-2} \tag{28}$$

By comparison with Eq. (11), α_t and β_t are identified as

$$\alpha_t = b_{0,t+k} \tag{29}$$

$$\beta_t = -a_{t+k} \hat{y}_{t+k-1|t} + b_{1,t+k} u_{t-1} + b_{2,t+k} u_{t-2} \tag{30}$$

Inserting Eq. (29) and (30) in Eq. (16) gives the controller

$$\begin{aligned} u_t \quad = \quad & \frac{-1}{\left(\frac{\lambda}{b_{0,t+k}} + b_{0,t+k}\right)} \left[\frac{-\lambda}{b_{0,t+k}} u_{t-1} + b_{1,t+k} u_{t-1} + b_{2,t+k} u_{t-2} \right. \\ & \left. - \, a_{t+k} \hat{y}_{t+k-1|t} - y_{t+k}^{\text{ref}} \right] \end{aligned} \tag{31}$$

or

$$\begin{aligned} u_t \quad = \quad & \frac{-1}{\left(\frac{\lambda}{b_{0,t+k}} + b_{0,t+k}\right)} \left[\left(b_{1,t+k} - \frac{\lambda}{b_{0,t+k}} \right) u_{t-1} + b_{2,t+k} u_{t-2} \right. \\ & \left. - \, a_{t+k} \hat{y}_{t+k-1|t} - y_{t+k}^{\text{ref}} \right] \end{aligned} \tag{32}$$

For $\lambda = 0$, the minimum variance controller presented in Søgaard (1988) appears.

Remarks: For the cost function in Eqs. (6) and (7) we obtain similarly

$$u_t = \frac{-1}{\left(\frac{\lambda}{b_{0,t+k}} + b_{0,t+k}\right)} \Big[b_{1,t+k} u_{t-1} + b_{2,t+k} u_{t-2} - a_{t+k} \hat{y}_{t+k-1|t} - y^{\text{ref}}_{t+k} \Big] \tag{33}$$

and

$$u_t = \frac{-1}{\left(\frac{\lambda_1}{b_{0,t+k}} + \frac{\lambda_2}{b_{0,t+k}} + b_{0,t+k}\right)} \Big[\left(b_{1,t+k} - \frac{\lambda_1}{b_{0,t+k}}\right) u_{t-1} + b_{2,t+k} u_{t-2} - a_{t+k} \hat{y}_{t+k-1|t} - y^{\text{ref}}_{t+k} - \frac{\lambda_2}{b_{0,t+k}} u^{\text{ref}}_t \Big] \tag{34}$$

respectively.

3 Supply Temperature Control

In some district heating systems, e.g. in Esbjerg in Denmark, the control strategy is to keep the supply temperature from the district heating plant as low as possible, but subject to the restriction that the required heat and minimum supply temperature at every consumer are delivered at any time. A lower supply temperature implies lower heat loss from the distribution network and lower production cost. Today, this is done by using a program package for optimal control of supply temperature called PRESS, developed at The Institute of Mathematical Statistics and Operations Research, The Technical University of Denmark.

The restriction is coped with by finding several representative (or critical) points (locations) in the distribution networks, defined in the way that, if the temperature requirements at these points are fulfilled, then every consumer is satisfied. These temperature requirements follow some minimum supply temperature curve, which expresses the temperature as a function of the outdoor air temperature (see Fig. 1). The supply temperature curve is a mutual agreement between the district heating company and the consumers.

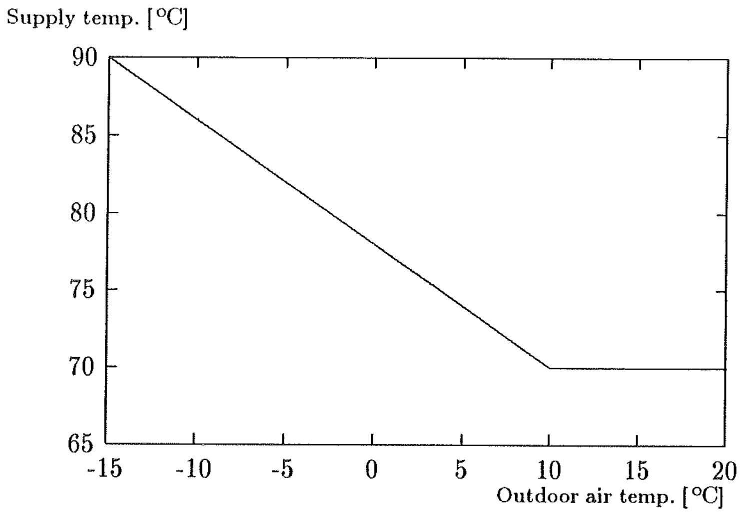


Fig. 1 An example of the minimum supply temperature at critical points in the network.

Each representative point has its own minimum curve, depending on its location in the network. As seen in Fig. 1 the minimum supply temperature is constant for outdoor air temperatures higher than a certain value and for lower outdoor temperature it is increasing linearly. The reference value is chosen so that the probability of the supply temperature less than the admissible minimum is less than, e.g., 1 percent. Hence, the reference value in the above controller is time-varying.

3.1 The Model

Transfer function models that describe the relation between the supply temperature of the water at the district heating plant and the supply temperature at selected location in the district heating network are shown in Madsen *et al.* (1990) and Søgaard (1988). The models are ARMAX model with time-varying parameters, see Eq. (1).

Furthermore, it is shown in Madsen *et al.* (1990) how the parameters can be estimated adaptively using a recursive forgetting factor method. It is assumed that the orders n, m and r of the polynomials are known and constant. Furthermore, it is assumed that the parameters $a_1, \ldots, a_n, b_0, \ldots, b_m$ and $c_1, \ldots, c_r$ and the time-delay k are unknown and time-varying.

The time-delay in district heating systems is relatively large. Since the time-delay is time-varying and a function of several variables, like distance, flow in the pipes, temperatures and intersection lay-out of the network, the identification of the delay is one of the major problems in modeling district heating systems. Methods for estimating or tracking such time-varying time-delay, are developed and described in details in Søgaard and Madsen (1991).

In the following, a model for a particular critical point in Esbjerg is considered. The parameters are estimated by PRESS using real data from the first months of 1990. The sampling interval is 1 hour.

From Eq. (26), with time-delay, $k = 2$,

$$y_t = -ay_{t-1} + b_{0,t}u_{t-2} + b_{1,t}u_{t-3} + b_{2,t}u_{t-4} + \varepsilon_t \tag{35}$$

where a is constant and $b_{j,t}$ are given by

$$\begin{aligned} b_{0,t} &= b_{01} + b_{02}\sin(\omega(t-2)) + b_{03}\cos(\omega(t-2)) \\ b_{1,t} &= b_{11} + b_{12}\sin(\omega(t-3)) + b_{13}\cos(\omega(t-3)) \\ b_{2,t} &= b_{21} + b_{22}\sin(\omega(t-4)) + b_{23}\cos(\omega(t-4)) \end{aligned} \tag{36}$$

with

$$\omega = \frac{2\pi}{24} = \frac{\pi}{12} \tag{37}$$

i.e., a diurnal variation of the parameters is expressed.

The 1-step prediction is then written as

$$\hat{y}_{t+1|t} = -ay_t + b_{0,t+1}u_{t-1} + b_{1,t+1}u_{t-2} + b_{2,t+1}u_{t-3} \tag{38}$$

Similarly the 2-step prediction is

$$\hat{y}_{t+2|t} = -a\hat{y}_{t+1|t} + b_{0,t+2}u_t + b_{1,t+2}u_{t-1} + b_{2,t+2}u_{t-2} \tag{39}$$

and the controller (see Eq. (32)) is

$$u_t = \frac{-1}{\left(\frac{\lambda}{b_{0,t+2}} + b_{0,t+2}\right)} \left[\left(b_{1,t+2} - \frac{\lambda}{b_{0,t+2}} \right) u_{t-1} + b_{2,t+2} u_{t-2} - a\hat{y}_{t+1|t} - y^{\text{ref}}_{t+2} \right] \quad (40)$$

Inserting Eq. (38) in Eq. (40) gives

$$u_t = \frac{-1}{\left(\frac{\lambda}{b_{0,t+2}} + b_{0,t+2}\right)} \left[\left(b_{1,t+2} - ab_{0,t+1} - \frac{\lambda}{b_{0,t+2}} \right) u_{t-1} + (b_{2,t+2} - ab_{1,t+1}) u_{t-2} - ab_{2,t+1} u_{t-3} + a^2 y_t - y^{\text{ref}}_{t+2} \right] \quad (41)$$

Remark: For the time-invariant parameter case the controller is given by

$$u_t = \frac{-1}{\left(\frac{\lambda}{b_0} + b_0\right)} \left[\left(b_1 - ab_0 - \frac{\lambda}{b_0} \right) u_{t-1} + (b_2 - ab_1) u_{t-2} - ab_2 u_{t-3} + a^2 y_t - y^{\text{ref}}_{t+2} \right] \quad (42)$$

3.2 Simulation Results

The above model, Eq. (35), was used in simulation studies. A white noise sequence, $\{\varepsilon_t\} \sim N(0,1)$ was generated by the IMSL routine RNNOR (see IMSL (1987)). For the parameters, a and b_{**}, in Eq. (36), following estimates were used (obtained by PRESS on data from the real system): $a = -0.5625, b_{01} = 0.3534, b_{02} = 0.3166, b_{03} = -0.2421, b_{11} = 0.1813, b_{12} = -0.3063, b_{13} = 0.2510, b_{21} = 0.1789, b_{22} = 0.0855, b_{23} = -0.2477$. The diurnal variation of the parameters is shown in Fig. 2.

In Fig. 3 the relations between the control change variance, $\hat{\sigma}^2_{\Delta u}$, and the output variance, $\hat{\sigma}^2_y$, are shown for various values of the penalty parameter

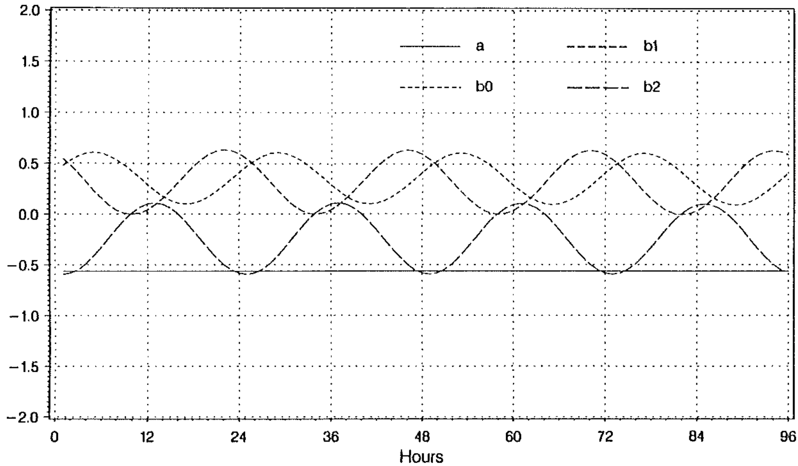


Fig. 2 Parameters used in the simulations.

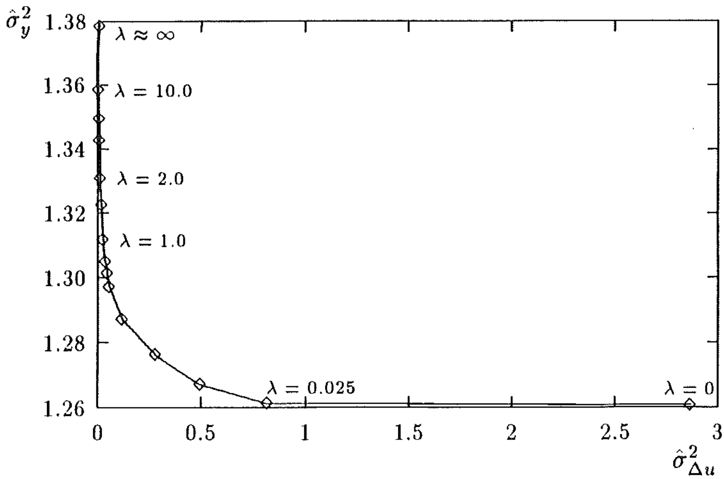


Fig. 3 Variance of the output signal, $\hat{\sigma}_y^2$, vs. variance of the change in the control signal, $\hat{\sigma}_{\Delta u}^2$, for various values of λ .

λ (y_t^{ref} was held constant). For $\lambda = 0$, $\hat{\sigma}_y^2$ is at minimum, corresponding to the minimum variance controller in Eq. (8). As expected, $\hat{\sigma}_{\Delta u}^2$ decreases and $\hat{\sigma}_y^2$ increases as λ is increased. However, for small values of λ, $\hat{\sigma}_y^2$ is not affected very much while $\hat{\sigma}_{\Delta u}^2$ decreases drastically. For $\lambda \to \infty$ the price of the control effort will be so high that there will be no control at all.

Figures 4 - 7 show the output, the reference output and the control signal for different values of λ. The changes in reference output, y_t^{ref}, are made in steps as indicated in the figures. Changing λ from 0 to 0.025 reduces the amplitude of the control signal noticeably, without any visible change in the output. For higher λ values the variations in the control is much smaller and slower. It is also noteworthy how well the output follows the reference output. For $\lambda = 10$ a larger gap between the output and the reference output is seen, as expected, due to the large penalty on changes in the control signal.

4 Summary and Conclusions

In this paper a modified version of the linear quadratic Gaussian controller is presented. The controller is based on predictions in terms of conditional expectations. The controller applies the optimal prediction of the output successively, and is therefore capable of handling non-stationarities, like time-varying model parameters. This is unlike the classical formulation of linear quadratic Gaussian controllers, where the controller is based on either the solution of the Diophantine or the Riccati Equation.

The model is on a polynomial (or difference equation) form, and the cost function keeps the output close to some predetermined reference output and at the same time keeps the change in the control signal small.

The controller is used in simulation studies together with a transfer function model which describes the relation between the supply temperature from a district heating plant and the supply temperature at some specific locations in the network. The model parameters vary trigonometrically with 24 hours as a period.

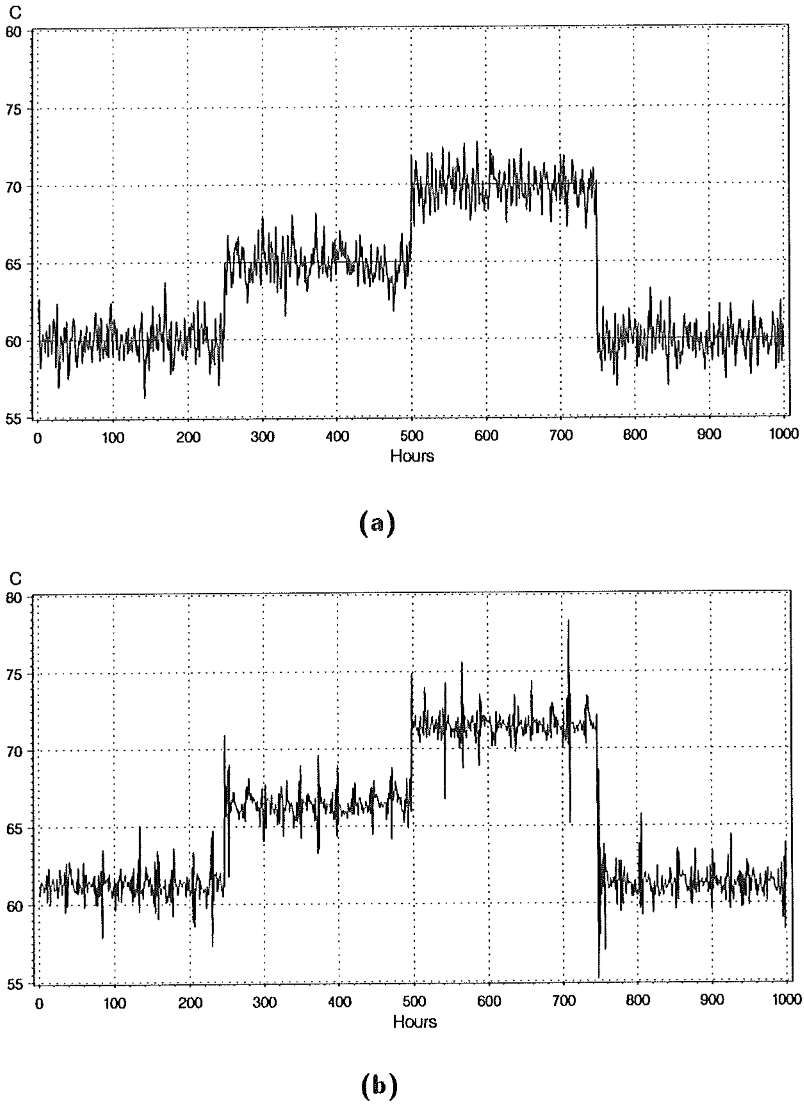
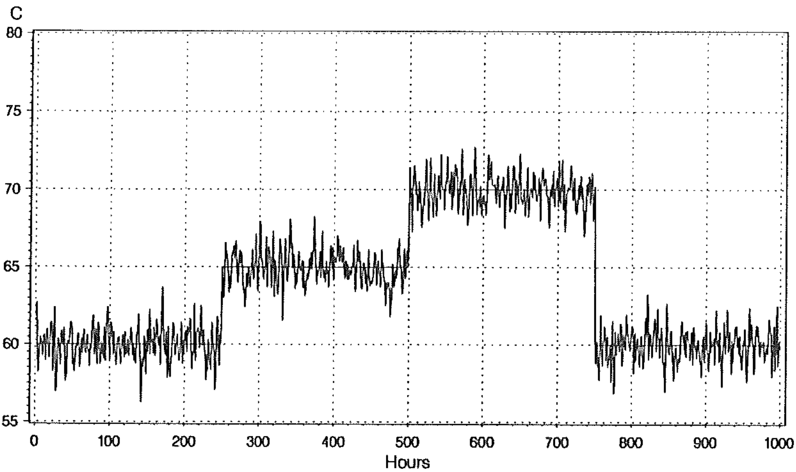
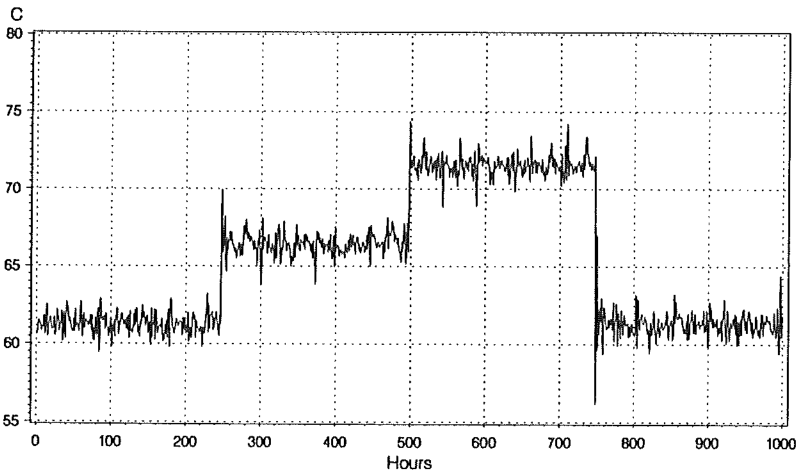


Fig. 4 (a) The output $\{y_t\}$ and the reference output $\{y_t^{\text{ref}}\}$. (b) The control signal $\{u_t\}$. $\lambda = 0.0$.

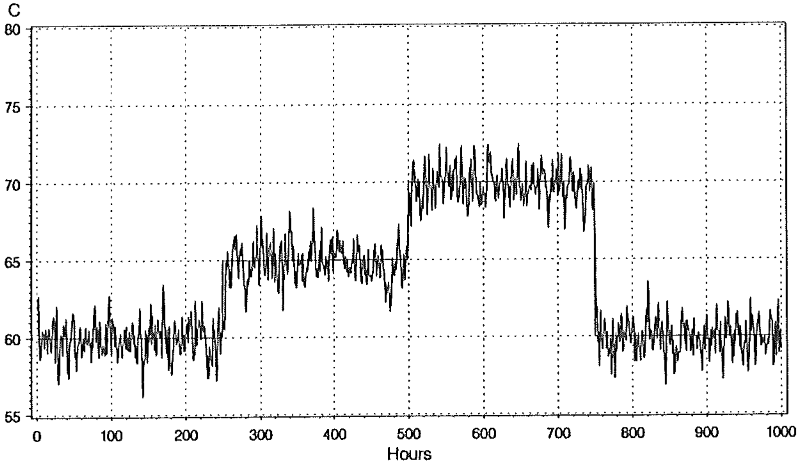


(a)

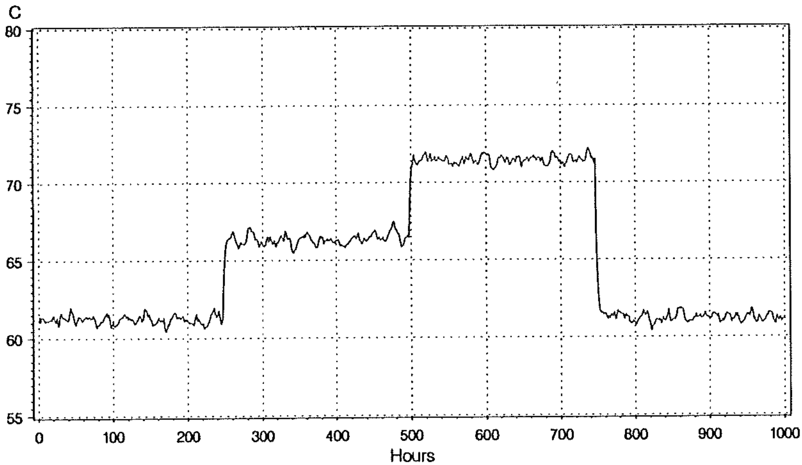


(b)

Fig. 5 (a) The output $\{y_t\}$ and the reference outpu t $\{y_t^{\text{ref}}\}$. (b) The control signal $\{u_t\}$. $\lambda = 0.025$.

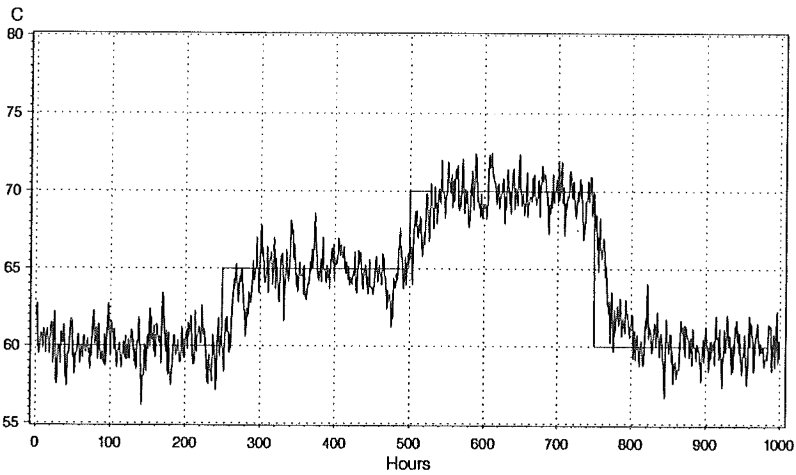


(a)

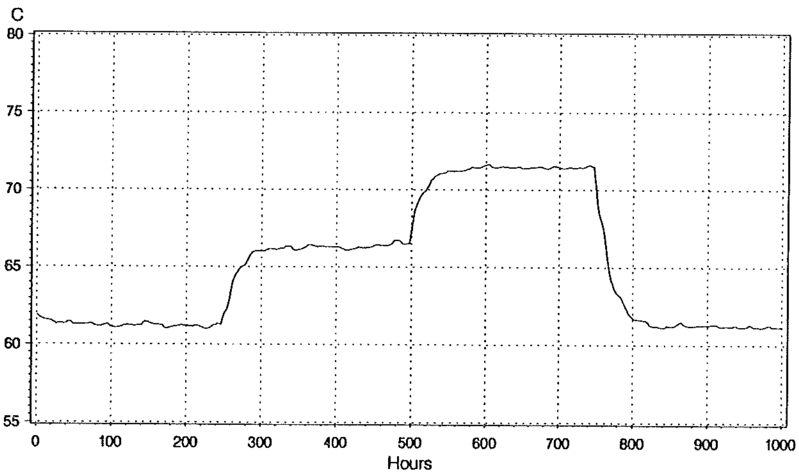


(b)

Fig. 6 (a) The output $\{y_t\}$ and the reference output $\{y_t^{\rm ref}\}$. (b) The control signal $\{u_t\}$. $\lambda = 1.0$.



(a)



(b)

Fig. 7 (a) The output $\{y_t\}$ and the reference output $\{y_t^{\mathrm{ref}}\}$. (b) The control signal $\{u_t\}$. $\lambda = 10.0$.

The simulation results show the effect of the penalty, λ, on the change in the control signal. For adequate values of the penalty the variance of the control changes decreases without noticeable higher output variance.

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Generalized Predictive Control for Non-Stationary Systems

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Abstract

This paper shows how the Generalized Predictive Control (GPC) can be extended to non-stationary (time-varying) systems. If the time-variation is slow, then the classical GPC can be used in context with an adaptive estimation procedure of a time-invariant ARIMAX model. However, in this paper prior knowledge concerning the nature of the parameter variations is assumed available.

The GPC is based on the assumption that the prediction of the system output can be expressed as a linear combination of present and future controls. Since the Diophantine Equation *cannot* be used due to the time-variation of the parameters, the optimal prediction is found as the general conditional expectation of the system output.

The underlying model is of an ARMAX type instead of an ARIMAX type as in the original version of the GPC (Clarke *et al.*, 1987) and almost all later references. This implies some further modifications of the classical GPC.

Key Words: Generalized Predictive Control (GPC); Time-varying Systems; Impulse Response; Filtering; Conditional Expectation.

1 Introduction

Most frequently, when Generalized Predictive Control (GPC) is used for time varying systems, an adaptive estimation procedure is used for an ordinary ARIMAX model with constant parameters. Hence, the model parameters are assumed constant over the prediction horizon applied. The predictions are then, beneficially, obtained by solving the Diophantine Equation, e.g., recursively as in Clarke *et al.* (1987). This procedure is reasonable only if the underlying time-variation is relatively slow. In general it is more reasonable to consider time-varying models and then extend the GPC to handle these models. This allows for control of both slow and fast changing systems, and the time-variation can be used for an improved prediction, and hence for an improved control.

In this paper a GPC for time-varying systems is proposed. This procedure allows for control of systems where, for instance, the parameter variations are so fast that the assumption of the parameters being constant over the prediction horizon is no longer valid. In that case the Diophantine Equation is useless. Instead the predictions of the future outputs are computed directly as the conditional expectations, conditioned on observations known.

In the literature, see e.g., Clarke *et al.* (1987) and Bitmead *et al.* (1990), the GPC is formulated by using an ARIMAX-type model. One of the main arguments for the integrating factor (integral part) is that it guarantees an offset free control, but it also gives a more straightforward formulation of the optimization problem. However, from a model building viewpoint this integral part may seem to be somewhat artificial. Therefore, in this paper the GPC is based on an ARMAX model rather than an ARIMAX model, and the offset free control is guaranteed by filtering the control signal.

The paper is organized as follows. In Section 2 the model structure is described and in Section 3 the optimal prediction for non-stationary systems is derived. Section 4 deals with the cost function and the optimization problem. In Section 5 some simulation experiments are presented to illustrate the performance of the controller, and finally the conclusions are drawn in Section 6.

2 Model Structure

It is assumed that the system can be described by a time-varying ARMAX model

$$A_t(q^{-1})y_t = B_t(q^{-1})u_t + C_t(q^{-1})e_t \quad (1)$$

where $\{y_t\}$ and $\{u_t\}$ are the output and the control signal, respectively. $\{e_t\}$ is white noise with mean zero and variance σ_e^2. A_t, B_t and C_t are polynomials in q^{-1} (the back shift operator) with time-varying coefficients:

$$\begin{aligned} A_t(q^{-1}) &= 1 + a_{1,t}q^{-1} + \cdots + a_{n_A,t}q^{-n_A} \\ B_t(q^{-1}) &= b_{1,t}q^{-1} + \cdots + b_{n_B,t}q^{-n_B} \\ C_t(q^{-1}) &= 1 + c_{1,t}q^{-1} + \cdots + c_{n_C,t}q^{-n_C} \end{aligned} \quad (2)$$

The nature of the parameter variations are assumed known. Consider, as an example, the periodical variation

$$\begin{aligned} a_{j,t} &= \alpha_{j,0} + \alpha_{j,1}\sin(\omega(t-j)) + \alpha_{j,2}\cos(\omega(t-j)) \\ b_{j,t} &= \beta_{j,0} + \beta_{j,1}\sin(\omega(t-j)) + \beta_{j,2}\cos(\omega(t-j)) \\ c_{j,t} &= \gamma_{j,0} + \gamma_{j,1}\sin(\omega(t-j)) + \gamma_{j,2}\cos(\omega(t-j)) \end{aligned} \quad (3)$$

which could, e.g., represent the diurnal variation in an energy system, see e.g., Madsen *et al.* (1992). The parameters α, β and γ may also be time-varying and then they can be estimated adaptively using recursive methods.

3 Output Prediction

The GPC is based on the assumption that the output predictions can be expressed as a linear combination of present and future controls. In Clarke *et al.* (1987) and many other references this is obtained by solving the Diophantine Equation (sometimes recursively). However, in the time-varying case the Diophantine Equation *cannot* be used due to the time-variation of the model parameters. Instead the j-step predictor, $\hat{y}_{t+j|t}$, is found as the conditional expectation of y_{t+j} conditioned on observations of the output up to time t (Madsen, 1989)

$$\begin{aligned}
\hat{y}_{t+j|t} &= -\sum_{i=1}^{n_A} a_{i,t+j}\hat{y}_{t+j-i|t} + \sum_{i=1}^{n_B} b_{i,t+j}u_{t+j-i} \\
&\quad + \sum_{i=0}^{n_C} c_{i,t+j}\hat{e}_{t+j-i|t}, \qquad j \geq 1 \qquad (4)\\
\hat{y}_{t+j|t} &= y_{t+j}, \qquad j < 1 \qquad (5)
\end{aligned}$$

where

$$\hat{e}_{t+l|t} = \begin{cases} 0, & \text{if } l \geq 1 \\ e_{t+l}, & \text{if } l < 1 \end{cases} \qquad (6)$$

A simple example illustrates the method.

Example

Consider the ARX(1,2) model ($n_A = 1$, $n_B = 2$)

$$y_t + a_{1,t}y_{t-1} = b_{1,t}u_{t-1} + b_{2,t}u_{t-2} + e_t \qquad (7)$$

The 1-step predictor is derived from Eqs. (4) – (6) as

$$\begin{aligned}
\hat{y}_{t+1|t} &= -a_{1,t+1}y_t + b_{1,t+1}u_t + b_{2,t+1}u_{t-1} \\
&= h_{1,t+1}u_t + v_{1,t} \qquad (8)
\end{aligned}$$

where $v_{1,t} = -a_{1,t+1}y_t + b_{2,t+1}u_{t-1}$ is known and $h_{1,t+1}u_t = b_{1,t+1}u_t$ is unknown until the control signal is chosen at time t. Note, that $h_{1,t+1} = b_{1,t+1}$ is the first weight of the time-varying impulse response function.

In general, the j-step predictor is given as

$$\begin{aligned}
\hat{y}_{t+j|t} &= -a_{1,t+j}\hat{y}_{t+j-1|t} + b_{1,t+j}u_{t+j-1} + b_{2,t+j}u_{t+j-2} \\
&= -a_{1,t+j}(h_{j-1,t+j-1}u_t + \cdots + h_{1,t+j-1}u_{t+j-2} + v_{j-1,t})
\end{aligned}$$

$$
\begin{aligned}
& +b_{1,t+j}u_{t+j-1} + b_{2,t+j}u_{t+j-2} \\
= \quad & h_{j,t+j}u_t + \cdots + h_{1,t+j}u_{t+j-1} + v_{j,t} \\
= \quad & \sum_{i=1}^{j} h_{i,t+j}u_{t+j-i} + v_{j,t}
\end{aligned} \tag{9}
$$

For the given model the $h_{i,t}$ and $v_{i,t}$ values can be computed recursively as

$$
h_{i,t} = \begin{cases} b_{i,t}, & \text{if } i = 1 \\ b_{i,t} - a_{1,t}h_{i-1,t-1}, & \text{if } i = 2 \\ -a_{1,t}h_{i-1,t-1}, & \text{if } i \geq 3 \end{cases} \tag{10}
$$

and

$$
v_{i,t} = \begin{cases} -a_{1,t+i}y_t + b_{2,t+i}u_{t-1}, & \text{if } i = 1 \\ -a_{1,t+i}v_{i-1,t}, & \text{if } i \geq 2 \end{cases} \tag{11}
$$

The coefficients $h_{i,t+j}$ $(i = 1, 2, \ldots)$ are the weights of the time-varying impulse response function describing the dynamic relation between the input and the output, i.e. $h_{i,t}$ is the marginal change of y_t changing u_{t-i}. □

It can be shown that a j-step predictor for an arbitrary ARMAX process can be written on the form shown in Eq. (8). This can, for instance, be shown by substitution as illustrated in the previous example.

An alternative scheme for calculating $h_{i,t+j}$ and $v_{j,t}$ follows directly from Eq. (8):

1. To find $v_{j,t}$, set the present and future control to zero ($u_{t+j-i} = 0$ for $i = 1, 2, \ldots, j$). Then compute $v_{j,t}$ as the conditional expectation, $\hat{y}_{t+j|t}$, using Eq. (4).
2. To find $h_{i,t+j}$, $(i = 1, 2, \ldots, j)$, set the present and past output and past control to zero ($v_{j,t} = 0$). Feed an impulse into the system at time $t + j - i$

$$
u_{t+j-l} = \begin{cases} 1, & \text{if } l = i \\ 0, & \text{otherwise} \end{cases} \tag{12}
$$

   Then compute $h_{i,t+j}$ as the conditional expectation, $\hat{y}_{t+j|t}$, using Eq. (4).

Now introduce a maximum prediction horizon $N(\geq 1)$. Considering Eq. (9) it is seen that the j-step predictions, j running from 1 up to N, can be written as a linear matrix expression:

$$\hat{\boldsymbol{y}}_t = \boldsymbol{H}_t \boldsymbol{u}_t + \boldsymbol{v}_t \tag{13}$$

where

$$\begin{aligned}
\hat{\boldsymbol{y}}_t &= [\hat{y}_{t+1|t}, \ldots, \hat{y}_{t+N|t}]^T \\
\boldsymbol{u}_t &= [u_t, \ldots, u_{t+N-1}]^T \\
\boldsymbol{v}_t &= [v_{1,t}, \ldots, v_{N,t}]^T \\
\boldsymbol{H}_t &= \begin{bmatrix}
h_{1,t+1} & 0 & \cdots & 0 & 0 \\
h_{2,t+2} & h_{1,t+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
h_{N-1,t+N-1} & h_{N-2,t+N-1} & \cdots & h_{1,t+N-1} & 0 \\
h_{N,t+N} & h_{N-1,t+N} & \cdots & h_{2,t+N} & h_{1,t+N}
\end{bmatrix}
\end{aligned}$$

In Eq. (13) it has been assumed that the same model is applied for all prediction horizons. Actually, this need not to be the case. If different models are used, then the j'th row of $\boldsymbol{H}_t$ and the j'th element of $\boldsymbol{v}_t$ belongs to a special model designed for j-step prediction. Making use of an individual model for each horizon is often relevant if a non-linear system is approximated by a family of linear models (e.g. threshold models). That is if the optimal linearization of the system depends on the prediction horizon.

4 Cost Function and Optimization

Consider a cost function of the form

$$J = E\left[\sum_{j=N_1}^{N_2} (y_{t+j} - y^{\text{ref}}_{t+j|t})^2 + \sum_{j=1}^{N_u} \lambda_{j,t} u^2_{t+j-1}\right] \tag{14}$$

where

- N_1 is the minimum costing horizon,
- N_2 is the maximum costing horizon,
- $\lambda_{j,t}$ is a control-weighting (or penalty) sequence,
- N_u is the control horizon,
- $y^{\text{ref}}_{t+j|t}$ is the future reference output.

The expectation in Eq. (14) is conditioned on observations available at time t.

The cost function expressed in Eq. (14) is almost identical to the original GPC cost function presented in Clarke *et al.* (1987). The only exception is that in Clarke *et al.* (1987) the control increments, Δu_t, is used instead of the absolute control values, u_t. The incremental version of the cost function will be discussed later.

Introducing matrix notation the cost function is written as

$$J = E[(\boldsymbol{y}_t - \boldsymbol{y}^{\text{ref}}_t)^T(\boldsymbol{y}_t - \boldsymbol{y}^{\text{ref}}_t) + \boldsymbol{u}_t^T \boldsymbol{\Lambda}_t \boldsymbol{u}_t] \tag{15}$$

where

$$\begin{aligned}
\boldsymbol{y}_t &= [y_{t+N_1}, \ldots, y_{t+N_2}]^T \\
\boldsymbol{y}^{\text{ref}}_t &= [y^{\text{ref}}_{t+N_1|t}, \ldots, y^{\text{ref}}_{t+N_2|t}]^T \\
\boldsymbol{u}_t &= [u_t, \ldots, u_{t+N_u-1}]^T \\
\boldsymbol{\Lambda}_t &= \begin{bmatrix} \lambda_{1,t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{N_u,t} \end{bmatrix}
\end{aligned}$$

By the projection theorem (Brockwell and Davis, 1987) the output vector, $\boldsymbol{y}_t$, is decomposed into two (stochastic) independent parts

$$\boldsymbol{y}_t = \hat{\boldsymbol{y}}_t + \tilde{\boldsymbol{y}}_t \tag{16}$$

where $\hat{\boldsymbol{y}}_t$ is a vector containing the output predictions described in the previous section, and $\tilde{\boldsymbol{y}}_t$ is a vector containing the prediction errors

$$\tilde{\boldsymbol{y}}_t = [\tilde{y}_{t+N_1|t}, \cdots, \tilde{y}_{t+N_2|t}]^T \tag{17}$$

The cost function can then be written as

$$J = [(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t^{\text{ref}})^T(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t^{ref}) + \boldsymbol{u}_t^T \Lambda_t \boldsymbol{u}_t] + E[\tilde{\boldsymbol{y}}_t^T \tilde{\boldsymbol{y}}_t] \tag{18}$$

Note that the last term in this expression does not depend on $\boldsymbol{u}_t$. Using Eq. (13)† the cost function becomes

$$J = [(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}})^T(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + \boldsymbol{u}_t^T \Lambda_t \boldsymbol{u}_t] + E[\tilde{\boldsymbol{y}}_t^T \tilde{\boldsymbol{y}}_t] \tag{19}$$

The GPC control law is then obtained by minimizing the resulting cost function. This is done by setting the derivative of the cost function w.r.t. $\boldsymbol{u}_t$ to zero, i.e.

$$\frac{\partial J}{\partial \boldsymbol{u}_t} = 0$$

That is

$$2\boldsymbol{H}_t^T(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + 2\Lambda_t\boldsymbol{u}_t = 0 \tag{20}$$

†The time-varying impulse response matrix is now written, according to the horizons N_1, N_2 and N_u (assuming that $N_2 > N_u + N_1$), as

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t+N_1} & 0 & \cdots & 0 \\ h_{2,t+N_1+1} & h_{1,t+N_1+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_u,t+N_1+N_u-1} & h_{N_u-1,t+N_1+N_u-1} & \cdots & h_{1,t+N_1+N_u-1} \\ h_{N_u+1,t+N_1+N_u} & h_{N_u,t+N_1+N_u} & \cdots & h_{2,t+N_1+N_u} + h_{1,t+N_1+N_u} \\ \vdots & \vdots & & \vdots \\ h_{N_2-N_1+1,t+N_2} & h_{N_2-N_1,t+N_2} & \cdots & \sum_{i=1}^{N_2-N_1-N_u+2} h_{i,t+N_2} \end{bmatrix}$$

The summation in the last column results from the assumption $\Delta u_{t+i-1} = 0, i > N_u$ (Bjerre, 1992). The vector $\boldsymbol{v}_t$ is now $\boldsymbol{v}_t = [v_{N_1,t}, \ldots, v_{N_2,t}]^T$.

or

$$2(\boldsymbol{H}_t^T \boldsymbol{H}_t + \boldsymbol{\Lambda}_t)\boldsymbol{u}_t + 2\boldsymbol{H}_t^T(\boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) = 0 \tag{21}$$

Solving for $\boldsymbol{u}_t$, results in

$$\boldsymbol{u}_t = -[\boldsymbol{H}_t^T \boldsymbol{H}_t + \boldsymbol{\Lambda}_t]^{-1}\boldsymbol{H}_t^T(\boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) \tag{22}$$

Only the first element of the control vector, $\boldsymbol{u}_t$, is implemented (receding horizon control) so the control law can be written as

$$u_t = -\boldsymbol{\theta}[\boldsymbol{H}_t^T \boldsymbol{H}_t + \boldsymbol{\Lambda}_t]^{-1}\boldsymbol{H}_t^T(\boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) \tag{23}$$

where

$$\boldsymbol{\theta} = [1, 0, \cdots, 0] \tag{24}$$

4.1 The Filtered Version

The control law in Eq. (23) may give an offset for $\boldsymbol{\Lambda}_t > \boldsymbol{0}$. This is coped with by introducing a filtered version of the control signal in the cost function.

The filter is defined as

$$\bar{u}_t = \frac{P_N(q^{-1})}{P_D(q^{-1})} u_t \tag{25}$$

where $P_N(q^{-1})$ and $P_D(q^{-1})$ are polynomials in the back shift operator q^{-1}.

The filtered control signal can be decomposed into a sum of two terms: one term containing past control signals, and another containing the present and future control signals. This can be done by means of solving the Diophantine Equation, (since the filter is time-invariant)

$$\bar{u}_{t+j} = \frac{D_j(q^{-1})}{P_D(q^{-1})} u_t + E_j(q^{-1}) u_{t+j} \tag{26}$$

By introducing matrix notation this can be rewritten as

$$\bar{\boldsymbol{u}}_t = \boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t \tag{27}$$

where

$$\bar{\boldsymbol{u}}_t = [\bar{u}_t, \ldots, \bar{u}_{t+N_u-1}]^T$$

$$\boldsymbol{u}_t = [u_t, \ldots, u_{t+N_u-1}]^T$$

$$\boldsymbol{g}_t = \bar{\boldsymbol{u}}_{t|\boldsymbol{u}_t=0}$$

$$\boldsymbol{F} = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ f_2 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{N_u} & f_{N_u-1} & \cdots & f_1 \end{bmatrix}$$

$\boldsymbol{F}$ contains the impulse response weights for the filter.

Using the filtered control signal Eq. (27) instead of the unfiltered in the cost function, Eq. (15), leads to

$$J = [(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}})^T(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + (\boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t)^T\Lambda_t(\boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t)] + E[\tilde{\boldsymbol{y}}_t^T\tilde{\boldsymbol{y}}_t] \quad (28)$$

Minimization of this expression results in the control vector

$$\boldsymbol{u}_t = -[\boldsymbol{H}_t^T\boldsymbol{H}_t + \boldsymbol{F}^T\Lambda_t\boldsymbol{F}]^{-1}[\boldsymbol{H}_t^T(\boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + \boldsymbol{F}^T\Lambda_t\boldsymbol{g}_t] \quad (29)$$

and the implemented control at time t

$$u_t = -\boldsymbol{\theta}[\boldsymbol{H}_t^T\boldsymbol{H}_t + \boldsymbol{F}^T\Lambda_t\boldsymbol{F}]^{-1}[\boldsymbol{H}_t^T(\boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + \boldsymbol{F}^T\Lambda_t\boldsymbol{g}_t] \quad (30)$$

with $\boldsymbol{\theta}$ as in Eq. (24).

For the important special case $\bar{u}_t = \Delta u_t = u_t - u_{t-1}$, $\boldsymbol{F}$ and $\boldsymbol{g}_t$ in Eq. (27) are given by

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \quad (31)$$

and

$$g_t = [-u_{t-1}, 0, \cdots, 0]^T \tag{32}$$

Remark: The main purpose of introducing the filter is to eliminate the offset, induced by the cost function in Eq. (14) for positive penalties, since the underlying model has not an integration.

The filter choice depends on which variations in the control signal should be penalized. For instance the high frequent variations are penalized by using the incremental filter as defined by $\boldsymbol{F}$ and $\boldsymbol{g}$ in Eqs. (31) and (32). The steady state gain of the chosen filter should though be equal to zero (in order to eliminate the offset).

5 Simulation Experiments

For the simulation studies the following ARX model is used

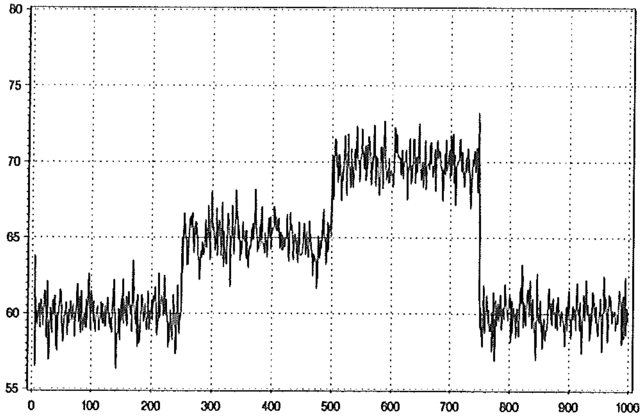
$$y_t = -a_1 y_{t-1} + b_{2,t} u_{t-2} + b_{3,t} u_{t-3} + b_{4,t} u_{t-4} + e_t \tag{33}$$

where a_1 is constant and $b_{l,t}$ $(l = 2, 3, 4)$ are given by

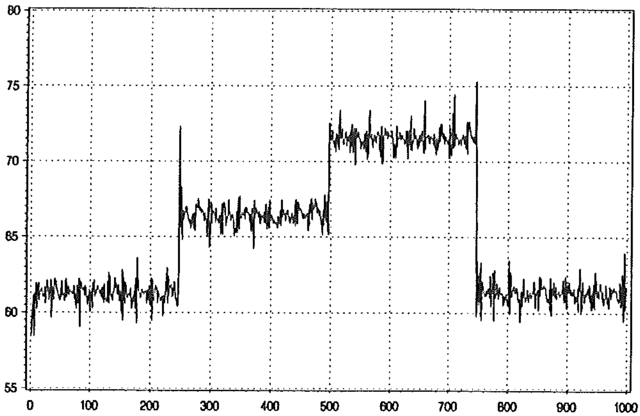
$$\begin{aligned} b_{2,t} &= \beta_{2,0} + \beta_{2,1} \sin(\omega(t-2)) + \beta_{2,2} \cos(\omega(t-2)) \\ b_{3,t} &= \beta_{3,0} + \beta_{3,1} \sin(\omega(t-3)) + \beta_{3,2} \cos(\omega(t-3)) \\ b_{4,t} &= \beta_{4,0} + \beta_{4,1} \sin(\omega(t-4)) + \beta_{4,2} \cos(\omega(t-4)) \end{aligned}$$

with $\omega = 2\pi/24 = \pi/12$. The following values are used: $a_1 = -0.56$, $\beta_{2,0} = 0.35$, $\beta_{2,1} = 0.32$, $\beta_{2,2} = -0.24$, $\beta_{3,0} = 0.18$, $\beta_{3,1} = -0.31$, $\beta_{3,2} = 0.25$, $\beta_{4,0} = 0.18$, $\beta_{4,1} = 0.09$, $\beta_{4,2} = -0.25$, and $\{e_t\} \sim N(0,1)$. The horizons N_1, N_2 and N_u are set to 2, 5 and 2, respectively. $\boldsymbol{\Lambda}_t$ is assumed constant and $\boldsymbol{\Lambda}_t = \lambda \boldsymbol{I}$.

Figures 1 and 2 show the output, the reference output and the control signal for $\lambda = 0.0$ and $\lambda = 0.05$, respectively. The incremental filter (Eqs. (31)

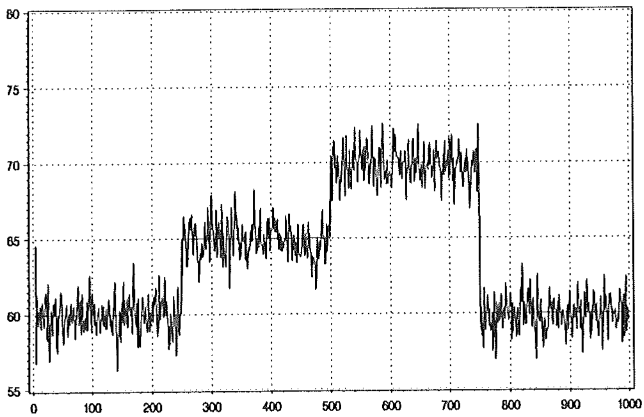


(a)

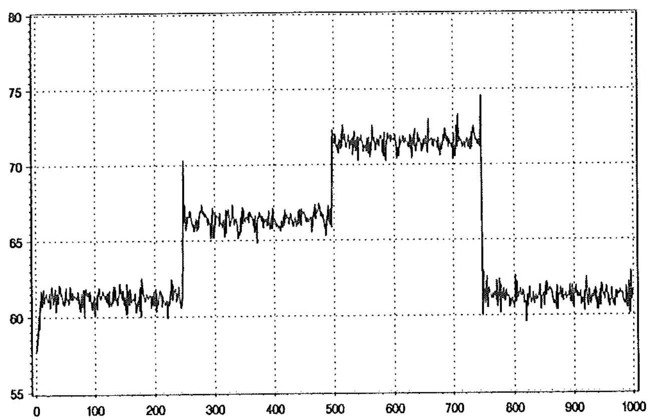


(b)

Fig. 1 **(a) The output** $\{y_t\}$ **and the reference output** $\{y_t^{\mathrm{ref}}\}$. **(b) The control signal** $\{u_t\}$. $\lambda = 0.0$.



(a)



(b)

Fig. 2 (a) The output $\{y_t\}$ **and the reference output** $\{y_t^{\mathrm{ref}}\}$**. (b) The control signal** $\{u_t\}$. $\lambda = 0.05$.

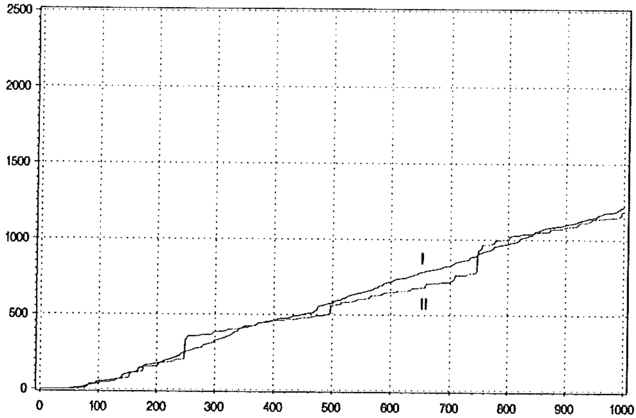


Fig. 3 $\sum_{i=50}^{t}(y_t - y_t^{\text{ref}})^2$ **(I) and** $\sum_{i=50}^{t}(\Delta u_t)^2$ **(II) as a function of the time** t. $\lambda = 0$.

and (32)) is used. The changes in the reference output, y_t^{ref}, are made in steps as indicated in the figures. Changing λ from 0 to 0.05 reduces the amplitude of the control signal noticeably, without any visible change in the output. This is also illustrated in Figs. 3 and 4, which expresses the control performance ($\sum_{i=50}^{t}(y_t - y_t^{\text{ref}})^2$) and the control effort ($\sum_{i=50}^{t}(\Delta u_t)^2$) as a function of the time t. It is seen that for $\lambda = 0.05$ the control effort is much lesser than for $\lambda = 0.0$, without any markable decrease in the control performance.

Figures 5 and 6 show simulations where the knowledge of the explicit time-variations in the model parameters are not used in the prediction Eq. (13), i.e., when computing the impulse response matrix $\boldsymbol{H}_t$ and the vector $\boldsymbol{v}_t$. The same time-varying model with the same parameters is used, but instead of using the knowledge of the explicit time variations, the parameters are assumed constant and equal to their value at time t, i.e., $b_{2,t+j} = b_{2,t}$, $b_{3,t+j} = b_{3,t}$ and $b_{4,t+j} = b_{4,t}$ for $j = N_1, \ldots, N_2$. It is seen that this increases the control effort for $\lambda = 0$, but the control performance is unchanged. Both the control performance and the control effort are nearly the same as before, for $\lambda = 0.05$. This is explained by the fact that the prediction is not as accurate as it is in the previous case (Figs. 1 - 4) and

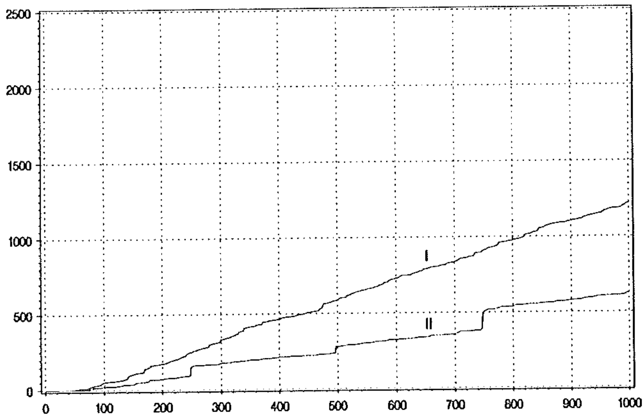


Fig. 4 $\sum_{i=50}^{t}(y_t - y_t^{\text{ref}})^2$ **(I) and** $\sum_{i=50}^{t}(\Delta u_t)^2$ **(II) as a function of the time** t. $\lambda = 0.05$.

therefore it affects the control increments, Δu_t in a negative way. This effect is deleted when the control increments are penalized, i.e., $\lambda > 0$.

The effect of using unfiltered control signal is shown in Fig. 7, i.e., the controller expressed in Eq. (23) is used. If Figs. 4 and 7 are compared it is clearly seen, as expected, that if the control signal is not filtered this may result in an offset control for $\lambda > 0$. Obviously there is no difference when $\lambda = 0$ (compare Eq. (23) and Eq. (30) for $\mathbf{\Lambda} = \mathbf{0}$), and therefore this case is not shown.

6 Conclusion

In this paper the classical GPC is extended to cope with time-varying systems, where the nature of the parameter variations is assumed known. This is done by using the knowledge of the explicit parameter variations when computing the prediction of the future output as a conditional expectation. This is reflected in the resulting time-varying impulse response matrix.

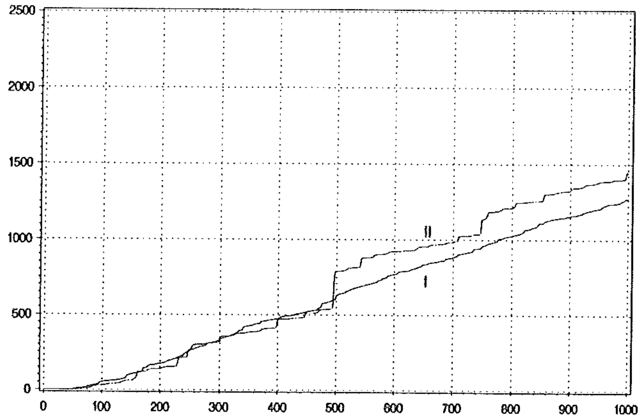


Fig. 5 $\sum_{i=50}^{t}(y_t - y_t^{\mathrm{ref}})^2$ **(I) and** $\sum_{i=50}^{t}(\Delta u_t)^2$ **(II) as a function of the time** t. $\lambda = 0$. **The parameters are assumed constant when computing the impulse response matrix.**

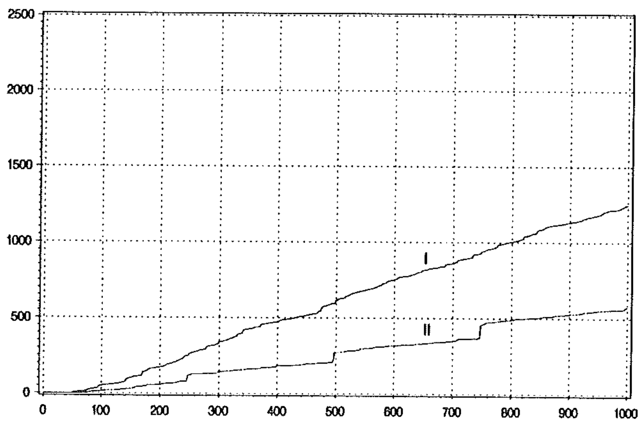


Fig. 6 $\sum_{i=50}^{t}(y_t - y_t^{\mathrm{ref}})^2$ **(I) and** $\sum_{i=50}^{t}(\Delta u_t)^2$ **(II) as a function of the time** t. $\lambda = 0.05$. **The parameters are assumed constant when computing the impulse response matrix.**

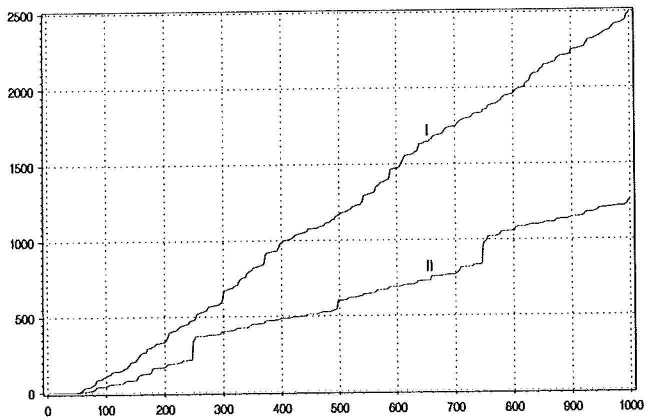


Fig. 7 $\sum_{i=50}^{t}(y_t - y_t^{\text{ref}})^2$ **(I) and** $\sum_{i=50}^{t}(\Delta u_t)^2$ **(II) as a function of the time** t. $\lambda = 0.05$. **Unfiltered control signal.**

The GPC is based on an ARMAX model instead of an ARIMAX model as in almost all GPC references. This is advantageous from a model building point of view if the system is not containing an integration. The offset free control is achieved by filtering the control signal.

A simulation study shows the effect of the above mentioned extensions and modification.

The proposed GPC for time-varying systems keeps all of the advantages of the classical GPC, but the stability and the convergence properties of the proposed GPC must be studied further.

Acknowledgements

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Application of Predictive Control in District Heating Systems

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Abstract

In district heating systems, and in particular if the heat production takes place at a combined heat and power (CHP) plant, a reasonable control strategy is to keep the supply temperature from the district heating plant as low as possible. However, the control is subject to some restrictions, e.g., that the total heat requirement for all consumers is supplied at any time, and each individual consumer is guaranteed some minimum supply temperature at any time. A lower supply temperature implies lower heat loss from the transport and the distribution network, and lower production cost.

A district heating system is an example of a non-stationary system, and the model parameters have to be time-varying. Hence, the classical predictive control theory has to be modified.

Simulation experiments are performed in order to study the performance of modified predictive controllers. The systems are, however, described by transfer function models identified from real data.

Key Words: Generalized Predictive Control (GPC); Time-varying Systems; District Heating; Distribution System Operation and Control.

1 Introduction

It can be shown (Madsen *et al.*, 1992) that considerable savings can be obtained by lowering the supply temperature from the CHP plant. These savings are due to both lower production cost at the plant, i.e., decreasing the supply temperature increases the ratio of power to heat output, as electricity is more valuable than heat, and so a more profitable operation is achieved, see e.g., IEA (1983), and lower heat loss in the transport and the distribution network. On the other hand, the CHP plant has to satisfy the total heat requirement for all consumers at any time, and guarantee each individual consumer some minimum supply temperature at any time.

Some other kind of constraints can be encountered, e.g., on the degree of time-variation of the supply temperature from the plant. Obviously this is some kind of trade off between the interests of the district heating company and its consumers. In this paper it is shown how optimal predictive control can be used to solve this problem.

In order to find an optimal control, a model for the district heating network is needed. If a perfect description of the entire network is available, a pure physical model can be obtained. Unfortunately, for large networks this type of model can be very difficult to establish, and if a model can be established it is most likely to complex for control purposes. Therefore, in this work statistical transfer functions are identified. These transfer functions describe the relations between the supply temperature from the district heating plant and the supply temperature at several representative locations in the network.

A district heating system is an example of a non-stationary system, and therefore time-varying parameters have to be introduced. The time-variations of the parameters are mainly of two kinds: slow annual variations which are, e.g., due to changes of the dynamic characteristics of the distribution network (induced by seasonal climate changes), and faster diurnal variations induced by varying heat consumption. The slow variations can be coped with by using adaptive estimation techniques, while the faster variations are modelled explicitly.

The time-delay in district heating systems is relatively large. Since the time-delay is time-varying and a function of several variables, like distance, flow in the pipes, temperatures and intersection lay-out of the network, the identification of the delay is one of the major problems in modelling district heating systems. Methods for estimating or tracking such a time-varying time-delay are developed and described in details in Søgaard and Madsen (1991) and Madsen *et al.* (1990). Hence, this problem will not be addressed in this paper.

Because of the large time-delays in the district heating systems, a predictive control is of major interest. Several types of predictive or long-range predictive controllers can be found in the literature, see e.g., Peterka (1984) and Clarke (1989), but in this paper only the Generalized Predictive Controller (GPC), first presented in Clarke *et al.* (1987) for stationary systems, will be considered. Due to the non-stationarity in district heating systems, some modifications to the ordinary GPC are required.

The paper starts with a discussion of the optimal control problem at hand. This is followed by the proposed solution of the problem. Then simulation experiments are performed in order to study the performance of the controllers. Finally some conclusions are drawn.

2 The Problem

If u_t denotes the supply temperature from the CHP plant (the control signal), and y_t denotes the supply temperature in the district heating network (the output signal), see Fig. 1, then the optimal control problem can be formulated as:

the objective is to find

$$\min_{u_t} E[g(y_t, u_t, e_t)] \tag{1}$$

subject to

$$y_t = f(u_t, e_t) \tag{2}$$

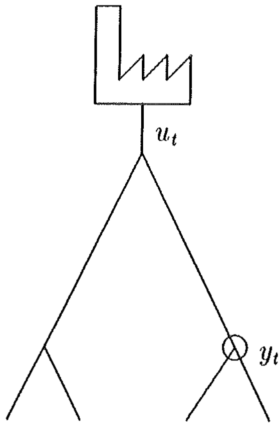

Fig. 1 A sketch of a simple district heating network.

$$P\{y_t \le y_t^{\min}\} = \alpha \tag{3}$$

$$\Delta u_t \qquad \text{small} \tag{4}$$

where E denotes the expected value, $g(\ )$ is an objective function, $f(\ )$ is a transfer function, $P\{\}$ is a probability measure, α is the probability of a too cold supply temperature (this is a kind of service parameter), and Δu_t denotes the change from time to time in the supply temperature from the district heating plant ($\Delta u_t = u_t - u_{t-1}$). Notice that $y_t^{\min}$ is a function of the ambient air temperature as illustrated in Fig. 2.

In the following the system equation Eq. (2) is assumed linear. The constraint in Eq. (3) is called the *Chance Constraint* in the stochastic optimization literature, see e.g., Ermoliev (1988). If the probability $\alpha = 0$ then the constraint is called the *Fat Formulation*. A reference value, y_t^{ref}, can be obtained as the deterministic equivalent for this constraint, assuming that the error in the predictions of the supply temperature in the network, and the ambient air temperature are normal distributed (and uncorrelated). This is illustrated in Fig. 2. The figure shows how the minimum supply temperature curve depends on the ambient air temperature, i.e., the minimum supply temperature is constant for ambient temperatures higher than a cer-

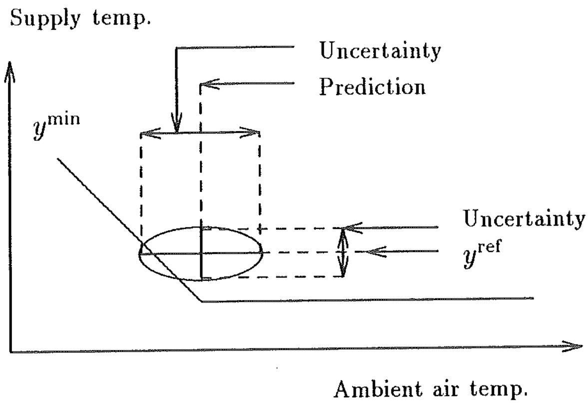

Fig. 2 The reference value determined from the prediction uncertainties. Probability mass over the minimum curve is $100(1-\alpha)$ per cent.

tain value and for lower ambient air temperature it is increasing linearly with falling ambient air temperature.

The soft constraint in Eq. (4) can be made more strict, i.e., $|\Delta u_t| \leq \Delta u_{\text{max}}$. This will be discussed later on.

In order to make use of the predictive control technique the object of the optimal control is written on the form

$$\min_{u_t} E[(y_{t+k} - y^{\text{ref}}_{t+k|t})^2 + \lambda(\Delta u_t)^2] \tag{5}$$

i.e., the controller keeps the output close to the predetermined reference output and keeps the change in the control signal small. The cost parameter λ can be adjusted to put more or less penalty on changes in the control signal.

It can be convenient to optimize the cost function in Eq. (5) over several time periods, this results in the GPC cost function

$$J = E\left[\sum_{j=N_1}^{N_2}(y_{t+j} - y^{\text{ref}}_{t+j|t})^2 + \sum_{j=1}^{N_u}\lambda_{j,t}(\Delta u_{t+j-1})^2\right] \tag{6}$$

where N_1 and N_2 are the minimum and the maximum costing horizon, respectively, $\lambda_{j,t}$ is a penalty sequence, N_u is the control horizon, and $y^{\text{ref}}_{t+j|t}$ is a reference temperature at time $t+j$.

The expectation, E, is conditioned on observations available at time t. Guidelines for determining the design parameters, N_1, N_2 and N_u, can be found in Clarke *et al.* (1987).

3 The Solution

First the model is introduced. Then it is shown how the predictions needed in the controller are obtained. This results in a modified GPC controller.

3.1 Transfer Function

The relation between the supply temperature from the CHP plant, u_t, and the supply temperature in the network, y_t, can be described by a time-varying ARMAX (Auto-Regressive-Moving-Average-eXtraneous) model

$$A_t(q^{-1})y_t = B_t(q^{-1})u_t + C_t(q^{-1})e_t \tag{7}$$

where $\{e_t\}$ is white noise with mean zero and variance σ_e^2. A_t, B_t and C_t are polynomials in q^{-1} (the back shift operator) with time-varying coefficients:

$$\begin{aligned} A_t(q^{-1}) &= 1 + a_{1,t}q^{-1} + \cdots + a_{n_A,t}q^{-n_A} \\ B_t(q^{-1}) &= b_{1,t}q^{-1} + \cdots + b_{n_B,t}q^{-n_B} \\ C_t(q^{-1}) &= 1 + c_{1,t}q^{-1} + \cdots + c_{n_C,t}q^{-n_C} \end{aligned} \tag{8}$$

The parameter variations are periodical

$$
\begin{aligned}
a_{j,t} &= \alpha_{j,0} + \alpha_{j,1}\sin(\omega(t-j)) + \alpha_{j,2}\cos(\omega(t-j)) \\
b_{j,t} &= \beta_{j,0} + \beta_{j,1}\sin(\omega(t-j)) + \beta_{j,2}\cos(\omega(t-j)) \\
c_{j,t} &= \gamma_{j,0} + \gamma_{j,1}\sin(\omega(t-j)) + \gamma_{j,2}\cos(\omega(t-j))
\end{aligned}
$$

where $\omega = 2\pi/24 = \pi/12$, i.e., the diurnal variation for sampling interval equal to 1 hour. If the parameters α, β and γ are time-varying as well, then they can be estimated adaptively using recursive methods, see e.g., Ljung (1987).

4 Output Prediction

The GPC is based on the assumption that the output predictions can be expressed as a linear combination of present and future controls. In Clarke *et al.* (1987) and many other references this is obtained by solving the Diophantine Equation (sometimes recursively). However, in the time-varying case the Diophantine Equation *cannot* be used due to the time-variation of the model parameters. Instead the j-step predictor, $\hat{y}_{t+j|t}$, is found as the conditional expectation of y_{t+j} conditioned on observations of the output up to time t (see e.g., Madsen (1989))

$$
\begin{aligned}
\hat{y}_{t+j|t} &= -\sum_{i=1}^{n_A} a_{i,t+j}\hat{y}_{t+j-i|t} + \sum_{i=1}^{n_B} b_{i,t+j}u_{t+j-i} \\
&\quad + \sum_{i=0}^{n_C} c_{i,t+j}\hat{e}_{t+j-i|t}, \qquad j \geq 1 \qquad (9) \\
\hat{y}_{t+j|t} &= y_{t+j}, \qquad j < 1 \qquad (10)
\end{aligned}
$$

where

$$
\hat{e}_{t+l|t} = \begin{cases} 0, & \text{if } l \geq 1 \\ e_{t+l}, & \text{if } l < 1 \end{cases} \qquad (11)
$$

A simple example illustrates the method.

Example Consider the ARX(1,2) model ($n_A = 1$, $n_B = 2$)

$$y_t + a_{1,t} y_{t-1} = b_{1,t} u_{t-1} + b_{2,t} u_{t-2} + e_t \tag{12}$$

The 1-step predictor is derived from Eqs. (9) – (11) as

$$\hat{y}_{t+1|t} = -a_{1,t+1} y_t + b_{1,t+1} u_t + b_{2,t+1} u_{t-1} = h_{1,t+1} u_t + v_{1,t} \tag{13}$$

where $v_{1,t} = -a_{1,t+1} y_t + b_{2,t+1} u_{t-1}$ is known and $(h_{1,t+1}) u_t$ is unknown before the control signal is chosen at time t. Note, that $h_{1,t+1} = b_{1,t+1}$ is the first weight of the time-varying impulse response function.

In general, the j-step predictor is given as

$$\begin{aligned} \hat{y}_{t+j|t} &= -a_{1,t+j} \hat{y}_{t+j-1|t} + b_{1,t+j} u_{t+j-1} + b_{2,t+j} u_{t+j-2} \\ &= -a_{1,t+j} (h_{j-1,t+j-1} u_t + \cdots + h_{1,t+j-1} u_{t+j-2} + v_{j-1,t}) \\ &\quad + b_{1,t+j} u_{t+j-1} + b_{2,t+j} u_{t+j-2} \\ &= h_{j,t+j} u_t + \cdots + h_{1,t+j} u_{t+j-1} + v_{j,t} \\ &= \sum_{i=1}^{j} h_{i,t+j} u_{t+j-i} + v_{j,t} \end{aligned} \tag{14}$$

For the given model the $h_{i,t}$ and $v_{i,t}$ values can be computed recursively as

$$h_{i,t} = \begin{cases} b_{i,t}, & \text{if } i = 1 \\ b_{i,t} - a_{1,t} h_{i-1,t-1}, & \text{if } i = 2 \\ -a_{1,t} h_{i-1,t-1}, & \text{if } i \geq 3 \end{cases} \tag{15}$$

and

$$v_{i,t} = \begin{cases} -a_{1,t+i} y_t + b_{2,t+i} u_{t-1}, & \text{if } i = 1 \\ -a_{1,t+i} v_{i-1,t}, & \text{if } i \geq 2 \end{cases} \tag{16}$$

□

The coefficients $h_{i,t+j}$ $(i = 1, 2, \ldots)$ are the weights of the time-varying impulse response function describing the dynamic relation between the input and the output, i.e., $h_{i,t}$ is the marginal change of y_t changing u_{t-i}.

An alternative way to find the $h_{i,t+j}$ and $v_{j,t}$ values is as follows:

1. To find $v_{j,t}$, set the present and future control to zero ($u_{t+j-i} = 0$ for $i = 1, 2, \ldots, j$). Then compute $v_{j,t}$ as the conditional expectation, $\hat{y}_{t+j|t}$, using Eq. (9).
2. To find $h_{i,t+j}, (i = 1, 2, \ldots, j)$, set the present and past output and past control to zero ($v_{j,t} = 0$). Feed an impulse into the system at time $t + j - i$

$$u_{t+j-l} = \begin{cases} 1, & \text{if } l = i \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Then compute $h_{i,t+j}$ as the conditional expectation, $\hat{y}_{t+j|t}$, using Eq. (9).

Considering Eq. (14) it is seen that the j-step predictions, j running from N_1 up to N_2, can be written as a linear matrix expression:

$$\hat{\boldsymbol{y}}_t = \boldsymbol{H}_t \boldsymbol{u}_t + \boldsymbol{v}_t \tag{18}$$

where

$$\begin{aligned} \hat{\boldsymbol{y}}_t &= [\hat{y}_{t+N_1|t}, \ldots, \hat{y}_{t+N_2|t}]^T \\ \boldsymbol{u}_t &= [u_t, \ldots, u_{t+N_u-1}]^T \\ \boldsymbol{v}_t &= [v_{N_1,t}, \ldots, v_{N_2,t}]^T \end{aligned}$$

$$\boldsymbol{H}_t = \begin{bmatrix} h_{1,t+N_1} & 0 & \cdots & 0 \\ h_{2,t+N_1+1} & h_{1,t+N_1+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_u,t+N_1+N_u-1} & h_{N_u-1,t+N_1+N_u-1} & \cdots & h_{1,t+N_1+N_u-1} \\ h_{N_u+1,t+N_1+N_u} & h_{N_u,t+N_1+N_u} & \cdots & h_{2,t+N_1+N_u} + h_{1,t+N_1+N_u} \\ \vdots & \vdots & & \vdots \\ h_{N_2-N_1+1,t+N_2} & h_{N_2-N_1,t+N_2} & \cdots & \sum_{i=1}^{N_2-N_1-N_u+2} h_{i,t+N_2} \end{bmatrix}$$

assuming that $N_2 > N_u + N_1$. The summation in the last column of the time-varying impulse response matrix, $\boldsymbol{H}_t$, results from GPC constraint $\Delta u_{t+i-1} = 0$, $i > N_u$ (Bjerre, 1992).

Remark In Eq. (18) it has been assumed that the same model is applied for all prediction horizons. Actually, this need not to be the case. If different models are used, then the j'th row of $\boldsymbol{H}_t$ and the j'th element of $\boldsymbol{v}_t$ belong to a special model designed for j-step prediction. Making use of an individual model for each horizon is often relevant if a non-linear system is approximated by a family of linear models (e.g. threshold models).

4.1 Controller

Introducing matrix notation the cost function Eq. (6) is written as

$$J = E[(\boldsymbol{y}_t - \boldsymbol{y}_t^{\mathrm{ref}})^T(\boldsymbol{y}_t - \boldsymbol{y}_t^{\mathrm{ref}}) + \Delta\boldsymbol{u}_t^T\Lambda_t\Delta\boldsymbol{u}_t] \tag{19}$$

where

$$\begin{aligned}
\boldsymbol{y}_t &= [y_{t+N_1}, \ldots, y_{t+N_2}]^T \\
\boldsymbol{y}_t^{\mathrm{ref}} &= [y_{t+N_1|t}^{\mathrm{ref}}, \ldots, y_{t+N_2|t}^{\mathrm{ref}}]^T \\
\Delta\boldsymbol{u}_t &= [\Delta u_t, \ldots, \Delta u_{t+N_u-1}]^T \\
\Lambda_t &= \begin{bmatrix} \lambda_{1,t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{N_u,t} \end{bmatrix}
\end{aligned}$$

By the projection theorem, (Brockwell and Davis, 1987), the output vector, $\boldsymbol{y}_t$, is decomposed into two (stochastic) independent parts

$$\boldsymbol{y}_t = \hat{\boldsymbol{y}}_t + \tilde{\boldsymbol{y}}_t \tag{20}$$

where $\hat{\boldsymbol{y}}_t$ is a vector containing the output predictions described in the previous section, and $\tilde{\boldsymbol{y}}_t$ is a vector containing the prediction errors

$$\tilde{\boldsymbol{y}}_t = [\tilde{y}_{t+N_1|t}, \ldots, \tilde{y}_{t+N_2|t}]^T \tag{21}$$

The cost function can then be written as

$$J = [(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t^{\text{ref}})^T(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t^{\text{ref}}) + \Delta\boldsymbol{u}_t^T\boldsymbol{\Lambda}_t\Delta\boldsymbol{u}_t] + E[\tilde{\boldsymbol{y}}_t^T\tilde{\boldsymbol{y}}_t] \tag{22}$$

Note that the last term in this expression does not depend on $\boldsymbol{u}_t$.

$\Delta\boldsymbol{u}_t$ is written as

$$\Delta\boldsymbol{u}_t = \boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t \tag{23}$$

where

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_t = [-u_{t-1}, 0, \cdots, 0]^T$$

Using Eqs. (18) and (23) the cost function becomes

$$\begin{aligned} J = & [(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}})^T(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) \\ & +(\boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t)^T\boldsymbol{\Lambda}_t(\boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t)] + E[\tilde{\boldsymbol{y}}_t^T\tilde{\boldsymbol{y}}_t] \end{aligned} \tag{24}$$

The GPC control law is then obtained by minimizing the resulting cost function, subject to the constraint

$$\Delta u_{t+i-1} = 0, \qquad i > N_u \tag{25}$$

This is done by setting the derivative of the cost function w.r.t. $\boldsymbol{u}_t$ to zero, i.e.

$$\frac{\partial J}{\partial \boldsymbol{u}_t} = 0 \tag{26}$$

In other words,

$$2\boldsymbol{H}_t^T(\boldsymbol{H}_t\boldsymbol{u}_t + \boldsymbol{v}_t - \boldsymbol{y}_t^{\text{ref}}) + 2\boldsymbol{F}^T\boldsymbol{\Lambda}_t(\boldsymbol{F}\boldsymbol{u}_t + \boldsymbol{g}_t) = 0$$

or

$$[H_t^T H_t + F^T \Lambda_t F] u_t + [H_t^T (v_t - y_t^{\text{ref}}) + F^T \Lambda_t g_t] = 0$$

Solving for u_t, results in

$$u_t = -[H_t^T H_t + F^T \Lambda_t F]^{-1} [H_t^T (v_t - y_t^{\text{ref}}) + F^T \Lambda_t g_t] \quad (27)$$

Only the first element of the control vector, u_t, is implemented so the control law is written as

$$u_t = [1, 0, \cdots, 0] u_t \quad (28)$$

5 Simulation Results

5.1 Preliminaries

Model and Parameters The performance of the controller is studied by simulations. Results for various values of the control parameters are compared.

The model used is the time-varying ARMAX model shown in Eqs. (7) and (8), where $n_A = 1$, $n_B = 4$, $n_C = 0$ and $k = 2$ (i.e., $b_{2,t}$ is the first nonzero parameter in $B_t(q^{-1})$). The time-variations are solely in the b parameters, i.e., a_1 is constant and $b_{l,t}$ are given by

$$\begin{aligned} b_{2,t} &= \beta_{2,0} + \beta_{2,1} \sin(\omega(t-2)) + \beta_{2,2} \cos(\omega(t-2)) \\ b_{3,t} &= \beta_{3,0} + \beta_{3,1} \sin(\omega(t-3)) + \beta_{3,2} \cos(\omega(t-3)) \\ b_{4,t} &= \beta_{4,0} + \beta_{4,1} \sin(\omega(t-4)) + \beta_{4,2} \cos(\omega(t-4)) \end{aligned} \quad (29)$$

with

$$\omega = \frac{2\pi}{24} = \frac{\pi}{12} \quad (30)$$

i.e., a diurnal variation of the parameters is expressed provided that the sampling interval is one hour.

For the parameters, a_1 and $\beta_{*,*}$, in Eq. (29), following estimates are used (obtained from data from the district heating system in Esbjerg, Denmark): $a_1 = -0.56$, $\beta_{2,0} = 0.35$, $\beta_{2,1} = 0.32$, $\beta_{2,2} = -0.24$, $\beta_{3,0} = 0.18$, $\beta_{3,1} = -0.31$, $\beta_{3,2} = 0.25$, $\beta_{4,0} = 0.18$, $\beta_{4,1} = 0.09$, $\beta_{4,2} = -0.25$ and $\sigma_e^2 = 1.0$.

Remark The above ARMAX model can, in some sense, be interpreted by physical terms. The $A(q^{-1})$ polynomial represents the dynamic behavior of the system, i.e., the time constant, which is induced by the heat capacity of the district heating water, the pipes and the surrounding media. The expression $(1 - B(1)/A(1))$ gives the relative heat loss between two points in the network, and the time-delay, k, is mainly due to the time that it takes to transport the water from one point to another.

This explains why a_1 is constant and $b_{l,t}$ are time-varying, viz., the heat capacity is constant, the mass is always the same, but the relative heat loss is time-varying, due to varying ambient temperature and varying heat consumption. Of course, the time delay is also time varying, see Søgaard and Madsen (1991), but in this simulation studies it is assumed constant.

Costing Horizons As the time-delay, k, is assumed known and constant, the minimum costing horizon, see Eq. (6), is set to $N_1 = k = 2$. The maximum costing horizon shall at least exceed the degree of $B(q^{-1})$, in this case 4, but Clarke *et al.* (1987) suggested a rather larger value of N_2, hence it is set to $N_2 = 10$.

Penalty matrix $\boldsymbol{\Lambda}_t = \lambda \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix.

Reference Temperature It is assumed that the ambient air temperature is a great deal right of the knee in the minimum curve shown in Fig. 2, hence the influence of the uncertainty of the prediction of the ambient air temperature is eliminated. (The knee is typically located at (10 °C, 70 °C).)

The reference temperature is determined in order to fulfill the constraint in Eq. (3), i.e.

$$P\{y_{t+j} \leq y^{\min}\} = \alpha, \qquad j = N_1, \ldots, N_2 \tag{31}$$

where α is a service parameter. Since, y_{t+j} is a sum of a prediction and a prediction error, $y_{t+j} = \hat{y}_{t+j|t} + \tilde{y}_{t+j|t}$, $y^{\text{ref}}_{t+j|t}$ is found by inserting this into Eq. (31) and then setting $\hat{y}_{t+j|t} = y^{\text{ref}}_{t+j|t}$

$$P\{y^{\text{ref}}_{t+j|t} + \tilde{y}_{t+j|t} \leq y^{\min}\} = \alpha, \qquad j = N_1, \ldots, N_2 \tag{32}$$

Assuming that $\tilde{y}_{t+j|t}$ is normal distributed white noise and that the predictions are computed as conditional expectations as described before, it is found that

$$y^{\text{ref}}_{t+j|t} = y^{\min} + \pi_{1-\alpha}\sigma_j, \qquad j = N_1, \ldots, N_2 \tag{33}$$

where $\pi_{1-\alpha}$ is the $100(1-\alpha)$ per cent quantile in the standardized normal distribution and σ_j^2 is the variance of the j-step prediction error. In a stochastic sense the model used is an AR(1) model, and in Madsen (1989) it is verified that

$$\sigma_j^2 = \sigma_e^2 \sum_{i=0}^{j-1} a_1^{2i}, \qquad j = N_1, \ldots, N_2 \tag{34}$$

$\alpha = 0.01$ is used in the main study, but the influence of various values of α was also examined. The minimum supply temperature is set to $y^{\min} = 70$ °C.

Number of Time Steps The simulations are performed over 1000 time steps (hours), but when computing variances, means, maximum, minimum and percentage the first and last 50 time steps are dismissed.

5.2 Results

The purpose of the simulation study is threefold: to investigate the effect of different control horizons N_u, different penalty parameters λ, and different service parameters α.

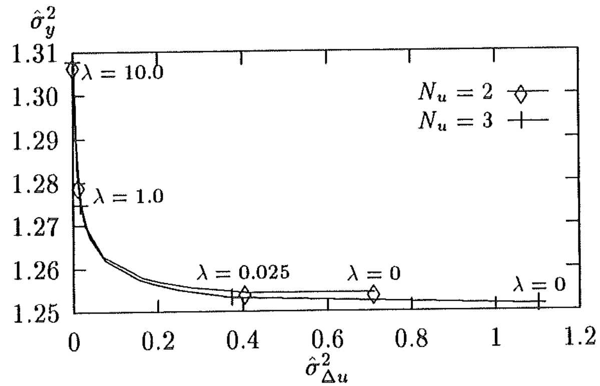

Fig. 3 Variance of the output signal, $\hat{\sigma}^2_y$, vs. variance of the change in the control signal, $\hat{\sigma}^2_{\Delta u}$, for N_u equal to 2 and 3 and for various values of λ. $\alpha = 0.01$.

N_u and λ ($\alpha = 0.01$) In Fig. 3 the relations between the control change variance, $\hat{\sigma}^2_{\Delta u}$, and the output variance, $\hat{\sigma}^2_y$, are shown for various values of the penalty parameter, λ, for $N_u = 2$ and 3 respectively. For $\lambda = 0$, $\hat{\sigma}^2_y$ is at minimum, for both $N_u = 2$ and 3. As expected, $\hat{\sigma}^2_{\Delta u}$ decreases and $\hat{\sigma}^2_y$ increases as λ is increased. However, for small values of λ, $\hat{\sigma}^2_y$ is not affected very much while $\hat{\sigma}^2_{\Delta u}$ decreases drastically. For $\lambda \to \infty$ the price of the control effort will be so high that there will be no control at all.

The difference between $N_u = 2$ and $N_u = 3$ is clearly seen for $\lambda = 0$, but the discrepancy vanished for higher values of λ. The results for $N_u = 4$ are very similar to those for $N_u = 3$, hence, they are not included in the figure. For $N_u = 1$ the results would lay concentrated in the upper left corner of the figure. For $\lambda = 0$ and $N_u = 1$ the results are nearly the same as for $\lambda = 10$ and $N_u = 2$ or 3, i.e., for $N_u = 1$ the controller is extremely passive, $\hat{\sigma}^2_{\Delta u}$ is low. This is due to the constraint of the GPC controller expressed in Eq. (25), i.e., the future control values are all equal to the present one, u_t. The control performance, $\hat{\sigma}^2_y$, for $N_u = 1$ is also very poor.

Table 1 The percentage of the time below the minimum curve and minimum and maximum changes in the control signal for various values of N_u and λ.

N_u	λ	Percentage	Δu_{min}	Δu_{max}
1	0.0	1.1	-0.25	0.24
	0.025	1.1	-0.24	0.24
	1.0	1.1	-0.22	0.21
	10.0	1.2	-0.12	0.11
2	0.0	1.1	-3.16	3.72
	0.025	1.2	-1.99	2.50
	1.0	1.3	-0.36	0.34
	10.0	1.1	-0.11	0.10
3	0.0	1.0	-4.64	4.59
	0.025	1.2	-1.91	2.25
	1.0	1.3	-0.42	0.37
	10.0	1.1	-0.11	0.10
4	0.0	1.0	-4.67	4.74
	0.025	1.1	-2.00	2.33
	1.0	1.3	-0.43	0.38
	10.0	1.1	-0.12	0.10

Table 1 shows how various values of N_u and λ influence the percentage of the time below the minimum curve, and the minimum, Δu_{min}, and the maximum change in the control signal, Δu_{max}. It is seen that the percentage compares nicely with the value of the service parameter, α, which in this case is set to 0.01. This compares in fact exactly for $N_u = 3$ and 4 with $\lambda = 0.0$. As expected, the percentage increases for increasing values of λ, though these increases are not very high.

In district heating plants, how fast the supply temperature from the plant

Table 2 The percentage of the time below the minimum curve, the mean of the output, $\bar{y}$, and the input, $\bar{u}$, for different α. $N_u = 3$ and $\lambda = 0.0$.

α	Percentage	$\bar{y}$	$\bar{u}$
0.0005	0.0	73.73	75.39
0.001	0.1	73.50	75.15
0.005	0.7	72.90	74.54
0.01	1.0	72.61	74.24
0.02	2.2	72.29	73.92
0.03	3.1	72.09	73.71
0.04	3.8	71.94	73.56
0.05	4.4	71.82	73.43

can be changed is limited. For the district heating system in Esbjerg these limits are $\Delta u_{\text{max}} = 5\,^\circ$C per hour (when rising the temperature) and $\Delta u_{\text{min}} = -2\,^\circ$C per hour (when lowering the temperature). Table 1 shows that these constraints are exceeded for $N_u = 2$, 3 and 4 and $\lambda = 0.0$. However, this can be coped with by tuning the penalty parameter λ.

α ($N_u = 3$ and $\lambda = 0.0$) The only effect that the service parameter α has on the results, is that the mean of the output and the input are shifted. Hence the percentage of the time below the minimum curve is changed.

Table 2 contains these results for different α values. It is seen that again the percentage compares favorably with the α values. As expected the $\bar{y}$ and $\bar{u}$ increase for decreasing values of α. The interesting point is to see how much can be gained by choosing a reasonable α. As mentioned in the introduction the saving potentials lie in lowering the supply temperature from the plant. It can for example be seen that changing α from 0.01 to 0.03 results in $0.5\,^\circ$C lower supply temperature from the plant, on average.

6 Conclusion

The following conclusions can be drawn based on the simulation results presented:

- The proposed modified GPC controller is well suited for controlling the supply temperature in district heating systems.
- By increasing the control horizon, N_u, the control becomes more active for low values of λ.
- It turns out that the difference between various values of N_u vanishes for higher λ.
- A controller with $N_u = 1$ is not interesting.
- Violations of the Δu constraint can be coped with by tuning the penalty parameter λ.
- The "best" controller is obtained with $N_u = 2$, 3, or 4 and for small non-zero λ values.
- Higher savings can be achieved by increasing the service parameter α, but this can result in more frequent complains from the consumers.

Implementation of a predictive control (however, not exactly like the one presented here) in the district heating system in Esbjerg, Denmark, has implied considerable energy savings (Madsen *et al.*, 1992).

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Notation

a, b, c	model parameters
A, B, C	polynomials
e	white noise
E	expected value
$f()$	transfer function
$\boldsymbol{F}$	$N_u \times N_u$ dimensional filter matrix
$g()$	objective function
$\boldsymbol{g}$	N_u dimensional vector
h	impulse response weight
$\boldsymbol{H}$	$(N_2 - N_1 + 1) \times N_u$ dimensional matrix of h
$\boldsymbol{I}$	identity matrix
J	cost function
n_A, n_B, n_C	order of the polynomial A, B, and C
N_1	minimum costing horizon
N_2	maximum costing horizon
N_u	control horizon
$P\{\}$	probability measure
q^{-1}	back shift operator
u	supply temperature from the CHP plant (control signal)
$\boldsymbol{u}$	N_u dimensional vector of u
v	known part of the prediction
$\boldsymbol{v}$	$(N_2 - N_1 + 1)$ dimensional vector of v
y	supply temperature in the district heating network (output signa
$\boldsymbol{y}$	$(N_2 - N_1 + 1)$ dimensional vector of y

Greek Letter

α	service parameter (probability of too cold supply temperature)
$\alpha_{\cdot}$, $\beta_{\cdot}$, $\gamma_{\cdot}$	model parameters
Δ	change from time to time (def. $\Delta u_t = u_t - u_{t-1}$)
λ	penalty parameter
Λ	$(N_u \times N_u)$ penalty matrix
π	quantile in the standardized normal distribution

σ^2	variance
ω	frequency

Superscripts and Subscripts

i	index
j	prediction horizon
k	time-delay
max	maximum value
min	minimum value
ref	reference value
t	time
$t+j\|t$	(prediction) at time $t+j$ given the information up to time t
T	transposed
$\hat{}$	prediction
$\tilde{}$	prediction error
$\bar{}$	mean value

Abbreviations

ARMAX	Auto-Regressive-Moving-Average-eXtraneous
CHP	Combined Heat and Power
GPC	Generalized Predictive Control

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On Flow and Supply Temperature Control in District Heating Systems

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Abstract

This paper discusses how the control of the flow and the supply temperature in district heating systems can be optimized, utilizing stochastic modeling, prediction and control methods. The main objective is to reduce heat production costs and heat losses in the transmission and distribution net by minimizing the supply temperature at the district heating plant. This control strategy is reasonable, in particular, if the heat production takes place at a combined heat and power (CHP) plant. The control strategy is subject to some restrictions, e.g., that the total heat requirement for all consumers is supplied at any time, and each individual consumer is guaranteed some minimum supply temperature at any time. Another important restriction is that the variation in time of the supply temperature is kept as small as possible.

This concept has been incorporated in the program package PRESS developed at the Technical University of Denmark. PRESS has been applied and tested, e.g., at Vestkraft in Esbjerg, Denmark, and significant saving potentials have been documented. PRESS is now distributed by the Danish District Heating Association.

Key Words: Heat Load Prediction; Supply Temperature; Predictive Control; District Heating.

1 Introduction – The Basic Ideas

In district heating systems, where the heat is produced at CHP plants, the simplest and most widely used is the central temperature control method, see Oliker (1980). In this case, the required amount of heat is provided to the consumer by varying the supply temperature at the power plant as a function of the current ambient air temperature. Such functions (control curves) can be found, e.g., in Madsen *et al.* (1992), Horlock (1987) and Oliker (1980). This means that the flow rate of water circulated in the district heating system is almost constant during the heating period, see e.g., Oliker (1980). This control strategy must necessarily be very cautious, i.e., frequently the supply temperature can be higher than it has to, since the ambient air temperature is the only meteorological factor considered, and the control curves do not take into account the dynamic characteristic of the district heating system and the diurnal variation of the heat consumption.

The main saving potential in district heating systems, supplied by CHP, lies in lowering the supply temperature from the plant, hence lowering the heat losses in the transmission and the distribution network and lowering the production costs. The savings in the production costs are due to the fact that decreasing the supply temperature implies an increase in the ratio of the power to heat output, and as electricity is more valuable than heat a more profitable operation is achieved, see e.g., IEA (1983).

Thus, a reasonable control strategy is to keep the supply temperature from the district heating plant as low as possible, but subject to the following restrictions:

1. The total heat requirement for all consumers is supplied at any time.
2. Each individual consumer is guaranteed some minimum supply temperature at any time. This supply temperature depends on the ambient air temperature.
3. The variation in time of the supply temperature from the district heating plant is kept reasonably small.

The main idea in the proposed control strategy is illustrated schematically in Fig. 1. The first restriction is coped with by using models for prediction of the heat load. An extensive research on predicting heat load has been done, and the main results are found in Sejling (1993), Madsen *et al.* (1992) and Madsen *et al.* (1990). By observing the return temperature of the water and assuming maximum water circulation, the lowest admissible supply temperature in the next time step (equal to one hour in Esbjerg, other time steps can equally well be applied) is computed from energy balance equations. The uncertainty of the prediction is taken into account in this calculation. A more advanced controller has been proposed in Søgaard (1993) and Madsen *et al.* (1992). In Fig. 1 this flow sub-controller is denoted as FSC.

The second restriction is coped with by finding several representative (or critical) points (locations) in the distribution networks, defined as follows: *If the temperature requirements at these points are met, then the temperature requirement are met for all consumers.* For each of the representative points a predictive type sub-controller (SC# in Fig. 1) is developed in order to compute the lowest supply temperature from the plant ensuring that the temperature at that particular point is minimized, but above an ambient air temperature dependent minimum. In the computation the time-delay of temperatures between the plant and the point is taken into account. The sub-controller also satisfies the third restriction which is due to the fact that rapidly changing the supply temperature from the plant is not desirable as it implies wearing of the pipelines and causes undesirable operation of the plant.

The overall controller (OC in Fig. 1) selects the highest supply temperature among required supply temperatures calculated from predictions of the heat load and from the above-mentioned sub-controllers for the representative points in the distribution network. This temperature is used as the supply temperature from the plant in the following hour, see Fig. 1.

The remainder of this paper is organized as follows: In Section 2 the models for prediction of the heat load are briefly described and it is discussed how they are used for minimizing the supply temperature, hence leading to the flow sub-controller. Section 3 is devoted to the supply temperature sub-

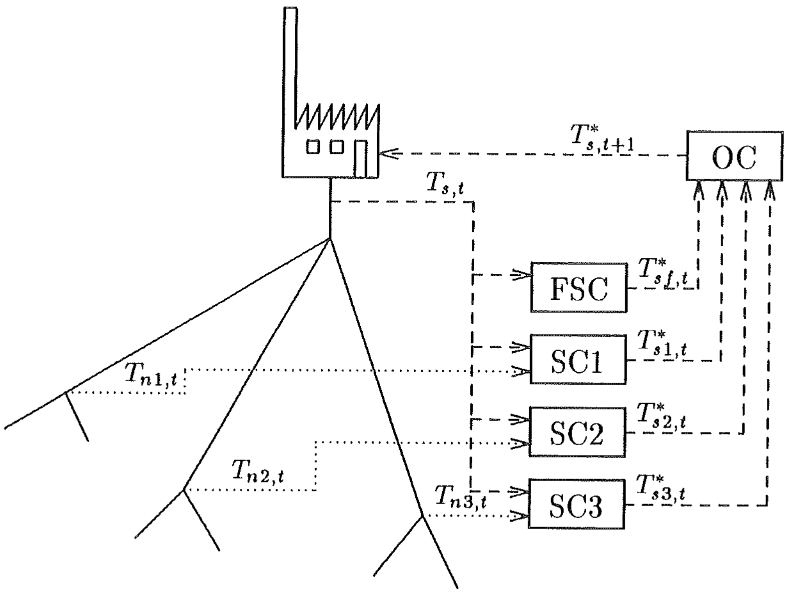

Fig. 1 A sketch of a district heating network, with 3 representative points and the controllers. OC is the overall controller, FSC is the flow sub-controller, SC# are the supply temperature sub-controllers, $T_{n\#,t}$ are the supply temperatures in the network, $T_{s,t}$ is the supply temperature from the plant, and $T^*_{s\#,t}$ are the supply temperatures required by the sub-controllers.

controller. Section 4 describes the overall controller, and Section 5 gives some practical experience. Finally, some conclusions are drawn in Section 6.

2 The Flow Sub-Controller

2.1 Prediction of the Heat Load

Forecasting the future heat load, using stochastic methods, has been the subject in Sejling (1993), Madsen *et al.* (1992), Madsen *et al.* (1990), and Sejling (1987). Forecasting of heat load has also been studied in Benonysson (1991), Wiklund (1991), and in Waldemark *et al.* (1992) where neural networks are applied.

The potential external variables in a model describing the heat load are, e.g., the ambient air temperature, the wind velocity (and direction), humidity/precipitation, the temperature of the supply water, the temperature of the return water, the water flow, the time of day and type of day (weekend and workdays).

Various types of models have been considered, e.g., general transfer function models, non-linear transfer function models, smooth threshold transfer function models, and neural network models. The first class contains linear models whereas the rest are classes of non-linear models.

The models found best suited for control purpose are the general transfer function models, see Sejling (1993) or Madsen *et al.* (1990). These models give rise to the one-step predictor

$$\begin{aligned}\hat{p}_{t+1|t} = \\ & \alpha_1 p_t + \alpha_2 p_{t-1} + \alpha_{24} p_{t-23} + \alpha_{25} p_{t-24} + \alpha_{26} p_{t-25} \\ & + \beta_{1,0} \Delta T_{s,t+1} + \beta_{1,1} \Delta T_{s,t} \\ & + \beta_{2,1} T_{a,t} + \beta_{2,2} T_{a,t-1} \\ & + \beta_{3,1} w_t + \beta_{3,2} w_{t-1}\end{aligned}$$

$$+\mu_{1,t+1}I_{\{\text{workday}\},t+1} + \mu_{2,t+1}(1 - I_{\{\text{workday}\},t+1}) + l \qquad (1)$$

The predictors proposed for prediction horizons, j, between 2 and 23 hour are

$$\begin{aligned}\hat{p}_{t+j|t} = {} & \alpha_j p_t + \alpha_{j+1} p_{t-1} \\ & +\alpha_{24} p_{t+j-24} + \alpha_{25} p_{t+j-25} + \alpha_{26} p_{t+j-26} \\ & +\beta_{1,0}\Delta T_{s,t+j} + \beta_{1,1}\Delta T_{s,t+j-1} \\ & +\beta_{1,2}\Delta T_{s,t+j-2} + \beta_{1,3}\Delta T_{s,t+j-3} \\ & +\beta_{2,j} T_{a,t} + \beta_{2,j+1} T_{a,t-1} + \beta_{2,j+2} T_{a,t-2} + \beta_{2,j+3} T_{a,t-3} \\ & +\beta_{3,j} w_t + \beta_{3,j+1} w_{t-1} + \beta_{3,j+2} w_{t-2} + \beta_{3,j+3} w_{t-3} \\ & +\mu_{1,t+j} I_{\{\text{workday}\},t+j} + \mu_{2,t+j}(1 - I_{\{\text{workday}\},t+j}) + l \qquad (2)\end{aligned}$$

Note that an individual model is set up for each prediction horizon. The parameters are re-estimated each hour.

2.2 The Flow Sub-Controller

In the present case the control variable is the supply temperature, and the process to be controlled is the mass flow. Actually the system will not allow the mass flow to exceed some maximum value due to physical limitations, but if it happens that the consumers demand of mass flow add up to exceed this maximum, the result will be that the most distant consumers will not have their heat demand fulfilled.

Normally, variations of the heat demand can be met either by varying the mass flow through the network or by varying the supply temperature. Since the object is to keep the supply temperature as low as possible the heat demand is met by varying the mass flow as far as possible. If the heat demand is suitably low the supply temperature is kept near its minimum, while the mass flow is subject to variations. For moderately high heat demand the mass flow occasionally reaches its upper limit during the day implying that an increase of the temperature is needed. In case the amount of heat required by the consumers becomes so large that the mass flow is around its maximum throughout the day (which is usually the case for non-summer situations), variations of the temperature is the only way to meet varying demands.

Since the process to be controlled is the mass flow, a conversion from predicted heat load to predicted water flow has to be carried out. In this conversion future supply and return temperatures are used.

The Reference Mass Flow

Suppose that future values of the mass flow, q_{t+j}, have to be belove the maximum flow, $q^{\max}$, with probability π (e.g., 99%),

$$P\{q_{t+j} \leq q^{\max}\} = \pi, \; 0 < N_1 \leq j \leq N_2 \tag{3}$$

where N_1 and N_2 are chosen such that $q_{t+N_1}, \ldots, q_{t+N_2}$ encompass future mass flow values significantly affected by the next control, $T_{s,t+1}$. A reference mass flow, $q^{\text{ref}}_{t+j|t}$, can then be obtained as the deterministic equivalent for this stochastic constraint, assuming that the error of the heat load predictions (and the return temperature predictions) are normal distributed, see Søgaard (1993) or Madsen *et al.* (1992).

A Simple Controller

By observing the return temperature of the water and assuming maximum water circulation, the lowest admissible supply temperature in the next hour is computed from the energy balance equation

$$p_t = c_w q_t (T_{s,t} - T_{r,t}) \tag{4}$$

The simplest possible controller is then

$$T_{s,t+1} = \hat{T}_{r,t+1|t} + \frac{\hat{p}_{t+1|t}}{c_w q^{\text{ref}}_{t+1|t}} \tag{5}$$

2.3 An Advanced Flow Sub-Controller - Weighted Predictive Control

It is obvious that a temporary change of the supply temperature affects the consumers after different time-delays. Therefore it is relevant to take heat load predictions with different horizons into consideration when controlling the supply temperature (the mass flow). Hence, a multi-step predictive controller is required. Such controllers (weighted predictive controller) are proposed in Søgaard (1993) and Madsen *et al.* (1992).

The main idea of the weighted-predictive controller is as follows : For each prediction horizon, j, ($j \in [N_1, \ldots, N_2]$, see Eq. (3)) the desired supply temperature, for that particular horizon, is found as

$$T_{s,t+1}^{(j)} = \hat{T}_{r,t+j|t} + \frac{\hat{p}_{t+j|t}}{c_w q_{t+j|t}^{\text{ref}}} \tag{6}$$

A reasonable j-step predictor for the return temperature is $\hat{T}_{r,t+j|t} = T_{r,t}$, due to minor variations. The desired supply temperature, for all the prediction horizons, is then constructed as a weighted average

$$T_{s,t+1} = \sum_{j=N_1}^{N_2} \gamma_j T_{s,t+1}^{(j)}, \qquad \left(\sum_{j=N_1}^{N_2} \gamma_j = 1 \right) \tag{7}$$

where each of the weights, γ_j, is chosen as the fraction of the heat consumed at the particular time-delay j.

3 The Supply Temperature Sub-Controller

3.1 The Transfer Function Model

In order to find an optimal control, a model of the district heating network is needed. If a perfect description of the entire network is available, a pure physical model can be obtained. Unfortunately, for large networks this type of models can be almost impossible to establish, and if a model can be established it is most likely too complex for control purposes. Therefore, in this work statistical transfer functions are identified. Each transfer function describes the relation between the supply temperature from the district heating plant and the supply temperature at a representative point in the network.

The transfer function model considered belongs to the single-input single-output ARX (Auto-Regressive-eXtraneous) structure, see Madsen (1989) for more details. In Søgaard (1988) it is shown how these models are identified and an example of a model for the district heating system in Esbjerg is

$$T_{n,t} = a_1 T_{n,t-1} + b_{2,t} T_{s,t-2} + b_{3,t} T_{s,t-3} + b_{4,t} T_{s,t-4} + \varepsilon_t \tag{8}$$

In this model example the time-delay is equal to two time steps (hours). The time-delay in the system actually show considerable variation due to the variation of the mass flow. The values of the time-delay and the parameters are estimated from hourly measurements of supply temperatures ($T_{s,t}$ and $T_{n,t}$) and the estimates are updated once each hour. See Søgaard (1988) or Madsen *et al.* (1990) for details.

The Time-Variation (The Non-Stationarity)

A district heating system is a non-stationary system, and therefore time-varying parameters have to be introduced. The time-variations of the parameters are mainly of two kinds: slow annual variations which are, e.g., due to changes of the dynamic characteristics of the distribution network (induced by seasonal climate changes), and faster diurnal variations induced by varying heat consumption. The slow variations can be coped with by using adaptive estimation techniques, while the diurnal variation is included as an explicit part of the model.

The Time-Delay

The time-delay in district heating systems can be relatively large compared to the sampling interval (in Esbjerg the time-delays are from 2 to 10 hours). Since the time-delay is time-varying and a function of several variables, like distance, flow in the pipes, and intersection lay-out of the network, the identification of the delay is one of the major problems in modeling district heating systems. Statistical methods for estimating or tracking such a time-varying time-delay are developed and described in details in Søgaard and Madsen (1991) and Madsen *et al.* (1990). Hence, this problem will not be addressed in this paper.

The Predictions

The optimal predictions are found using conditional expectations, see e.g., Madsen (1989) or Abraham and Ledolter (1983), conditioned on the information available at the present time.

The predictions are decomposed into two parts, one depending on information known at the present time t (the past controls $(T_{s,t-1}, T_{s,t-2}, \ldots)$ and the present and past outputs $(T_{n,t}, T_{n,t-1}, \ldots)$), and another depending on information not available (known) at the present time t (the present and the future controls $(T_{s,t}, T_{s,t+1}, \ldots)$).

Furthermore, the controllers utilize that the predictions and the prediction errors are uncorrelated, whereby the problem, to be minimized, can be decomposed into a deterministic part and a stochastic part.

3.2 The Reference Temperature

Typically, the temperature requirements are given by a minimum supply temperature curve, which expresses the supply temperature as a function

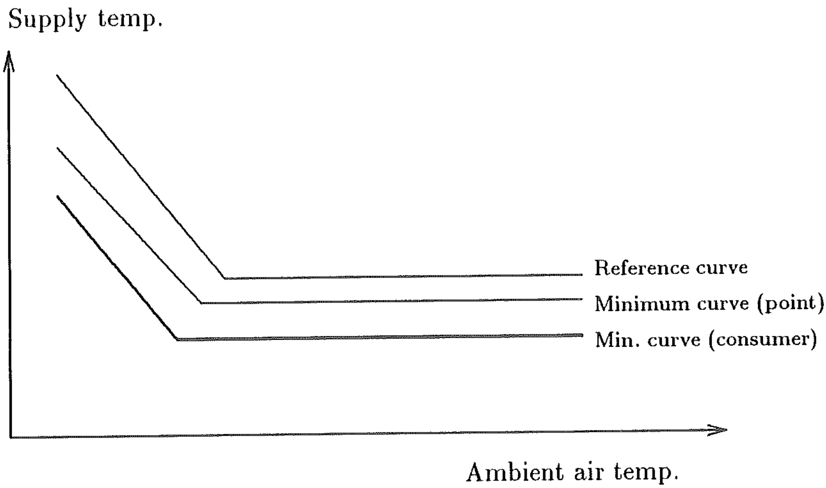

Fig. 2 The minimum temperature curve at the consumers, at the representative point, and the reference temperature curve.

of the ambient air temperature (see Fig. 2). The supply temperature curve may be a mutual agreement between the district heating company and the consumers. Each representative point has its own minimum curve, depending, e.g., on its location in the network. The minimum curves used are all of the same form as shown in Fig. 2: The minimum temperature is constant for ambient air temperatures higher than a certain value and for lower ambient air temperature it increases linearly with falling ambient air temperature.

The reference value is determined such that the supply temperature is over the minimum supply temperature curve, for the particular point, with some pre-specified probability - see Madsen *et al.* (1992).

Since the guaranteed minimum temperature depends on the ambient air temperature and due to the time-delay in the system, it is necessary to have predictions of the future ambient air temperature. The computer program PRESS uses models for ambient air temperature described in Madsen *et al.* (1990). The models are based on exponential smoothing. In the calculation of the reference temperature both the uncertainty of the prediction of the supply temperature at the representative points and the uncertainty of the prediction of the ambient air temperature are taken into account.

3.3 The Sub-Controller

A Simple Controller

In the present case the aim is to minimize the supply temperature from the district heating plant observing that the supply temperatures in the network (at the representative points) should be kept above the minimum curve in at least 99%, say, of the time. Therefore, it is desirable to keep the reference supply temperature (set point) as low as possible. This can be done by minimizing the variance of the supply temperature from the plant (illustrated in the Fig. 3), leading to a Minimum Variance (MV) control.

Minimum Variance Control A simple description of the MV controller is as follows: Assume that the time-delay (the transport-delay of the water) is k time steps. The k-step prediction of the supply temperature is then simply set equal to the reference supply temperature for the same time step

$$\hat{T}_{n,t+k|t} = T^{\text{ref}}_{n,t+k|t} \qquad (9)$$

The supply temperature from the plant (the control variable), $T_{s,t}$, can then easily be solved from Eq. (9), as $\hat{T}_{n,t+k|t}$ is a function of $T_{s,t}$, cf., Eq. (8).

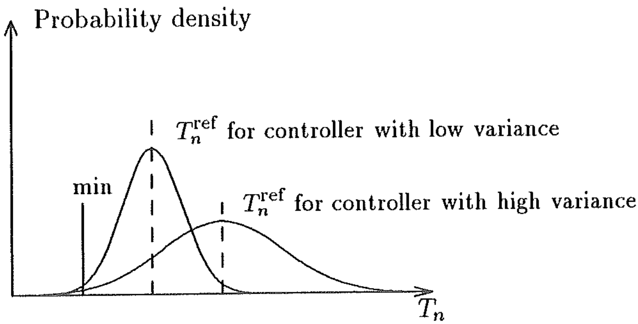

Fig. 3 Expressing regulation performance in terms of variation in output (error) variables.

Extended Controllers

In spite of its simplicity the MV controller has some disadvantages, such as it highly depends on that the underlying process is accurately represented by the identified model, i.e., the order of the model and the time-delay are accurate, and instability may be encountered if the model is non-minimum-phase. Furthermore, as the controller solely concerns about the output, $T_{n,t}$, and not about the control signal, $T_{s,t}$, this may result in an undesirable variation of the control signal. It is obvious that changing the supply temperature rapidly is not desirable as it implies wearing of the pipelines and causes an undesirable operation of the plant.

In order to circumvent this problem some extensions to the controller are proposed.

Linear Quadratic Gaussian Control (LQG) This type of controller is presented and analyzed in Palsson *et al.* (1992). By using this controller it is possible to penalize variations in the supply temperature from the plant by means of a penalty parameter. It is shown in Palsson *et al.* (1992) how various values of the penalty parameter influence the control performance and the control effort. The main result of the simulations in Palsson *et al.* (1992) is that for small positive values of the penalty the variations in the supply temperature from the plant are reduced considerably without aggravating the control performance (the variance of the supply temperature at the particular point in the network).

This controller eliminates some of the drawbacks of the MV controller, however, it is still sensitive to inaccurate time-delay and model order.

Generalized Predictive Control for Non-Stationary Systems Due to, e.g., the thermal capacity of the district heating system the changes in the supply temperature, $T_{s,t}$, do not only influence the supply temperature at the representative point after k time steps, $T_{n,t+k}$, (k is still the time-delay), but also at subsequent time-delays, $T_{n,t+k+1}, T_{n,t+k+2}, \ldots$. Therefore, is it natural to extend the controller in order to take care of this, i.e., the controller endeavor, not only, to keep $T_{n,t+k}$ close to the reference $T^{\text{ref}}_{n,t+k|t}$ (see MV and LQG), but also to keep $T_{n,t+j}$ close to $T^{\text{ref}}_{n,t+j|t}$ for all relevant time-delays j, $j \in [N_1, \ldots, N_2]$.

Furthermore, it is not only desirable to keep the change in the present control signal, $T_{s,t}$, as is done in the previous described LQG controller, but also to keep the variations of the subsequent controls, $\Delta T_{s,t}, \ldots, \Delta T_{s,t+N_u-1}$, where N_u is the control horizon, down.

These extensions, in fact, lead to a controller called Generalized Predictive Controller (GPC), which was first presented, for time-invariant systems, in Clarke *et al.* (1987). In order to circumvent the problem, concerning the time-varying system, some modifications, to the classical GPC, are proposed in Palsson *et al.* (1993a). Furthermore, the underlying model, in the present case, is an ARMAX model instead of an ARIMAX model as in the original version of the GPC, Clarke *et al.* (1987), and almost all later references. This implies some further modifications, see Palsson *et al.* (1993a).

The proposed modified GPC controller is studied in Palsson *et al.* (1993b) in the district heating context by simulation. The conclusions in Palsson *et al.* (1993b) are that the effect of penalizing the changes in the supply temperature is similar as for the previous described LQG controller. The effect of various control horizons, N_u, is also studied. The "best" controller is obtained with $N_u = 2,3$, or 4 and for small positive values of the penalty parameter.

Furthermore, it has been shown that the Generalized Predictive Controller is quite robust, i.e., non-sensitive to inaccurately estimated time-delays and incorrect model order.

4 The Overall Controller

As previously mentioned the overall controller selects the highest supply temperature among required supply temperatures calculated from the flow sub-controller and from the supply temperature sub-controllers. This temperature is used as the supply temperature from the plant in the following hour. (The model parameters, the predictions and the controller signals are updated each hour). See again Fig. 1.

5 Practical Experience

The computer program PRESS has been in use for several years at Vestkraft, in Esbjerg, where five representative points in the network have been used, with excellent results. In order to assess the savings the following rules of thumb have been used:

- the heat energy price at the CHP plant is assumed to decrease with falling plant supply temperature (about 0.7%/°C)
- the heat loss from the network is assumed to decrease with falling plant supply temperature (about 0.5%/°C)

The savings are analyzed in Madsen *et al.* (1992). It is concluded that in the ambient air temperature interval, where data is available, the use of PRESS has implied a lowering of the supply temperature of about 9 °C compared with the control used before PRESS was brought into operation. Consequently, the fuel consumption has been lowered considerably in this ambient air temperature interval, and hereby implied large savings in production costs.

6 Summary and Conclusion

This paper describes a concept to minimize the supply temperature from a CHP plant, where the pumping costs are considered to be of minor importance compared to the heat loss. This is done by several sub-controllers. A flow sub-controller that utilizes the predictions of the future heat load to keep the mass flow near, but with given probability below, the maximum flow limit, hence, keeps the supply temperature from the plant as low as possible. A number of supply temperature sub-controllers, which endeavor is to keep the supply temperatures at several representative locations in the network close to, but with given probability over, a given ambient air temperature dependent minimum. Hence, the supply temperature sub-controllers keep the supply temperature from the plant as low as possible, and with some extensions penalize large variations of the plant supply temperature. The highest supply temperature among the optimal supply temperatures from the sub-controllers is then used as the supply temperature from the plant in the following hour.

The conclusion that can be drawn is that stochastic modeling and predictive control techniques are extremely well suited for controlling supply temperature and mass flow in district heating systems, as they take care of the uncertainties, that are embedded in every system, in an appropriate way and they do not depend on a huge amount of physical data, just on relevant measurement data.

Experience has shown that use of these methods has resulted in a considerably lower supply temperature, at least in the major part of the year, and hence considerably lower fuel consumption.

Acknowledgements

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Nomenclature

a	=	model parameters
b	=	model parameters
c_w	=	specific heat of the water [J/(kg °C)]
$I_{\{\text{workday}\},t}$	=	indicator function being 1 on workdays and 0 else
j	=	prediction horizon
l	=	adjusting parameter accounting for the variables having mean values different from zero [J/s]
p_t	=	heat load [J/s]
$\hat{p}_{t+1\|t}$	=	one-step prediction, at time t, of the heat load at time $t+1$ [J/s]
P	=	probability measure
N_1	=	minimum horizon
N_2	=	maximum horizon
N_u	=	control horizon
q	=	mass flow [kg/s]
t	=	time (in units of hours)
$T_{a,t}$	=	ambient air temperature [°C]
$T_{n,t}$	=	supply temperature in the network [°C]
$T_{r,t}$	=	return temperature at the plant [°C]
$T_{s,t}$	=	supply temperature from the plant

		$(\Delta T_{s,t} = T_{s,t} - T_{s,t-1})$ [°C]
w_t	=	wind speed [m/s]

Greek Letters

α	=	model parameters
β	=	model parameters
γ	=	weights
ε	=	mean zero model error
π	=	probability
$\mu_{1,t}$	=	diurnal heat load profile for workdays [J/s]
$\mu_{2,t}$	=	diurnal heat load profile for weekends [J/s]

Superscripts and Subscripts

a	=	ambient
max	=	maximum
n	=	network
s	=	supply
ref	=	reference value
$\hat{}$	=	prediction
$t+1\|t$	=	(prediction) at $t+1$ given the information at t

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Scenario Analysis and the Progressive Hedging Algorithm - Simple Numerical Examples

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Abstract

The Progressive Hedging Algorithm (PHA) is a new method which may be applied to multiperiod optimization problems under uncertainty. It was first presented in Rockafellar and Wets (1991). In their paper no examples are given and the main emphasize is on the proofs and convergence analysis. Therefore, in order to understand the idea behind scenario analysis and PHA very simple numerical examples are given in this paper and the results are compared to results derived from classical stochastic optimization methods.

Key Words: Progressive Hedging; Scenario Analysis; Stochastic Optimization.

1 Introduction

Many optimization problems with practical origin incorporate stochastic elements. In general it is very difficult to solve stochastic problems. This is due to either the mathematical properties (in particular if the space for stochasticity is infinite) or to the computational burden due to the combinatorial explosion (in the case with discrete stochastic events). Only few specific models can be solved easily, in particular the unconstrained Quadratic-Linear model with normally distributed stochastic elements. For an introduction to the classical material the readers are referred to Jensson (1975) (for a Mathematical Programming approach) and to Bertsekas (1987) (for the Dynamic Programming approach).

Recently, stochastic optimization problems have received renewed interest. One of the promising approaches is based on scenario analysis, which is a stochastic programming technique employing discrete scenarios with known probabilities, usually covering several time periods. Figure 1 shows a scenario tree with eight scenarios. Each of these eight scenarios are shown in Fig. 2.

The Progressive Hedging Algorithm (PHA) is a method to solve scenario problems. The PHA was first presented in Rockafellar and Wets (1991). Since then this method has been studied in several papers, e.g., Robinson (1991), Dembo (1991), Helgason and Wallace (1991), Wallace and Helgason (1991) and Palsson and Ravn (1993). In Helgason and Wallace (1991) it is shown how PHA can be combined with approximate solution of the individual scenario problems, resulting in a computationally efficient algorithm where two individual Lagrangian-based procedures are merged into one. This method is used to study examples from fisheries management. In Wallace and Helgason (1991) the structural properties of the PHA is discussed. In Palsson and Ravn (1993) the PHA is applied to the management of a heat storage.

The purpose of this paper is to give an easy introduction to the PHA. The paper starts with the description of the scenario analysis in connection with the decision problem. Then the PHA and some definitions needed are given,

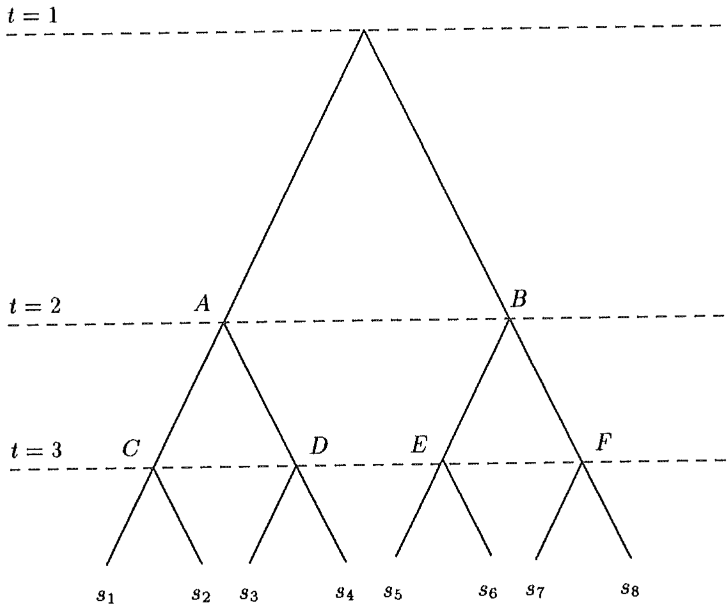

Fig. 1 A scenario tree with eight scenarios. There are three time periods, i.e., $T = 3$, and the stochastic variable can take two values each time.

simple numerical examples are studied, and finally some concluding remarks are given.

2 Scenario Analysis

The present paper considers decision problems that are sequential in nature. This means that the decision vector $X \in \mathbb{R}^n$ is departed: $X = (x_1', x_2', \ldots, x_T')'$. Thus, there are T *stages* and at each stage t one decision x_t must be made, first x_1, then x_2, etc. A sequence of decisions is called a *policy*.

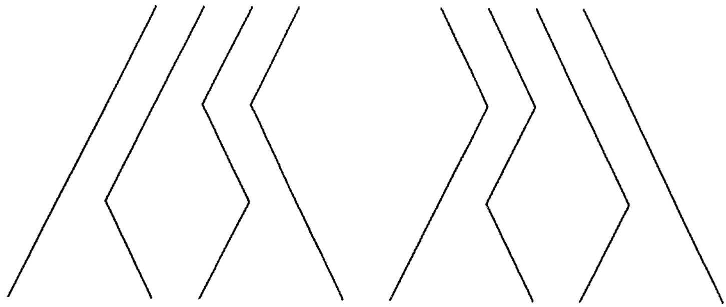

Fig. 2 The eight scenarios from the scenario tree shown in Fig. 1.

The information structure is also sequential. Thus, each time a decision is made some information is gained.

The information structure is described by scenarios. The uncertainty about parameters or components of the system is modeled by a small number of subproblems. Each version is called a *scenario*. The set of scenarios is denoted by S.

Each scenario s is supposed to take place with probability p_s:

$$p_s > 0, \qquad \sum_{s \in S} p_s = 1 \tag{1}$$

Eight scenarios are indicated in Fig. 1. The sequential nature of the problem is indicated by the sequence of stages $t = 1$, $t = 2$ and $t = 3$. Thus, x_t must be chosen three times. Information is also gained three times, after a decision is made. Thus, at the time x_1 must be chosen, it is not known which of the eight scenarios, $s_1, \ldots, s_8$, will take place. The choice of x_1 must be made, and then the state will be in one of the two points A or B. If the actual scenario was s_1, s_2, s_3 or s_4 the state will be in A, if the actual scenario was s_5, s_6, s_7 or s_8 the state will be in B.

It is assumed that after x_1 is chosen it can be observed, whether the state is in A or B, and therefore information is gained.

Now x_2 must be chosen, and then the state will end up in C or D (if the state is in A) or E or F (if the state is in B), and so on.

This decision process has two kinds of requirement to the choice of x_t at each stage. Firstly, there is the requirement that x_t must be *admissible*. This means that x_t must be chosen from some set C_s.

Secondly, there is the requirement that x_t must be *implementable*. This refers to the information structure. Thus, when x_t is chosen at stage t only the knowledge which has been gained until then is available.

This can be formulated more specifically as follows. As seen from Fig. 1 - 2 there are relations between the scenarios. Thus, at stage $t = 2$ the scenarios fall in two groups: one contains scenarios $s_1, \ldots, s_4$ the other scenarios $s_5, \ldots, s_8$. Similarly at stage $t = 3$ there are four groups of scenarios, each group with two scenarios.

Such a group is denoted as a *scenario bundle*. As seen, the scenario bundles at time t are such that the scenarios in each bundle are observationally indistinguishable at time t.

Therefore, the requirement of implementability can now be stated as follows:

> *The policy is implementable if for all t, x_t is dependent on the scenario bundle but not on the particular scenario in the bundle.*

At any time t, it is known what scenario bundle is relevant for the decision. Thus, at stage $t = 2$ on Fig. 1 it is known if the relevant scenario bundle contains $s_1, \ldots, s_4$ (this is the case if the state is at A) or $s_5, \ldots, s_8$ (if the state is at B).

The decision problem can now be formulated as follows. There is a sequential decision and information gaining process, defined by scenarios $s \in S$ with probabilities p_s, see Eq. (1). To each decision $X \in \mathbb{R}^n$ and each scenario $s \in S$ there is a constant set C_s and an objective function value

$f_s(X(s))$ can be calculated.

The purpose is to minimize the expectation, i.e., to solve

$$\min \sum_{s \in S} p_s f_s(X(s)) \tag{2}$$

subject to

$$X(s) \in C_s \qquad (X(s) \text{ is admissible}) \tag{3}$$

and

$$X(s) \text{ is implementable} \tag{4}$$

This problem is difficult to solve in general. The algorithm described below is based on the assumption that if it is known which scenario $s \in S$ is relevant then the optimal solution to this *deterministic* problem:

$$\min f_s(X(s)) \tag{5}$$

$$X(s) \in C_s \tag{6}$$

is easily found for any $s \in S$.

If this is true the following problem can also easily be solved

$$\min f_s(X(s)) + X'(s)W + \frac{1}{2}r||X(s) - \hat{X}||^2 \tag{7}$$

$$X(s) \in C_s \tag{8}$$

for any $s \in S$. In Eqs. (7) - (8) $\hat{X}$ is a given value in $\mathbb{R}^n$, W is a given value in $\mathbb{R}^n$, $X'W$ is the scalar product of X and W, $r \in \mathbb{R}$, $r > 0$, and $||\cdot||$ is the Euclidean norm.

The relation between the problems in Eqs. (5) - (6) and in Eqs. (7) - (8) may be interpreted as follows.

To each of the potentially relevant bundles at stage t the policy may be calculated

$$\hat{X}_t = \sum p_s X_t(s) / \sum p_s \tag{9}$$

In this equation, the summations are over the scenarios in the relevant bundle, and $X_t(s)$ is the solution at time t to a problem like Eqs. (7) - (8). It is seen that $\hat{X}_t$ in Eq. (9) can be interpreted as a conditional expectation, relative to the relevant scenario bundle, i.e., relative to the information available. It is also observed that by this definition $\hat{X}_t$ is implementable. Moreover, if C_s is convex and independent of the particular scenario in the bundle, then $\hat{X}_t$ is admissible.

The vector $W \in \mathbb{R}^n$ in Eq. (7) may now be interpreted as a Lagrange multiplier vector relative to the constraint in Eq. (9) (or similarly Eq. (4)) and the last term in Eq. (7) may be interpreted as a penalty term which is introduced in order to attain convergence stability in an algorithmic sense.

In summary the project can be stated as follows: the decision problem in Eqs. (2) - (4) is solved by solving the following problem

$$\min \sum p_s f_s(X(s)) \tag{10}$$

$$X(s) \in C_s \tag{11}$$

$$X(s) = \hat{X} \tag{12}$$

where $\hat{X}$ is defined in Eq. (9), and Eq. (12) ensures that the solution is implementable.

In Rockafellar and Wets (1991) an algorithm is given for solution of the problem in Eqs. (10) - (12). This is described in the next section.

3 Progressive Hedging Algorithm

In this section the PHA is given as in Rockafellar and Wets (1991) and just the definitions needed are given. The readers are referred to Rockafellar and Wets (1991) for the details.

PROGRESSIVE HEDGING ALGORITHM. In iteration ν (where $\nu = 0, 1, \ldots$) one has an *admissible* but not necessarily *implementable* policy $X^\nu \in C$

and a *price system* $W^\nu \in \boldsymbol{M}$. (Initially one can take X^0 to be the policy obtained by letting $X^0(s)$ be for each scenario $s \in S$ an optimal solution to the given scenario subproblem $(\boldsymbol{P}_s)$. One can take $W^0 = 0$.)

Step 1. Calculate the policy $\hat{X}^\nu = JX^\nu$, which is implementable but not necessarily admissible. (If ever one wishes to stop, this policy $\hat{X}^\nu$ is to be offered as the best substitute yet available for a solution to $\boldsymbol{P}$.)

Step 2. Calculate as $X^{\nu+1}$ an (approximately) optimal solution to the subproblem

$$(\boldsymbol{P}^\nu) \quad \text{minimize}[F(X) + < X, W^\nu > + \tfrac{1}{2}r||X - \hat{X}^\nu||^2] \quad \text{over all } X \in \boldsymbol{C}$$

This decomposes into solving (approximately) for each scenario $s \in S$ the subproblem

$$(\boldsymbol{P}_s^\nu) \quad \text{minimize}[f_s(x) + x \cdot W^\nu(s) + \tfrac{1}{2}r|x - \hat{X}^\nu(s)|^2] \quad \text{over all } x \in C_s$$

in order to get $X^{\nu+1}(s)$. The policy $X^{\nu+1}$ will again be admissible but not necessarily implementable.

Step 3. Update from W^ν to $W^{\nu+1}$ by the rule $W^{\nu+1} = W^\nu + rKX^{\nu+1}$. The price system $W^{\nu+1}$ will again be in $\boldsymbol{M}$. Return to Step 1 with ν replaced by $\nu + 1$. □

The definitions and some comments are given in order of appearance:

- ν is an iteration number,
- X^ν is the decision to be made at iteration ν, $X^\nu \in \mathbb{R}^n$,
- $\boldsymbol{C}$ is a set of admissible polices, $\boldsymbol{C} \subseteq \mathbb{R}^n$,
- W^ν is a price system (Lagrange multiplier), $W^\nu \in \boldsymbol{M} \subset \mathbb{R}^n$,
- $\boldsymbol{M}$ is a set, $\boldsymbol{M} \subset \mathbb{R}^n$. $\boldsymbol{M}$ is such that for each scenario bundle A at stage t, $\sum_{s \in A} p_s W_t(s) = 0$. The algorithm secures that this is fulfilled, if W^0 fulfills it; the initial choice $W^0 = 0$ fulfills it,

- $X(s)$ is the decision to be made given the scenario s,
- s is the scenario,
- S is the scenario set,
- $\boldsymbol{P}_s$ is the scenario subproblem,
- $\hat{X}$ is a conditional expectation, defined as in Eq. (9) using $X = X^{\nu}$. Observe that if C_s is convex for all $s \in S$ then $\hat{X}$ is admissible,
- J is an aggregation operator (or conditional expectation operator relative to the given information structure and values p_s, i.e., JX^{ν} defines $\hat{X}$ as in Eq. (9) using $X = X^{\nu}$, where p_s is the probability attached to s),
- $\boldsymbol{P}$ is the scenario problem,
- $F(X)$ is equal to $E\{f_s(X(s))\}$, where E denotes the expectation value, see Eq. (10),
- $<,>$ is the inner product on Euclidean vector space,
- r is a penalty parameter (> 0),
- $||\cdot||$ is a norm,
- $f_s(x)$ is the objective function value for the scenario subproblem s and decision X,
- x is the decision to be made for the scenario subproblem,
- $|\cdot|$ is the ordinary Euclidean norm ($|x| = \sqrt{\sum x_i^2}$),
- C_s is a set of admissible polices for the scenario s, $C_s \subseteq \mathbb{R}^n$,
- K is an operator ($KX = X - \hat{X}$), such that $KX^{\nu+1}$ in step 3 is equal to $X^{\nu+1} - \hat{X}^{\nu+1}$.

Remark: As noted, $\hat{X}^{\nu+1}$ is needed in the update of W^{ν} to $W^{\nu+1}$, but in the way the algorithm is written in Rockafellar and Wets (1991), $\hat{X}^{\nu+1}$ is not available at that moment (first at Step 1 in the next iteration). Therefore, it is recommended that the algorithm is rewritten as:

Step 0. Initialize.

Step 1. Solve (approximately) for each scenario $s \in S$ the subproblem (in order to get $X^{\nu+1}(s)$).

Step 2. Calculate the policy $\hat{X}^{\nu+1} = JX^{\nu}$.

Step 3. Update from W^{ν} to $W^{\nu+1}$. Return to Step 1 with ν replaced by $\nu + 1$. □

In Rockafellar and Wets (1991) it is shown, that if for any $s \in S$, C_s is convex and f_s is convex, and some regularity conditions apply, then the algorithm will converge to an optimal solution to Eqs. (10) - (12). The regularity conditions are what can be expected from basic mathematical programming theory. The rate of convergence is linear. It is also shown that the algorithm will converge if Eqs. (7) - (8) are solved only approximately, and the error bound is specified.

4 Numerical Example

In this section the PHA is used on simple stochastic optimization problems, the original algorithm is used.

4.1 Example 1

The problem is to find

$$\min_{x} E[(x-d)^2] \tag{13}$$

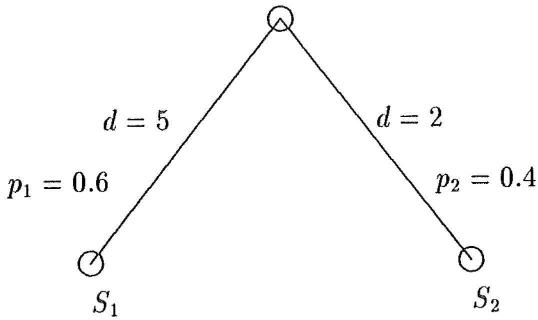

Fig. 3 A small scenario tree.

subject to the constraint

$$3 \leq x \leq 6, \; x \in \mathbb{R} \tag{14}$$

d can takes two values $[5, 2]$ with probabilities $[0.6, 0.4]$, respectively. Thus, there are two scenarios, one with $d = 5$ and one with $d = 2$. This is illustrated in Fig. 3. It is observed that this problem is not sequential (except in a trivial sense) since only one choice is made.

It is clearly seen from Eqs. (13) - (14), that the optimal solutions for the deterministic problem corresponding to each of the two scenarios are $X^*(s_1) = 5$ and $X^*(s_2) = 3$, respectively. Now the Progressive Hedging Algorithm is applied to this problem.

4.1.1 PHA

$\underline{\nu = 0}$:

$X^0 = (5, 3)$ (i.e. $X^0(s_1) = x_1^0 = 5$ and $X^0(s_2) = x_2^0 = 3$)
$W^0 = (0, 0)$ and $r = 1$ (chosen).

STEP 1:
$\hat{X}^0 = p_{s,1}X^0(s_1) + p_{s,2}X^0(s_2) = 0.6 \times 5 + 0.4 \times 3 = 4.2$
(this is admissible, i.e., $3 \leq \hat{X}^0 \leq 6$)

STEP 2:
$s_1 : \min[(x_1 - 5)^2 + x_1 0 + \frac{1}{2}1(x_1 - 4.2)^2] \Rightarrow X^1(s_1) = 4.73$
$s_2 : \min[(x_2 - 2)^2 + x_2 0 + \frac{1}{2}1(x_2 - 4.2)^2] \Rightarrow X^1(s_2) =$ ~~2.73~~
the solution must be admissible, i.e., $X^1(s_2) = 3$

STEP 3:
$W^1 = W^0 + r(X^1 - \hat{X}^1) = (0,0) + 1(4.73 - 4.04, 3.0 - 4.04)$
$= (0.69, -1.04)$

$\underline{\nu = 1:}$

STEP 1:
$\hat{X}^1 = p_{s,1}X^1(s_1) + p_{s,2}X^1(s_2) = 0.6 \times 4.73 + 0.4 \times 3.0 = 4.04$

STEP 2:
$s_1 : \min[(x_1 - 5)^2 + x_1 0.69 + \frac{1}{2}1(x_1 - 4.04)^2] \Rightarrow X^2(s_1) = 4.45$
$s_2 : \min[(x_2 - 2)^2 + x_2(-1.04) + \frac{1}{2}1(x_2 - 4.04)^2] \Rightarrow X^2(s_2) = 3.03$

STEP 3:
$W^2 = (0.69, -1.04) + 1(4.45 - 3.88, 3.03 - 3.88) = (1.26, -1.89)$

$\underline{\nu = 2:}$

STEP 1:
$\hat{X}^2 = p_{s,1}X^2(s_1) + p_{s,2}X^2(s_2) = 0.6 \times 4.45 + 0.4 \times 3.03 = 3.88$

STEP 2:
$s_1 : \min[(x_1 - 5)^2 + x_1 1.26 + \frac{1}{2}1(x_1 - 3.88)^2] \Rightarrow X^3(s_1) = 4.21$
$s_2 : \min[(x_2 - 2)^2 + x_2(-1.89) + \frac{1}{2}1(x_2 - 3.88)^2] \Rightarrow X^3(s_2) = 3.26$

STEP 3:

$W^3 = (1.26, -1.89) + 1(4.21 - 3.83, 3.26 - 3.83) = (1.64, -2.46)$

Table 1 Results from the iterations.

ν	$X(s_1)$	$X(s_2)$	$\hat{X}$	w_1	w_2	$\hat{z}$	$\hat{z}'$
0	5.00	3.00	4.20	0.00	0.00	0.40	2.32
1	4.73	3.00	4.04	0.69	-1.04	0.44	2.22
2	4.45	3.03	3.88	1.26	-1.89	0.60	2.17
3	4.21	3.26	3.83	1.64	-2.46	1.01	2.16
4	4.06	3.43	3.81	1.89	-2.84	1.35	2.16
5	3.97	3.55	3.80	2.06	-3.09	1.60	2.16
6	3.91	3.63	3.80	2.18	-3.26	1.77	2.16
7	3.88	3.69	3.80	2.25	-3.38	1.90	2 16
8	3.85	3.73	3.80	2.30	-3.45	1.98	2.16
9	3.83	3.75	3.80	2.33	-3.50	2.04	2.16

$\underline{\nu = 3:}$

STEP 1:
$\hat{X}^3 = p_{s,1}X^3(s_1) + p_{s,2}X^3(s_2) = 0.6 \times 4.21 + 0.4 \times 3.26 = 3.83$

STEP 2:
$s_1 : \min[(x_1 - 5)^2 + x_1 1.64 + \frac{1}{2}1(x_1 - 3.83)^2] \Rightarrow X^4(s_1) = 4.06$
$s_2 : \min[(x_2 - 2)^2 + x_2(-2.46) + \frac{1}{2}1(x_2 - 3.83)^2] \Rightarrow X^4(s_2) = 3.43$

STEP 3:
$W^4 = (1.64, -2.46) + 1(4.06 - 3.83, 3.43 - 3.83) = (1.89, -2.84)$

The results from 9 iterations are found in Table 1. The table also shows the results for the objective function, i.e., $\hat{z} = p_{s,1}(X(s_1) - d_1)^2 + p_{s,2}(X(s_2) - d_2)^2$ and $\hat{z}' = p_{s,1}(\hat{X} - d_1)^2 + p_{s,2}(\hat{X} - d_2)^2$. Observe that $\sum p_s w_s \simeq 0$, i.e., $0.6 \cdot 2.33 + 0.4 \cdot (-3.50) \simeq 0$.

4.1.2 Mathematical Programming Approach

The stochastic problem could also be solved by solving

$$\min_x [p_{s,1}(x-d_1)^2 + p_{s,2}(x-d_2)^2] = \min_x [0.6(x-5)^2 + 0.4(x-2)^2] \quad (15)$$

with the constraint in Eq. (14). This gives $x^* = 3.8$, i.e., the same solution as before.

4.1.3 Choice of the Penalty Parameter

In Helgason and Wallace (1991) it is pointed out that the choice of the penalty parameter r is of vital importance and the conclusion drawn there is that the penalty should be as small as possible, provided it is large enough to guarantee convergence.

The error can be measured as the expected value (with respect to the distribution of the scenarios), see Helgason and Wallace (1991)

$$m^\nu = E\left[\sum_{t=0}^{T}\left\{||\hat{X}_t^\nu(s) - \hat{X}_t^{\nu-1}(s)||^2 + \frac{1}{r^2}||W_t(s)^\nu - W_t(s)^{\nu-1}||^2\right\}\right] \quad (16)$$

Hence the termination criterion is

$$m^\nu \leq \varepsilon \quad (17)$$

For the simple problem in Example 1 results for various values of the penalty parameter, r, were investigated. For this case the error measure is simply

$$m^\nu = (\hat{X}^\nu - \hat{X}^{\nu-1})^2 + \frac{1}{r^2}(p_{s,1}(w_1^\nu - w_1^{\nu-1})^2 + p_{s,2}(w_2^\nu - w_2^{\nu-1})^2) \quad (18)$$

The results are demonstrated in Table 2. It is seen that m^ν converges faster for large values of r, but it converges to a wrong value of $\hat{X}$. This is explained by the fact that for large values of r the solutions of the subproblems

Table 2 Number of iterations to fulfill the criterion in Eq. (17) for different r values and the corresponding implementable solution, $\hat{X}$.

	$\varepsilon = 5 \cdot 10^{-3}$		$\varepsilon = 5 \cdot 10^{-5}$	
r	ν	$\hat{X}^\nu$	ν	$\hat{X}^\nu$
0.1	75	3.80	123	3.80
0.5	17	3.80	28	3.80
1.0	10	3.80	16	3.80
2.0	6	3.81	10	3.81
3.0	5	3.85	9	3.81
4.0	5	3.86	10	3.81
5.0	5	3.90	11	3.81
10.0	4	4.03	15	3.83
100.0	2	4.19	8	4.15

are pressed together very quickly and then $\hat{X}$ converges slowly to the right value. On the other hand, for lower values of r, the solutions of the subproblems converge to the right solutions from both sides. This is illustrated in Figs. 4 – 6.

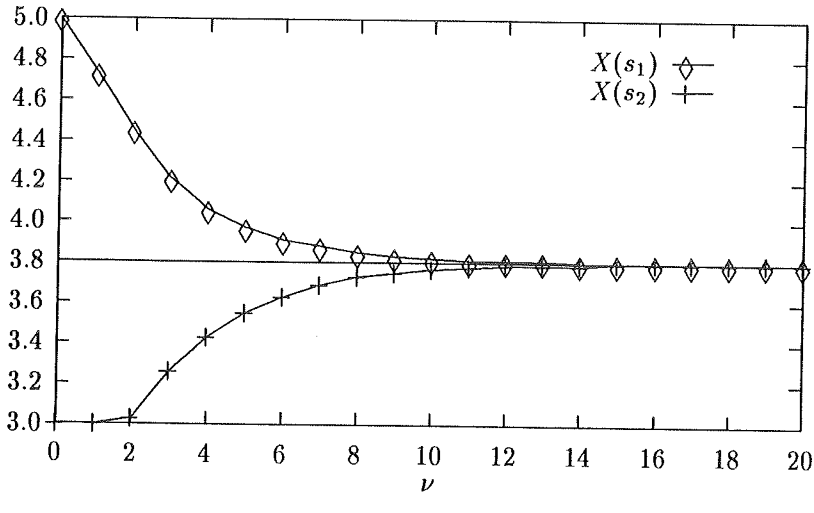

Fig. 4 Convergence of the subproblems. $r = 1.0$.

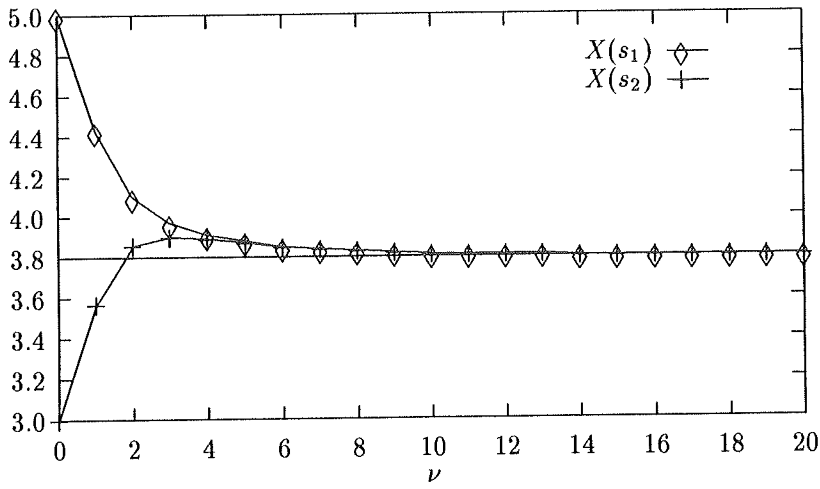

Fig. 5 Convergence of the subproblems. $r = 5.0$.

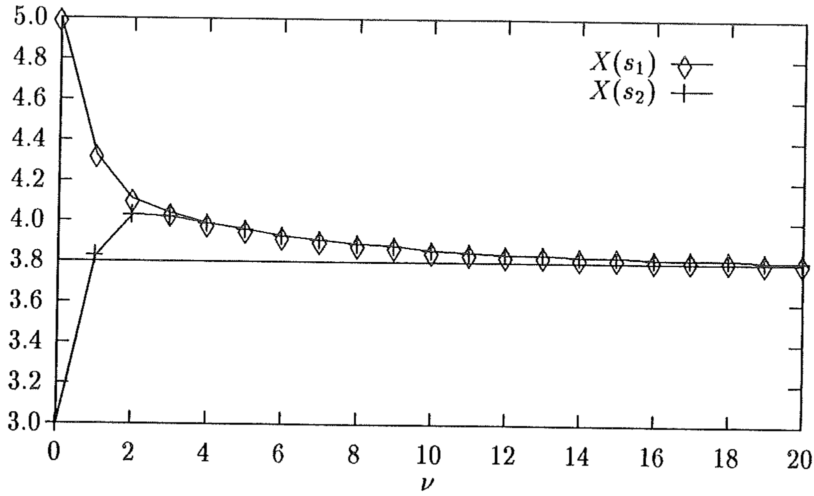

Fig. 6 Convergence of the subproblems. $r = 10.0$.

4.2 Example 2

The problem is now to find

$$\min_x E\left[\sum_{t=1}^{3}\left\{\frac{1}{2}(x_t - x_{t-1})^2 + (x_t - d_t)^2\right\}\right] \tag{19}$$

subject to the constraints

$$3 \le x_t \le 6, \qquad 1 \le t \le T \tag{20}$$

d and p take the values shown in Fig. 7. $x_0 = 4$ and T is equal to two or three, i.e., $T = 2$ for scenarios s_1 to s_4 and $T = 3$ for scenarios s_5 and s_6. This problem is sequential. It is observed that at time $t = 1$ there is one scenario bundle containing all 6 scenarios. At time $t = 2$ there are two bundles, the bundle containing s_1, s_2 and s_3, and the bundle containing s_4, s_5 and s_6. At time $t = 3$ there are 5 bundles, 4 of them are "empty", and the last one contains the two scenarios s_5 and s_6.

Clearly this example is more complex than Example 1, therefore, not all intermediate results will be shown.

4.2.1 PHA

First the optimal solutions to the given scenario subproblems are found and the price system is set to zero. This is shown in Table 3. In this example r is chosen equal to 1.

$\underline{\nu = 0}$:

STEP 1

The conditional expectations for the four decision states, labeled I, II, III, and IV in Fig. 7 are obtained from data in Tables 3 and 4

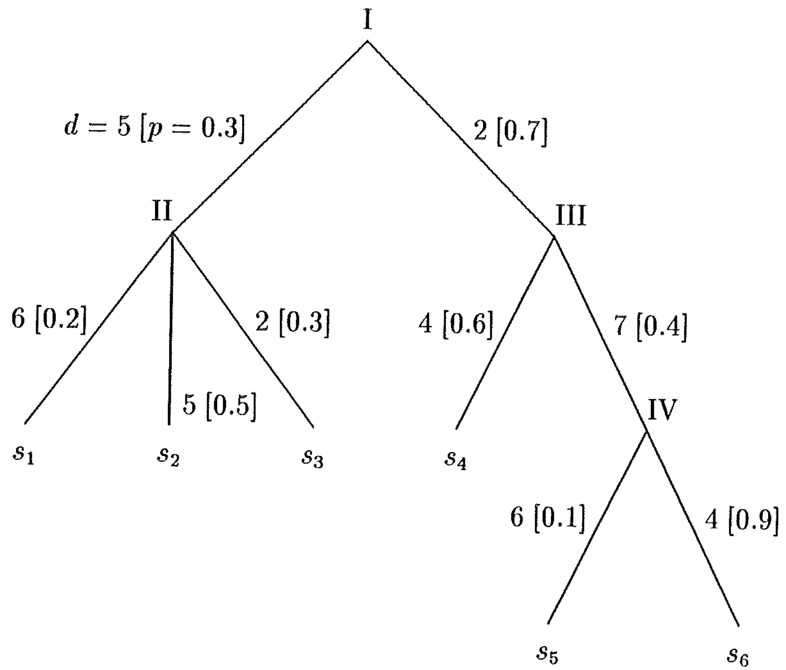

Fig. 7 The scenario tree used in Example 2, the probabilities are in parentheses $[p]$ and d is the stochastic variable.

Table 3 Initial solutions. * means that the solution lies on the lower boundary.

S	$X^0(s)$			$W^0(s)$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
s_1	4.91	5.64		0.0	0.0	
s_2	4.73	4.91		0.0	0.0	
s_3	4.25	3.00*		0.0	0.0	
s_4	3.00*	3.67		0.0	0.0	
s_5	3.46	5.85	5.95	0.0	0.0	0.0
s_6	3.37	5.46	4.49	0.0	0.0	0.0

Table 4 The probabilities, see also Fig. 7.

S	$p(s)$			$\hat{X}$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
s_1	0.06	0.20		I	II	
s_2	0.15	0.50		I	II	
s_3	0.09	0.30		I	II	
s_4	0.42	0.60		I	III	
s_5	0.028	0.04	0.1	I	III	IV
s_6	0.252	0.36	0.9	I	III	IV

$$\begin{aligned}\hat{X}_I^0 &= 0.06*4.91+0.15*4.73+0.09*4.25\\ &\quad +0.42*3.00+0.028*3.46+0.252*3.37=3.59\\ \hat{X}_{II}^0 &= 0.20*5.64+0.50*4.91+0.30*3.00=4.48\\ \hat{X}_{III}^0 &= 0.60*3.67+0.04*5.85+0.36*5.46=4.40\\ \hat{X}_{IV}^0 &= 0.10*5.95+0.90*4.49=4.63\end{aligned}$$

STEP 2 & 3

As an example, solution of subproblem s_1 is found as follows ($x_1 = X(s_1)_{t=1}$, $x_2 = X(s_1)_{t=2}$, $w_1 = W(s_1)_{t=1}$, and $w_2 = W(s_1)_{t=2}$)

$$\begin{aligned}f &= \frac{1}{2}(x_1-x_0)^2+(x_1-d_1)^2+\frac{1}{2}(x_2-x_1)^2+(x_2-d_2)^2\\ &\quad +x_1w_1+x_2w_2+\frac{1}{2}r(x_1-\hat{X}_I)^2+\frac{1}{2}r(x_2-\hat{X}_{II})^2 \qquad (21)\end{aligned}$$

$\partial f/\partial x_1 = 0$ and $\partial f/\partial x_2 = 0$ results in

$$\begin{aligned}(4+r)x_1-x_2 &= x_0+2d_1-w_1+r\hat{X}_I\\ -x_1+(3+r)x_2 &= 2d_2-w_2+r\hat{X}_{II} \qquad (22)\end{aligned}$$

respectively. This can easily be solved, the solutions are $x_1 = 4.57$ and 5.26, which also satisfy Eq. (20).

The price system is updated as

$$\begin{aligned}w_1^1 &= w_1^0+r(x_1^1-\hat{X}_I^1)=0.0+1(4.57-3.54)=1.03\\ w_2^1 &= w_2^0+r(x_2^1-\hat{X}_{II}^1)=0.0+1(5.26-4.37)=0.89 \qquad (23)\end{aligned}$$

The results for the other subproblems are found in Table 5.

Table 5 $X^1(s)$ **and** $W^1(s)$.

S	$X^1(s)$			$W^1(s)$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
s_1	4.57	5.26		1.03	0.89	
s_2	4.47	4.74		0.93	0.37	
s_3	4.15	3.16		0.61	-1.21	
s_4	3.09	3.85		-0.45	-0.51	
s_5	3.34	5.35	5.41	-0.20	0.96	0.93
s_6	3.30	5.14	4.37	-0.25	0.75	-0.10

$\nu = 5$:

STEP 1

$\hat{X}_I^5 = 3.53$, $\hat{X}_{II}^5 = 4.16$, $\hat{X}_{III}^5 = 4.48$ and $\hat{X}_{IV}^5 = 4.29$

STEP 2 & 3

$X^6(s)$ and $W^6(s)$ are shown in Table 6.

Summary: Table 7 shows the conditional expectations for the four decision stages. It is noted that $\hat{X}_I$, the first decision, converges already after 2 iterations while the others converges much slower or after approximately 15 iterations.

Table 7 also shows that the first decision is 3.53, then if d_1 happens to be

Table 6 $X^6(s)$ and $W^6(s)$.

S	$X^6(s)$			$W^6(s)$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
s_1	3.83	4.42		3.08	2.62	
s_2	3.84	4.27		2.90	1.08	
s_3	3.87	3.81		2.35	-3.55	
s_4	3.41	4.32		-1.30	-1.57	
s_5	3.36	4.72	4.57	-0.98	2.68	2.74
s_6	3.37	4.74	4.26	-1.01	2.32	-0.31

equal $x_1 = 5$ then the second decision is 4.05, else it is 4.55 and finally if $d_1 = 2$ and $d_2 = 7$ then the optimal decision is 4.23.

4.2.2 Classical Stochastic Optimization

Dynamic Programming: The stochastic problem may be solved by Dynamic Programming as follows.

The expected value at stage $t = 3$, state IV is

$$V(3, IV, x_3, x_2) = \frac{1}{2}(x_3 - x_2)^2 + 0.1(x_3 - 6)^2 + 0.9(x_3 - 4)^2 \qquad (24)$$

Differentiating and equating to zero gives the optimal x_3:

$$\frac{\partial V(3, IV, x_3, x_2)}{\partial x_3} = (x_3 - x_2) + 0.2(x_3 - 6) + 1.8(x_3 - 4) = 0 \qquad (25)$$

or

$$x_3 = \frac{x_2}{3} + 2.8 \qquad (26)$$

Table 7 The conditional expectation, $\hat{X}$, for the four decision stages.

ν	$\hat{X}_I$	$\hat{X}_{II}$	$\hat{X}_{III}$	$\hat{X}_{IV}$
0	3.59	4.48	4.40	4.63
1	3.54	4.37	4.39	4.48
2	3.53	4.29	4.41	4.40
3	3.53	4.24	4.44	4.35
4	3.53	4.19	4.46	4.32
5	3.53	4.16	4.48	4.29
10	3.53	4.08	4.54	4.25
15	3.53	4.05	4.55	4.23
20	3.53	4.05	4.55	4.23

Here and in the sequel Eq. (20) is disregarded, which will be fulfilled. Thus the optimal expected value V is

$$V(3, IV, x_2) = \frac{1}{9}\left[\frac{1}{2}(8.4 - 2x_2)^2 + 0.1(x_2 - 9.6)^2 + 0.9(x_2 - 3.6)^2\right] \quad (27)$$

The expected value at stage $t = 2$, state II is

$$V(2, II, x_2, x_1) = \frac{1}{2}(x_2 - x_1)^2 + 0.2(x_2 - 6)^2 + 0.5(x_2 - 5)^2 + 0.3(x_2 - 2)^2 \quad (28)$$

In the same way as before the optimal x_2 is

$$x_2 = \frac{1}{3}\left[x_1 + 8.6\right] \quad (29)$$

and the optimal expected value V:

$$V(2, II, x_1) = \frac{1}{9}\left[\frac{1}{2}(8.6 - 2x_1)^2 + 0.2(x_1 - 9.4)^2 + 0.5(x_1 - 6.4)^2 + 0.3(x_1 + 2.6)^2\right] \quad (30)$$

The expected value of stage $t = 2$, state III is, using optimal decision at stage $t = 3$ (see Eq. (27))

$$V(2, III, x_2, x_1) = \frac{1}{2}(x_2 - x_1)^2 + 0.6(x_2 - 4)^2 + 0.4(x_2 - 7)^2 + 0.4V(3, IV, x_2) \quad (31)$$

This gives the optimal x_2:

$$x_2 = \frac{1}{9.8}[3x_1 + 34.56] \quad (32)$$

and the optimal expected value V:

$$V(2, III, x_1) = \quad (33)$$
$$\frac{1}{9.8^2}\left\{\frac{1}{2}(34.56 - 6.8x_1)^2 + 0.6(3x_1 - 4.64)^2 + 0.4(x_1 - 34.04)^2 + 0.4\frac{1}{9}\left[\frac{1}{2}(13.2 - 6x_1)^2 + 0.1(3x_1 - 59.52)^2 + 0.9(3x_1 - 0.72)^2\right]\right\}$$

Finally, the expected value of stage $t = 1$, state I is, using the optimal decision at stage $t = 2$ (see Eqs. (30) and (33))

$$V(1, I, x_1) = \frac{1}{2}(x_1 - 4)^2 + 0.3(x_1 - 5)^2 + 0.7(x_1 - 2)^2 + 0.3V(2, I, x_1) + 0.7V(2, II, x_1) \quad (34)$$

This gives the optimal $x_1 = 3.5620$, Eqs. (29) and (32) give the optimal $x_2^{II} = 4.0540$ and $x_2^{III} = 4.6169$, respectively. Finally, x_2^{III} and Eq. (26) give the optimal $x_3 = 4.3390$.

Mathematical Programming Method: An alternative solution to the stochastic problem is as follows.

The expected value is

$$V(x_1, x_2^{II}, x_2^{III}, x_3) =$$
$$\frac{1}{2}(x_1 - 4)^2 + 0.3(x_1 - 5)^2 + 0.7(x_1 - 2)^2$$

$$
\begin{aligned}
&+ \quad 0.3\left[\frac{1}{2}(x_2^{II}-x_1)^2+0.2(x_2^{II}-6)^2+0.5(x_2^{II}-5)^2+0.3(x_2^{II}-2)^2\right] \\
&+ \quad 0.7\left[\frac{1}{2}(x_2^{III}-x_1)^2+0.6(x_2^{III}-4)^2+0.4(x_2^{III}-7)^2\right] \\
&+ \quad 0.7\cdot 0.4\left[\frac{1}{2}(x_3-x_2^{III})^2+0.1(x_3-6)^2+0.9(x_3-4)^2\right] \qquad (35)
\end{aligned}
$$

$\partial V/\partial x_1 = \partial V/\partial x_2^{II} = \partial V/\partial x_2^{III} = \partial V/\partial x_3 = 0$ gives

$$
\begin{array}{rrrrcl}
4.0x_1 & -0.3x_2^{II} & -0.70x_2^{III} & & = & 9.8 \\
-0.3x_1 & +0.9x_2^{II} & & & = & 2.58 \\
-0.7x_1 & & +2.38x_2^{III} & -0.28x_3 & = & 7.28 \\
 & & -0.28x_2^{III} & +0.84x_3 & = & 2.352
\end{array}
\qquad (36)
$$

respectively. The solutions are easily found as: $x_1 = 3.5620$, $x_2^{II} = 4.0540$, $x_2^{III} = 4.6169$ and $x_3 = 4.3390$. It is verified that these solutions satisfy Eq. (20). These are exactly the same as before.

4.2.3 Note (Choice of the Penalty Parameter)

It is noted that there are some discrepancies between these optimal solutions and the solutions obtained by the PHA, found in Table 7.

Table 8 compares the PHA solutions for various penalty r. In all cases $|\hat{X}^{\nu+1} - \hat{X}^{\nu}| < 5\cdot 10^{-8}$ at least. It is seen that for decreasing r the solutions get closer to the exact ones. For $r = 0.001$ the solutions are almost exactly the same as those obtained by the classical methods.

Table 8 The conditional expectation, $\hat{X}$, for the four decision stages.

r	$\hat{X}_I$	$\hat{X}_{II}$	$\hat{X}_{III}$	$\hat{X}_{IV}$
1.0	3.5304	4.0435	4.5557	4.2281
0.1	3.5572	4.0524	4.6079	4.3229
0.01	3.5615	4.0538	4.6156	4.3373
0.005	3.5618	4.0539	4.6165	4.3381
0.001	3.5620	4.0541	4.6168	4.3389

5 Discussion

It has been demonstrated that the scenario technique is feasible and allows the modeling of stochastic elements. The PHA is explained and illustrated by numerical examples, thus it is very easy to understand and implement. The only requirement is that the scenario subproblems, which are deterministic optimization problems, can be solved efficiently.

It will also be clear that the weak part of the PHA is the rate of convergence. The penalty parameter r must be chosen with great care. This further indicates that the number of scenarios should be kept at a reasonable small number.

However, there are no other easy ways. Dynamic Programming and Mathematical Programming have been illustrated (Section 4.2.2). These techniques were easy to apply and the optimal solution were found directly. But the reason for this success was that the inequality constraints were not active. In general these techniques are difficult (or even impossible) to apply in case of constrained problems.

Conclusion: scenario analysis and the PHA may be an attractive alternative to the classical methods.

Acknowledgements

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Stochastic Heat Storage Problem – Solved by the Progressive Hedging Algorithm

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Abstract

In the present study a problem concerning operation of a heat storage tank in connection with a combined heat and power (CHP) plant is considered. The heat storage is used to supply the district heating system when the CHP plant is producing electric power alone and also to optimally redistribute over time the required heat production. Stochasticity is assumed attached to the future power production and it is assumed that accurate predictions of the future heat consumption are available. A receding horizon idea is used. The problem is solved by the Progressive Hedging Algorithm (PHA), a new method to deal with multi-period optimization problems under uncertainty. The application of the method is explained in detail.

Key Words: Heat Storage; Combined Heat and Power; Stochastic Optimization; Scenario Analysis; Progressive Hedging Algorithm.

1 Introduction

One of the most efficient and economical ways of producing electric power (P) and district heat (Q) is by applying combined heat and power (CHP), as the utilization of the fired energy is much higher than producing electric power and district heat separately. The CHP plants are normally situated at a reasonable distance from the heat consumers (near larger cities). For a general description of CHP, see e.g., Horlock (1987).

Due to the fact that consumption of electricity and heat varies during the day, and not in parallel, the installation of a heat storage tank may be of great advantage. The heat storage technique is described in, e.g., Lorentzen (1993). Optimal operation of a heat storage has been discussed in, e.g., Mosbech (1982), Nielsen (1991), Ravn (1987) and Ravn and Rygaard (1992) where the problem is formulated and solved as a deterministic optimization problem.

In the present study stochasticity is assumed attached to the future power production and it is assumed that the future power production takes two values at each time, with a given probability. This is represented by a scenario tree. The introduction of stochastic elements makes the model much more realistic.

In this paper the problem is formulated as a stochastic optimization problem and the solution method used is the new Progressive Hedging Algorithm (PHA), first presented in Rockafellar and Wets (1991). This method is based on scenario analysis, which is a stochastic programming technique employing discrete scenarios with known probabilities, usually covering several time periods. This solution technique is embedded in a receding horizon structure to obtain a system which will be suitable for on-line operation.

The paper starts with a discussion of the CHP system and the problem specification. This is followed by a brief description of the scenario aggregation and the PHA. Then some technical and implementation aspects are discussed, such as how the scenario tree grows. Simulation experiments are performed in order to study the performance of the method. Finally the

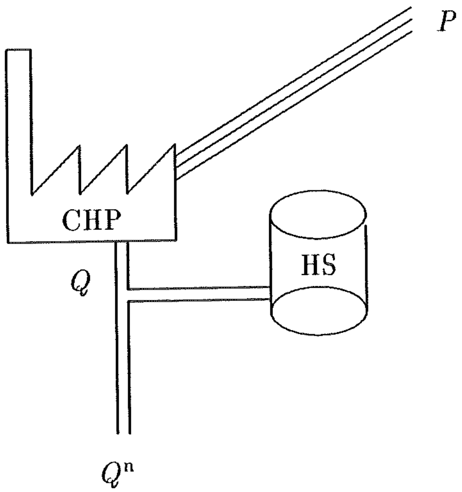

Fig. 1 A sketch of a combined heat and power (CHP) plant with heat storage (HS).

conclusions are drawn.

2 The Problem

2.1 Preliminaries

It is assumed in the present study that there is one CHP plant connected to the district heating network and that this CHP plant is the only heat producer in the network.

The electric power is fed to a power transmission network. There is one heat storage tank (HS) attached to the district heating network and it is

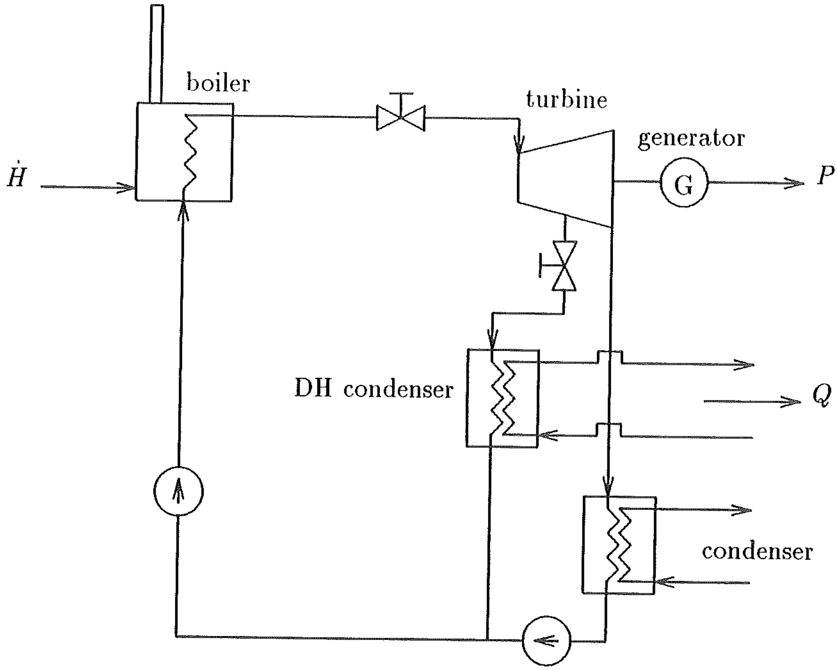

Fig. 2 Extraction unit.

located near the CHP plant. The heat storage tank is pressureless and the heat loss from it is assumed to be negligible. A sketch of a CHP plant with HS is shown in Fig. 1.

The steam power plant at hand is assumed to be an extraction unit, in which some or all of the steam can be extracted from the turbines before it is fully expanded, and condensed in a district heating condenser (see Fig. 2). With this arrangement, the heat and power outputs can be chosen independently within certain limits. Figure 3 shows a sketch of an operating region for a typical extraction unit.

When the unit is solely producing electric power, the operating point lies on the P axis, see Fig. 3, and all the steam is condensing in the low pres-

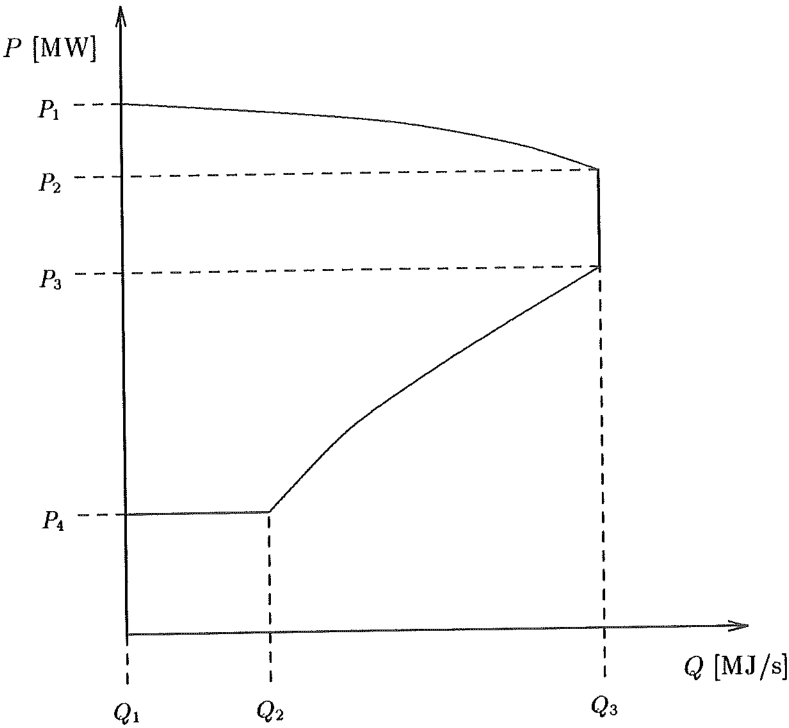

Fig. 3 A sketch of an operating region for an extraction unit.

sure condenser. In this mode the thermal efficiency is around 0.40. The lower right hand boundary of the region corresponds to all the heat of the condensation being transferred to the district heating water. Producing in this mode is called back-pressure mode and it is the most efficient mode of an extraction unit (with a thermal efficiency of about 0.85 or above). The upper boundary of the operating region is given by the boiler capacity.

The operating region, shown in Fig. 3, is defined by P_1, P_2, P_3, P_4, $Q_1 = 0$, Q_2, Q_3, and the upper and lower right hand (back-pressure) boundary, which are approximated by third order polynomials

$$Q = k_{u,3}P^3 + k_{u,2}P^2 + k_{u,1}P + k_{u,0} \tag{1}$$

and

$$Q = k_{b,3}P^3 + k_{b,2}P^2 + k_{b,1}P + k_{b,0} \tag{2}$$

respectively.

2.2 The Objective

The objective is to minimize the expected operating (fuel) cost over the T periods, $t = 1, \ldots, T$, within the planning horizon, i.e.

$$\min E\left[\sum_{t=1}^{T} g(t, Q_t)\right] \tag{3}$$

where

$$\begin{aligned} g(t, Q_t) &= C^{\text{fuel}}[a_t Q_t^2 + b_t Q_t + c_t] \qquad (4) \\ a_t &= \gamma_3 \\ b_t &= \gamma_1 + \gamma_4 P_t + \gamma_6 P_t^2 \\ c_t &= \gamma_0 + \gamma_2 P_t + \gamma_5 P_t^2 + \gamma_7 P_t^3 \end{aligned}$$

$E[\sum g(t, Q_t)]$ is the expected fuel cost within the planning horizon considered, i.e., the fuel price, C^{fuel}, times the fuel consumption. The fuel consumption is approximated by a second order polynomial in Q_t, where the coefficients are polynomials in P_t, see Nielsen (1991), the γ coefficients are unit and supply temperature dependent constants.

The minimization (optimization) is subject to the following constraints:

$$\begin{aligned} E_{t+1} &= E_t + (Q_t - Q_t^{\text{n}})\Delta t && (5) \\ E^{\text{min}} &\leq E_t \leq E^{\text{max}} && (6) \\ Q^{\text{min}} &\leq Q_t \leq Q^{\text{max}}(P_t) && (7) \\ E_1 &= E^{\text{start}} && (8) \\ E_{T+1} &= E_t^{\text{final}} && (9) \\ P^{\text{min}} &\leq P_t \leq P^{\text{max}} && (10) \end{aligned}$$

- E_t denotes the energy content in the storage at the beginning of the time interval number t,
- E^{min} and E^{max} are the lowest and highest value that E_t can take, determined in such a way that the physical limits of the heat storage are not violated (it is also possible to incorporate uncertainty of the predictions of the future heat consumption in these limits),
- E^{start} is the initial storage content,
- E^{final} is the final storage content,
- Δt is the time interval (with no loss of generality it is hereafter assumed equal to 1 hour),
- Q_t is the amount of heat produced at the extraction unit in the time interval,
- Q^{min} and $Q^{\text{max}}(P_t)$ are physical limits on Q_t, $Q^{\text{min}} = 0$, and Q^{max} depends on P_t, see Fig. 3,
- Q_t^{n} is a prediction of the future heat consumption, (assumed known in this study),
- P^{min} and P^{max} are physical limits on P_t, (on Fig. 3 $P^{\text{min}} = P_4$ and $P^{\text{max}} = P_1$).

The dimension of E_t is [MWh], Q_t is in [MJ/s], and P_t is in [MW] ([MJ/s] is the same as [MW], but [MJ/s] is commonly used to distinguish between heat and electric power). In the sequel vector notation is sometimes used, such that, e.g., Q means $(Q_1, \ldots, Q_T)$.

2.3 The Stochasticity

It is assumed in the present study that the future power production (in the next few time steps) is known with some uncertainty and that the future heat consumption (for the time being) is fairly well predicted by suitable models, see e.g., Sejling (1993).

Furthermore, it is assumed that the stochasticity in the future power production can be described by a finite number of scenarios, where each scenario is assumed to have a finite, known probability of occurrence.

As P_t is stochastic, the parameters in the objective function, Eq. (4), and the maximum heat production, Eq. (7), are also stochastic. In the next sections it will be shown how the scenario analysis and the Progressive Hedging Algorithm are used to decompose the stochastic optimization problem into a finite number of deterministic subproblems and then, in an iterative way, find implementable and admissible (hence, feasible) policy for the heat production in the next few time steps.

The intention is then to implement the solution for the first time step, then wait for the outcome of the power production (to see which way in the scenario tree to go) and then repeat the computation at the next time step (a receding horizon type of control). This will be discussed further in Section 5.

3 Scenario Analysis and Scenario Aggregation

As previously mentioned the Progressive Hedging Algorithm is based on scenario analysis. The information structure in the scenario analysis can be represented in terms of a tree, with each scenario corresponding to a path from the root of the tree to a leaf. Fig. 4 illustrates a scenario tree with four time steps, $T = 4$, and where the stochastic variable, e.g., the power production, can take two values each time. Hence, there are sixteen (2^4) scenarios. Each scenario is assumed to have a fixed, known probability of occurrence, p_s.

The scenarios are aggregated into scenario bundles, where the scenarios in each bundle share a common history, until that time. In Fig. 4, there are fifteen scenario bundles, where the first seven are numbered in Roman. For instance, scenarios $s_1 - s_4$ in bundle IV share the decisions made at time

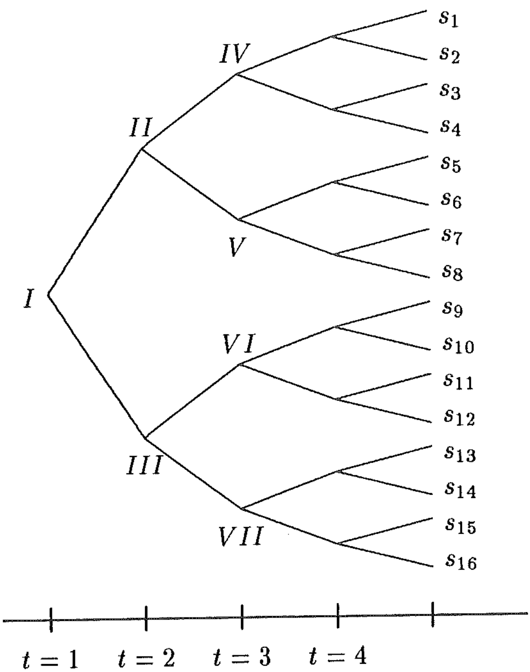

Fig. 4 A scenario tree with sixteen scenarios.

$t = 1$, $t = 2$, and $t = 3$ and the outcome of the stochastic events at time $t = 1$ and $t = 2$.

The aim of the Progressive Hedging Algorithm is to find implementable and admissible decisions (a policy). Admissible means that the constraints Eqs. (5) - (10) are not violated. The requirement of implementability, on the other hand, is stated as follows: *The decision is implementable if it solely depends on the scenario bundle, but not on the particular scenario in the bundle.* I.e., the decision to be made at $t = 1$ for scenarios $s_1 - s_{16}$ must not depend on the particular scenario, as it is not known, at that time, which scenario is to be followed. In other words only *one* decision, common to all 16 scenarios, is to be made at time $t = 1$. In the same way, the decision to be made at time $t = 3$ for scenarios $s_1 - s_4$ solely depends on the scenario bundle IV.

4 Progressive Hedging Algorithm

As previously mentioned, the PHA is proposed by Rockafellar and Wets and first presented in Rockafellar and Wets (1991). In this paper the algorithm is briefly described and the readers are referred to the references Rockafellar and Wets (1991), Helgason and Wallace (1991) or Palsson and Ravn (1993), for the details.

4.1 The Algorithm

The basic algorithm is as follows, see also Fig. 5. The different elements will be specified below.

STEP 0: (Initialize)

- Find the optimal solution $(\boldsymbol{Q}(s))$ to any one of the deterministic scenario subproblems
- Compute $\hat{\boldsymbol{Q}}^0(s) = \sum p_s \boldsymbol{Q}(s) / \sum p_s$ (conditional expectation)
- Set $\boldsymbol{W}^0(s) = 0$, the price system
- Set $\nu = 0$, the iteration number

REPEAT

STEP 1: Solve for each scenario, see Section 4.2

$$\min_{Q_t(s)} \sum_{t=1}^{T} \left\{ g(t, Q_t(s)) + Q_t(s) W_t^\nu(s) + \frac{1}{2} r \cdot (Q_t(s) - \hat{Q}_t^\nu(s))^2 \right\} \tag{11}$$

subject to the constraints in Eqs. (5) – (10) $\Rightarrow \boldsymbol{Q}^{\nu+1}(s)$

STEP 2: Compute $\hat{\boldsymbol{Q}}^{\nu+1}(s)$, see Section 4.3

STEP 3: Update

$$\boldsymbol{W}^{\nu+1}(s) = \boldsymbol{W}^\nu(s) + r \cdot (\boldsymbol{Q}^{\nu+1}(s) - \hat{\boldsymbol{Q}}^{\nu+1}(s)) \tag{12}$$

set $\nu = \nu + 1$

UNTIL $(m^{\nu+1} < \epsilon$ or $\nu > N)$

where

$$m^{\nu+1} = E\left[\sum_{t=1}^{T} \left\{ (\hat{Q}_t^{\nu+1}(s) - \hat{Q}_t^\nu(s))^2 + \frac{1}{r^2} (W_t^{\nu+1}(s) - W_t^\nu(s))^2 \right\} \right] \tag{13}$$

where W is a price system, r is a penalty parameter, ϵ is a small positive number, and N is a maximum number of iterations.

4.2 The Deterministic Subproblem

The PHA decomposes the stochastic optimization problem into a finite number of deterministic subproblems one for each scenario, which then can be solved by any suitable method (STEP 1 in Section 4.1). In the present study use is made of the Forward Maximum Principle described in Ravn (1987).

This method, as it is proposed in Ravn (1987), assumes that the state vector is one-dimensional (here the storage content E, i.e., one storage) and

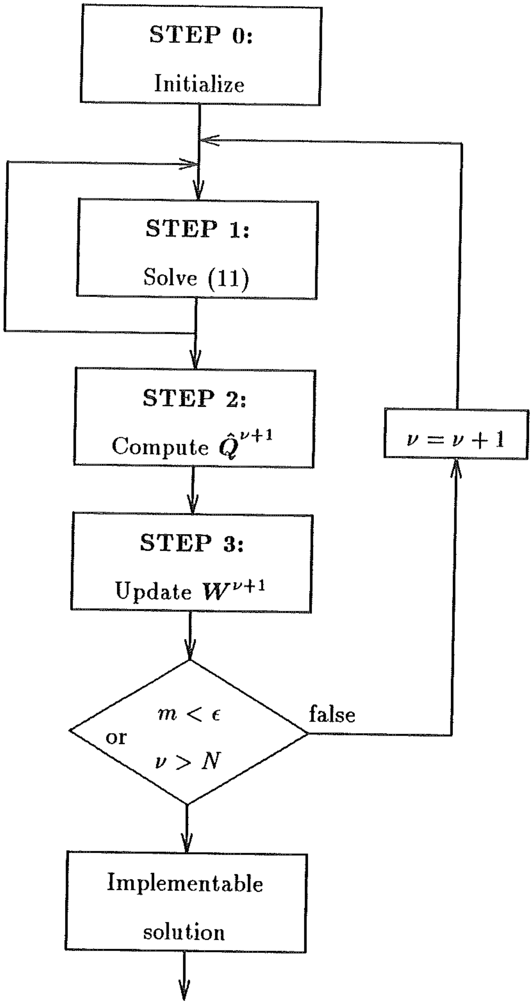

Fig. 5 The Progressive Hedging Algorithm.

the control is m-dimensional (here the heat production at the CHP block, Q_t, i.e., $m = 1$). The objective function is assumed to be separable in the states and the controls (E and Q). The state-part of the objective function is assumed to be concave (in the present problem the objective function is minimized, hence concave is to be red as convex) and continuously differentiable and the control-part is assumed to be strictly concave.

All these assumptions are fulfilled in the present study. The problem is then solved by the maximum principle in a forward manner, see Ravn (1987) for the details. In the simulation study use is made of a computer program (called ANSGAR), that makes use of this method, for solving the deterministic subproblems.

The deterministic problems are to be solved, for each scenario, both in STEP 0 and in STEP 1. In the former case the problems to be minimized are convex second order polynomials, see Eq. (4)

$$\sum_{t=1}^{T} \left\{ a_t Q_t^2 + b_t Q_t + c_t \right\} \tag{14}$$

On the other hand, in the latter case, terms that ensure the implementability and convergence of the algorithm are incorporated. The objective function is in this case

$$\sum_{t=1}^{T} \left\{ a_t Q_t^2 + b_t Q_t + c_t + W_t Q_t + \frac{1}{2} r \cdot (Q_t - \hat{Q}_t)^2 \right\} \tag{15}$$

which can be rewritten as

$$\sum_{t=1}^{T} \left\{ \left(a_t + \frac{1}{2} r \right) Q_t^2 + \left(b_t + W_t + r\hat{Q}_t \right) Q_t + \left(c_t + \frac{1}{2} r \hat{Q}_t^2 \right) \right\} \tag{16}$$

and then used in the same optimization procedure as before. Equations (14) and (16) are equivalent for $r = 0$ and $W_t = 0$.

4.3 The Conditional Expectation (The Aggregation)

As previously mentioned, each scenario has a fixed probability of occurrence, p_s. In order to find an implementable decision, the conditional expectation, $\hat{Q}_t$, is calculated (cf. STEP 1 in Section 4.1). The expectation is conditioned on the information available at that time, viz., the particular scenario bundle.

The conditional expectation can be expressed as

$$\hat{Q}_t(s) = \sum_{\text{bundle}} p_s Q_t(s) / \sum_{\text{bundle}} p_s \tag{17}$$

4.3.1 Example:

For scenario bundle VII in Fig. 4 the conditional expectation is written as

$$\begin{aligned}\hat{Q}_{VII} &= \sum_{VII} p_s Q_{t=2}(s) / \sum_{VII} p_s \\ &= [Q_{t=2}(s_{13})p_{s,13} + Q_{t=2}(s_{14})p_{s,14} + Q_{t=2}(s_{15})p_{s,15} \\ &\quad + Q_{t=2}(s_{16})p_{s,16}] / [p_{s,13} + p_{s,14} + p_{s,15} + p_{s,16}]\end{aligned} \tag{18}$$

or more conveniently

$$\begin{aligned}\hat{Q}_{VII} &= \sum_{VII} p'_s Q_{t=2}(s) \\ &= Q_{t=2}(s_{13})p'_{s,13} + Q_{t=2}(s_{14})p'_{s,14} + Q_{t=2}(s_{15})p'_{s,15} + Q_{t=2}(s_{16})p'_{s,16}\end{aligned} \tag{19}$$

where p'_s is the normalized probability $p'_s = p_s / \sum_{VII} p_s$, and $\sum_{VII} p'_s = 1$.

□

4.4 The Price System

The vector $\boldsymbol{W}$ in Eq. (11) may be interpreted as a Lagrange multiplier relative to the implementability constraint in Eq. (17). It is obvious, from the update equation Eq. (12), that when implementability is achieved, then W has converged to a fixed value. The number of elements in W is equal to the number of scenarios times the number of time steps.

4.5 The Penalty Parameter

The main task of the parameter r is to ensure convergence of the algorithm. It is easily observed that it has to be strictly positive (otherwise, there is no update of W and Eq. (11) is always equal to the initial deterministic subproblem). On the other hand, it can be difficult to choose the value of r. In Helgason and Wallace (1991) it is pointed out that the choice of the parameter r is of vital importance and the conclusion drawn there is that the penalty should be as small as possible, provided it is large enough to guarantee convergence. Similar conclusions are drawn in Palsson and Ravn (1993) where the algorithm is tested on simple problems.

4.6 The Error Measure

In order to find a criterion for when to terminate the algorithm the error measure in Eq. (13) is used, see Helgason and Wallace (1991). I.e., the error measure is the expected value with respect to the distribution of the scenarios. This measure consists of two terms, the first term determines the convergence of the implementable policy and the second term measures the implementability (insert Eq. (12) in Eq. (13)).

5 Growing Trees and Implementation Aspects

In this section it is described how the scenario tree is constructed and how it is updated according to the receding horizon control idea. The whole optimization procedure is then described by a flow chart. It is also shown

Table 1 The power production matrix.

S	$t = 1$	$t = 2$
s_1	P_{AB}	P_{BD}
s_2	P_{AB}	P_{BE}
s_3	P_{AC}	P_{CF}
s_4	P_{AC}	P_{CG}

how the scenarios are, advantageously, represented by matrices.

With no loss of generality, for the time being, assume that the optimization horizon is two time steps (hours), and that the power production can take two values each time, hence there are four scenarios, see Fig. 6.

At A, in Fig. 6, the objective is to determine the heat productions, Q_t, in the next two hours, such that the expected fuel cost (for the whole decision period, here 2 hours) is minimized.

The power production can, for the first decision horizon, scenarios s_1 - s_4, advantageously be written as a matrix, see Table 1. I.e., the power production in the first time step is P_{AB} for scenarios s_1 and s_2 and P_{AC} for scenarios s_3 and s_4, for the second time step, the power production are P_{BD}, P_{BE}, P_{CF}, and P_{CG} for the four scenarios, respectively.

The probability attached to this matrix (the four scenarios) is assumed to be as in Table 2 (the notation p'_s is adapted from Section 4.3 and means the normalized probability).

As the maximum heat production and the coefficients in the objective function depend on the power production (see Section 2.3) they can also be expressed as matrices, see Table 4 and 5 in Appendix B, respectively. Hence, there are four deterministic optimization problems to be solved. The opti-

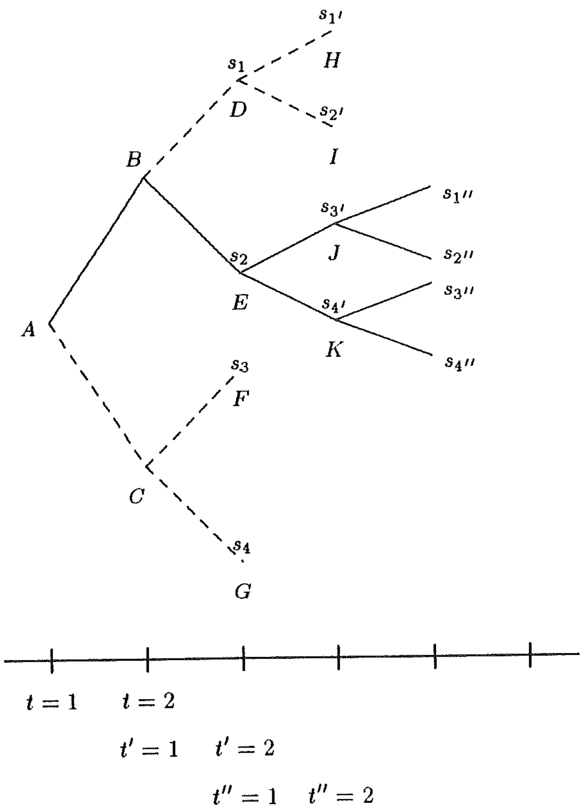

Fig. 6 Growing scenario tree.

Table 2 The probability matrix.

S	$t=1$	$t=2$		S	$t=1$	$t=2$
s_1	$p_{s,1}$	$p'_{s,1}$		s_1	$(1-\alpha)^2$	$(1-\alpha)$
s_2	$p_{s,2}$	$p'_{s,2}$	=	s_2	$(1-\alpha)\alpha$	α
s_3	$p_{s,3}$	$p'_{s,3}$		s_3	$\alpha(1-\alpha)$	$(1-\alpha)$
s_4	$p_{s,4}$	$p'_{s,4}$		s_4	α^2	α

mal solution is written in Table 6, in Appendix B.

The PHA finds an implementable decision to be made at $t = 1$. This solution is then implemented. Then it is observed whether the power production, in the next hour, actually was P_{AB} or P_{AC}. Assume that is was P_{AB}, hence the scenarios s_3 and s_4 are of no further interest. Now four scenario paths are added to D and E (labeled $s_{1'}$ - $s_{4'}$ in Fig. 6).

These new paths are given by the recursive equation (from time $T-1$ to T)

$$P_T = (1-\lambda)P_{T-1} + \lambda \cdot \text{profile}_T \tag{20}$$

where profile_T is the diurnal (or weekly) power production profile, and

$$\lambda = \begin{cases} \beta_1, & \text{with probability } p = \alpha \\ \beta_2, & \text{with probability } p = (1-\alpha) \end{cases} \tag{21}$$

where $\alpha \in]0,1[$ and $\beta_1,\ \beta_2 \in [0,1]$.

Hence, the power production matrix is updated, see Table 3, and so on. The whole procedure can be described by the flow chart in Fig. 7.

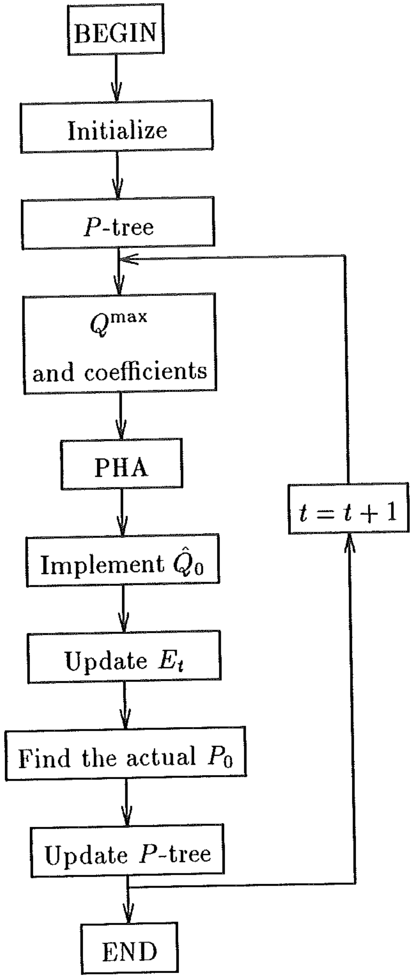

Fig. 7 Flow chart of the optimization procedure.

Table 3 The power production matrix updated.

S	$t=1$	$t=2$
s_1	P_{AB}	P_{BD}
s_2	P_{AB}	P_{BE}
s_3	P_{AC}	P_{CF}
s_4	P_{AC}	P_{CG}

$\Rightarrow$

S	$t=1'$	$t=2'$
$s_{1'}$	P_{BD}	P_{DH}
$s_{2'}$	P_{BD}	P_{DI}
$s_{3'}$	P_{BE}	P_{EJ}
$s_{4'}$	P_{BE}	P_{EK}

Remark: The recursion in Eq. (20) is just one of many that could be proposed. The main idea is that the new scenario paths are functions of the scenario paths they are to be connected to and also of some profile, and that there are two stochastic outcomes. The T-step prediction of the power production could, advantageously, be incorporated in the equation.

6 The Simulation Study

In order to study the performance of the Progressive Hedging Algorithm some simulations were performed. The decision horizon used is $T = 6$, and the power production can take two values each time, hence there are 64 (2^6) scenarios within the decision horizon.

6.1 The Data

Most of the data originate from Nielsen (1991). The data are used directly whenever possible. Minor adaption and supplements have been made, but in general the model presented here is representative for the so called Block 2 in the CHP plant in Esbjerg, Denmark. In this section the most relevant data are described. The remaining data are listed in the appendix.

Power Production Profile The diurnal power production profile used is shown in, e.g., Fig. 8. The parameters used in the recursive equation (see Eqs. (20) and (21)) are chosen as $\alpha = 0.2$, $\beta_1 = 0.4$, and $\beta_2 = 0.7$.

Heat Consumption The heat consumption, Q_t^n, e.g., shown in Fig. 9, are the real heat consumption observed at Vestkraft in Esbjerg, 01-03.02.92 (reduced by 35 MJ/s, in order to adjust consumption to the extraction unit data).

Final Heat Storage Content As shown in Eq. (9), the minimization is subject to a constraint on the desired content of the heat storage at the end of the decision horizon, viz., $E_{T+1} = E_t^{\text{final}}$. The final heat storage content, E_t^{final}, used in this study follows the diurnal profile shown in, e.g., Fig. 10. The profile is our own construction.

PHA Parameters In this study no emphasis is on optimizing the penalty parameter, hence, it is chosen to $r = 1$. ϵ is 10^{-3} and the maximum number of iterations is $N = 300$.

6.2 The Results

In Fig. 6 it is shown how the scenario tree is reconstructed each time information of the stochastic output has been gained. In the following simulation four cases are considered, one where the stochastic outcome is determined by uniformly distributed random numbers, this is compared to simulations where the most and least probable scenarios, respectively, are followed all the time and furthermore compared to simulations obtained for the deterministic case (the stochastic variable is replaced with its expected value), i.e.,

Case 1 the stochastic outcome,

Case 2 the most probable scenario,

Case 3 the least probable scenario,

Case 4 the deterministic case.

Case 1 The simulated power production is shown in Fig. 8, for 72 hours. The power production was generated in each time period using a random number generator to determine whether β_1 or β_2 should be selected. As shown, the simulated power production is close to the profile since the β_2 parameter, in Eq. (21), is more probable than β_1. The simulated heat production is shown in Fig. 9, and furthermore the heat consumption and the maximum heat production are also shown. It is noted that the heat demand can not be satisfied by the extraction unit alone in a period around noon. In this period the storage is discharged and then recharged again in the evening. The variations in $Q^{\max}$ are due to the dependency of the power production, see Figs. 3 and 8. The non-smoothness in the simulated Q curve is due to the fact that the parameters in the objective function in Eq. (4) depend on the power production. It is observed that the right hand side of the restriction in Eq. (7), $Q_t \geq Q^{\max}(P_t)$, is fulfilled. Most of the time actually $Q_t < Q^{\max}(P_t)$, time $t = 36$ is one of the few exceptions.

Figure 10 shows the simulated content of the storage and the desired storage content at the end of the decision period. It is noted that there is some difference between these curves. The difference is due the receding horizon principle, i.e., the optimization is subject to the constraint in Eq. (9), and at the next time another desired final heat storage content is used.

Case 2 In this case the most probable scenario is chosen ($\lambda = \beta_2$) (see Fig. 6). As shown is Fig. 11, 12 and 10 the results are similar to the results in Case 1. It is observed that the right hand side of Eq. (7) is now fulfilled as equality, i.e., $Q_t = Q^{\max}(P_t)$, in more periods, particular around noon (time periods around $t = 12$, $t = 36$ and $t = 60$). This indicates that the production capacity is "expensive", and that the two products, viz., heat and power, "compete" for using the production capacity.

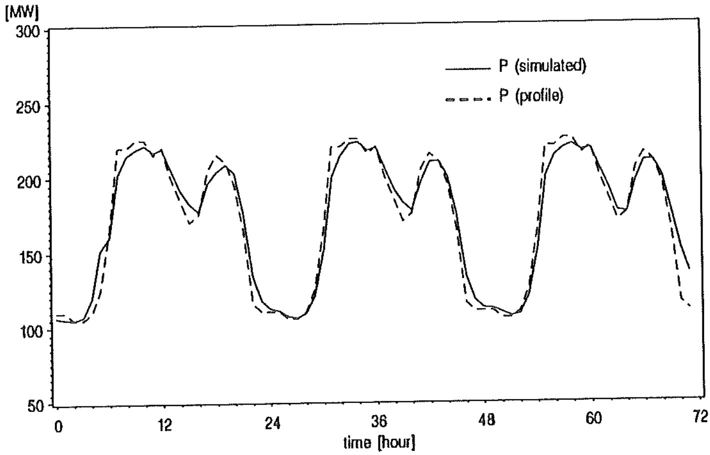

Fig. 8 The simulated power production and the power production profile. Case 1.

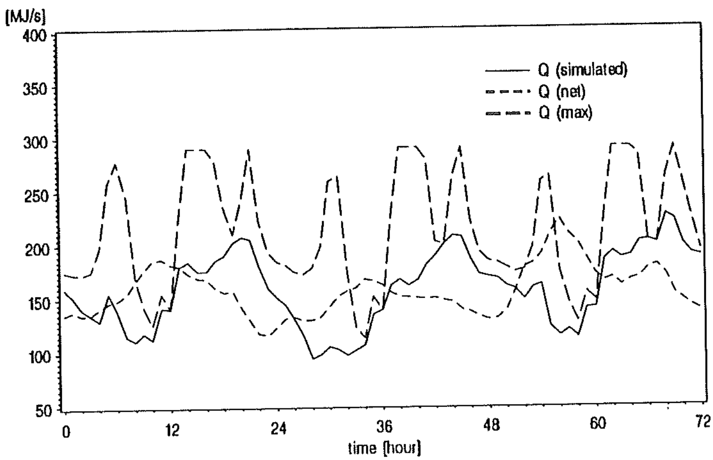

Fig. 9 The simulated heat production, the heat consumption, Q^n, and the maximum heat production Q^{max}. Case 1.

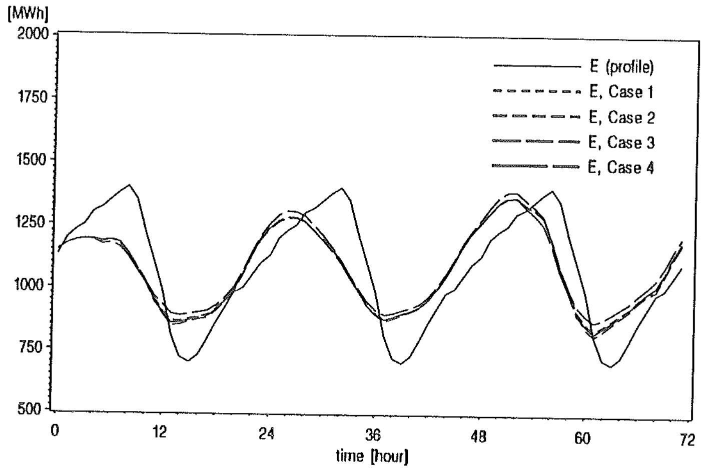

Fig. 10 The final heat storage content profile (own construction), E_t^{final} and the simulated content of the storage E_t. Case 1, 2, 3 and 4.

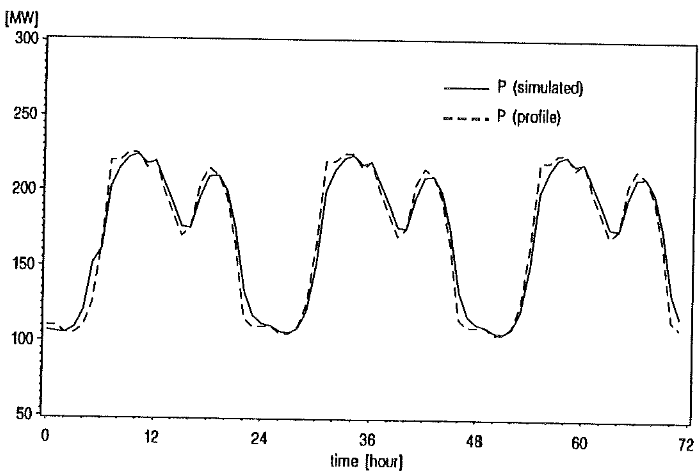

Fig. 11 The simulated power production and the power production profile. Case 2.

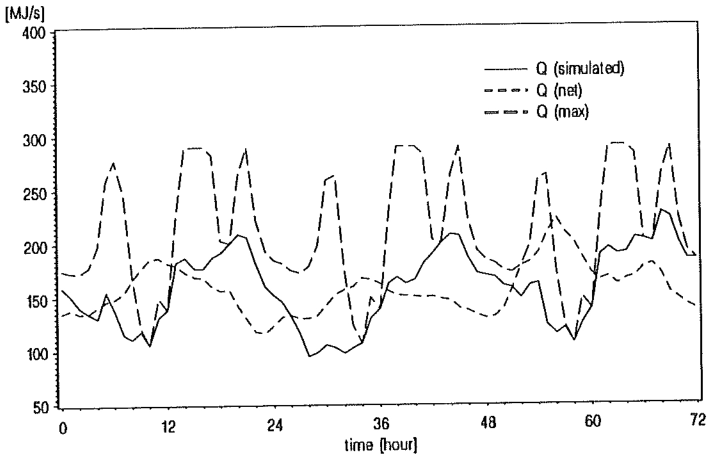

Fig. 12 The simulated heat production, the heat consumption, Q^n, and the maximum heat production Q^{max}. Case 2.

Case 3 In this case the least probable scenario is chosen each time, and here there is more weight on the last power production than the profile (i.e., $\lambda = \beta_1 = 0.4$) this is clearly observed in Fig. 13. Q_t is more smooth in this case, see Fig. 14. This is mainly because $Q^{max}(P_t)$ is higher, and in fact $Q_t < Q^{max}(P_t)$, cf. Eq. (7), in all periods. However, the simulated heat storage content, E_t, is similar as before, see Fig. 10.

Case 4 The deterministic case is obtained by replacing the stochastic variable, λ in Eqs. (20) and (21), with its expected value, $\lambda = \alpha\beta_1+(1-\alpha)\beta_2$. The simulated power production in this case is similar, as expected, to the simulated power production in Case 1, see Fig. 15 and 8. However, it is observed that the simulated heat production is at its maximum, $Q_t = Q^{max}(P_t)$, in longer periods than in the stochastic case (Case 1), compare Figs. 16 and 9. This indicates that the solution in the stochastic case is more cautious in keeping clear of the maximum limit, since the future is uncertain. The simulated heat storage content, E_t, is similar as before, see Fig. 10.

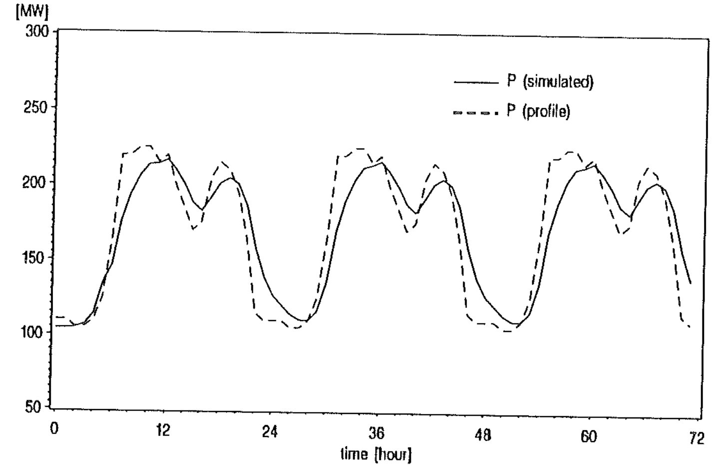

Fig. 13 **The simulated power production and the power production profile. Case 3.**

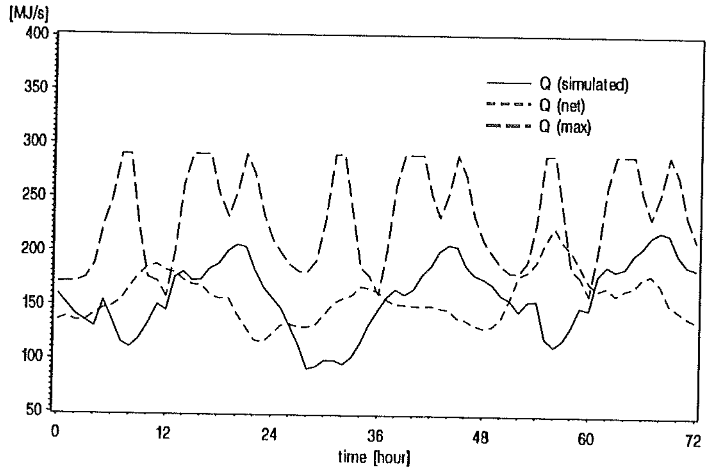

Fig. 14 **The simulated heat production, the heat consumption, Q^{n}, and the maximum heat production Q^{max}. Case 3.**

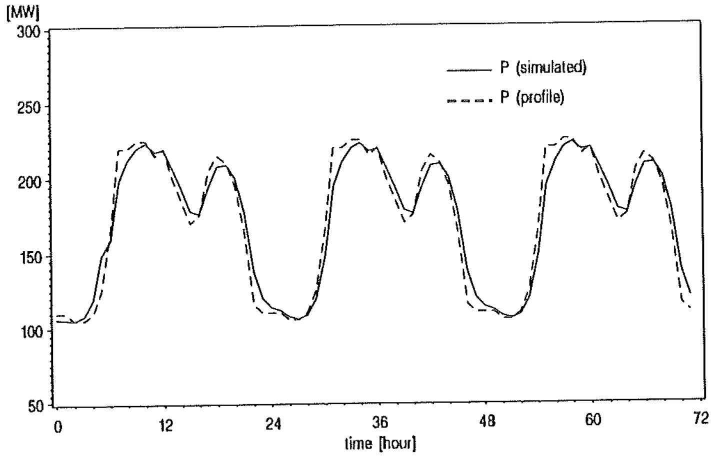

Fig. 15 The simulated power production and the power production profile. Case 4.

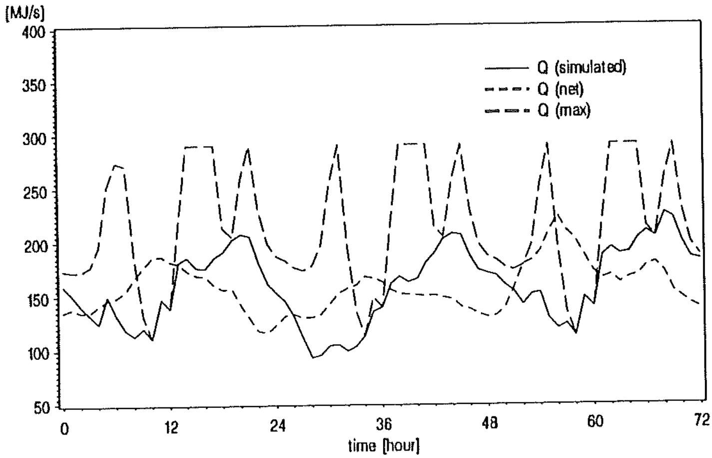

Fig. 16 The simulated heat production, the heat consumption, Q^n, and the maximum heat production Q^{max}. Case 4.

7 Summary and Conclusions

This paper discusses the optimal operation of a heat storage tank connected to a CHP plant. The problem is formulated as a stochastic optimization problem where the stochasticity is assumed attached to the power production. The problem is solved by scenario analysis and the Progressive Hedging Algorithm, where the stochastic problem is decomposed into deterministic subproblems.

A new feature in the paper is that the scenario analysis and PHA are used in a receding horizon manner, i.e., the scenario tree is updated each time the stochastic outcome is realized.

It is assumed, in the present study, that the stochasticity is attached to the power production, this implies stochastic parameters in the objective function and stochastic maximum on the heat production. Although the future power production may be known with good accuracy, emergency situations may occur and these have consequences for the acceptable ranges of heat productions.

The future heat demand is, of course, also stochastic, this could be taken into account in a similar way by representing the uncertainty by scenarios and then solve a "3-dimensional" PHA. This will have more obvious consequences in the simulations. The 3-dimensional PHA will be the subject of a later publication.

The four cases have demonstrated the feasibility of incorporating stochastic elements in the modeling of heat storage operation. This permits more realistic modeling.

The paper concludes by two observations.

By studying Fig. 10, firstly it is observed that the heat content in the storage tends to be delayed in relation to the desired profile. It is suspected that this is due to the horizon applied, viz., 6 hours, being too short. However, it should be noted, that although it is conceptually straightforward to increase the horizon, it is computationally demanding. This is because the number of individual deterministic scenario problems grows exponentially. Thus, while here $2^6 = 64$ scenarios are solved in each iteration, a doubling of the horizon to 12 periods calls for solution of $2^{12} = 4096$ scenarios. This is a main difficulty in optimization of stochastic systems.

Secondly, it is observed that the heat production characteristics are different in the four cases presented, cf. Figs. 9, 12, 14, and 16. In particular this is seen with respect to the upper heat production limit, see Eq. (7). However, the variation in the heat storage content is much less outspoken, cf. Fig. 10. This in fact illustrates one of the rationalities for having a heat storage, viz., the possibility of smoothing production over time and - as demonstrated here - stochastic variations.

Acknowledgements

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A Data used in the Simulation Study

A.1 The Operating Region

The operating region used in this simulation study is found in Nielsen (1991) and it is from Block 2 in Esbjerg CHP. The notation is according to Fig. 3.

$$
\begin{aligned}
P_1 &= 237.9 \text{ MW} \\
P_2 &= 194.9 \text{ MW} \\
P_3 &= 166.9 \text{ MW} \\
P_4 &= 50.0 \text{ MW} \\
Q_1 &= 0.0 \text{ MJ/s} \\
Q_2 &= 96.9 \text{ MJ/s} \\
Q_3 &= 290.0 \text{ MJ/s}
\end{aligned}
$$

The upper boundary, see Eq. (1)

$$
\begin{aligned}
k_{u,3} &= -3.6680439 \cdot 10^{-4} \\
k_{u,2} &= 2.1053620 \cdot 10^{-1} \\
k_{u,1} &= -4.6161617 \cdot 10^{1} \\
k_{u,0} &= 4.005 \cdot 10^{3}
\end{aligned}
$$

The back-pressure boundary, see Eq. (2)

$$
\begin{aligned}
k_{b,3} &= -1.4866497 \cdot 10^{-5} \\
k_{b,2} &= 9.4837165 \cdot 10^{-3} \\
k_{b,1} &= 1.707279 \cdot 10^{-1} \\
k_{b,0} &= 6.65 \cdot 10^{1}
\end{aligned}
$$

A.2 The Objective Function

See Eq. (4), the supply temperature is assumed to be 95 °C.

$$\begin{aligned}
\gamma_0 &= 1.105135 \cdot 10^{2} \\
\gamma_1 &= 2.035534 \\
\gamma_2 &= 9.214590 \\
\gamma_3 &= 1.35074 \cdot 10^{-3} \\
\gamma_4 &= -9.04633 \cdot 10^{-3} \\
\gamma_5 &= -9.65249 \cdot 10^{-3} \\
\gamma_6 &= 2.18858 \cdot 10^{-5} \\
\gamma_7 &= 2.88695 \cdot 10^{-5}
\end{aligned}$$

$$C^{\text{fuel}} = 13 \text{ DKK/GJ}$$

A.3 The Storage Content

$$\begin{aligned}
E^{\text{max}} &= 2111 \text{ MWh (constant)} \\
E_t^{\text{min}} &= (Q_{t+1}^{\text{n}} + Q_{t+2}^{\text{n}})\text{MWh}
\end{aligned}$$

B Matrices

Table 4 The $Q^{\max}$ matrix.

S	$t=1$	$t=2$
s_1	$Q_{AB}^{\max}$	$Q_{BD}^{\max}$
s_2	$Q_{AB}^{\max}$	$Q_{BE}^{\max}$
s_3	$Q_{AC}^{\max}$	$Q_{CF}^{\max}$
s_4	$Q_{AC}^{\max}$	$Q_{CG}^{\max}$

Table 5 The coefficient matrices.

S	$t=1$	$t=2$
s_1	b_{AB}	b_{BD}
s_2	b_{AB}	b_{BE}
s_3	b_{AC}	b_{CF}
s_4	b_{AC}	b_{CG}

S	$t=1$	$t=2$
s_1	c_{AB}	c_{BD}
s_2	c_{AB}	c_{BE}
s_3	c_{AC}	c_{CF}
s_4	c_{AC}	c_{CG}

Table 6 The solutions to the deterministic problems.

S	$t=1$	$t=2$
s_1	$Q_{1,1}$	$Q_{1,2}$
s_2	$Q_{2,1}$	$Q_{2,2}$
s_3	$Q_{3,1}$	$Q_{3,2}$
s_4	$Q_{4,1}$	$Q_{4,2}$

B.1 PHA Matrices

Table 7 The price system and the implementable solutions.

S	$t=1$	$t=2$
s_1	$W_{1,1}$	$W_{1,2}$
s_2	$W_{2,1}$	$W_{2,2}$
s_3	$W_{3,1}$	$W_{3,2}$
s_4	$W_{4,1}$	$W_{4,2}$

S	$t=1$	$t=2$
s_1	$\hat{Q}_{1-4,1}$	$\hat{Q}_{1-2,2}$
s_2	$\hat{Q}_{1-4,1}$	$\hat{Q}_{1-2,2}$
s_3	$\hat{Q}_{1-4,1}$	$\hat{Q}_{3-4,2}$
s_4	$\hat{Q}_{1-4,1}$	$\hat{Q}_{3-4,2}$

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