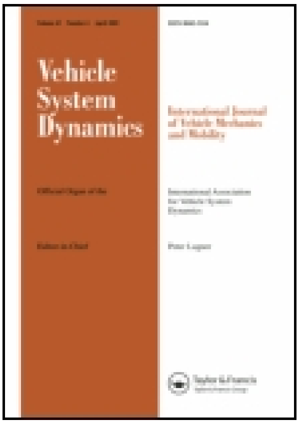

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Multiple attractors and critical parameters and how to find them numerically: the right, the wrong and the gambling way

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Multiple attractors and critical parameters and how to find them numerically: the right, the wrong and the gambling way

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In recent years, several authors have proposed 'easier numerical methods' to find the critical speed in railway dynamical problems. Actually, the methods do function in some cases, but in most cases it is really a gamble. In this article, the methods are discussed and the pros and contras are commented upon. I also address the questions when a linearisation is allowed and the curious fact that the hunting motion is more robust than the ideal stationary-state motion on the track. Concepts such as 'multiple attractors', 'subcritical and supercritical bifurcations', 'permitted linearisation', 'the danger of running at supercritical speeds' and 'chaotic motion' are addressed.

Keywords: bifurcation analysis; stability analysis; state estimation

1. Introduction

The calculation of critical parameters leads to the mathematical problem of finding multiple solutions to a nonlinear initial value problem. The mathematical problem is an *existence problem*. In order to solve this problem, we shall use a geometrical description of the solutions, and the practical problem is to then find the appropriate solutions in the state space.

The *state* of a dynamical multibody system at time t_0 is defined as the pair of the position and the velocity vectors of each and every body in the physical Euclidean space at time t_0. In a three-dimensional physical space, this definition yields six scalars for the description of the state of a body. For a multibody system with N bodies, the $6N$ scalars define the state of the multibody system at time t_0 in the $6N$-dimensional *state space*. The solution of the dynamical system $\mathbf{x}(t)$ with a given set of appropriate $6N$ initial values will, as a function of time t, create a curve in the state space with time t as the curve parameter. This curve is called a *trajectory* or often a *phase trajectory*. If the first derivative $\mathrm{d}\mathbf{x}/\mathrm{d}t$ is defined for all t, then the trajectory is uniquely defined. In that case, trajectories can never cross each other or themselves. If the second derivative $\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$ exists for all t, then the system is called *smooth*; otherwise it is *non-smooth*. We consider dissipative dynamical systems on the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \lambda), \tag{1}$$

where $\mathbf{F}$ is a vector function of the state variables $\mathbf{x}(t)$ and the set of parameters λ.

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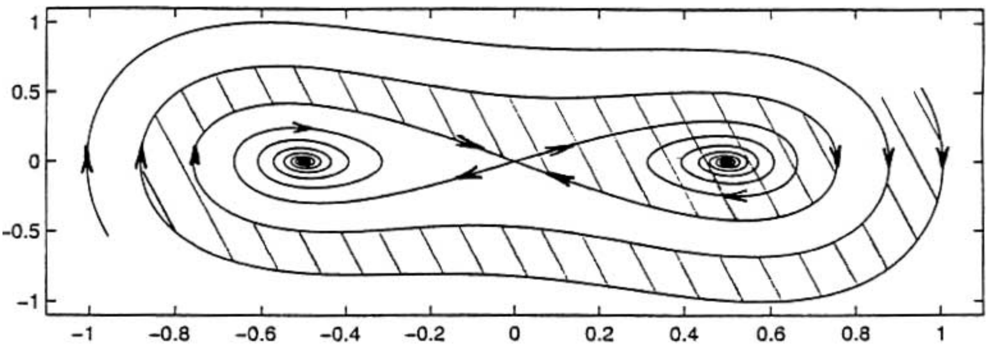

Figure 1. Two stable spirals separated by a saddle in the state plane. The 'incoming' asymptotic trajectories of the saddle – the inset – split the domains of attraction of the two spirals.

The solutions of Equation (1) with $\dot{\mathbf{x}} = \mathbf{0}$ are called *equilibrium points*. They are calculated from the following equation:

$$\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}. \quad (2)$$

An equilibrium point is also a trajectory. It corresponds to a stationary solution in the physical space. An equilibrium point is called *asymptotically stable* if all trajectories in an infinitesimally small open neighbourhood of the point approach the point asymptotically for $t \to \infty$. Otherwise, the equilibrium point is unstable. Depending on the way the trajectories approach the equilibrium point in dissipative systems, we call the point a *spiral point* or a *node*. In the high-dimensional state spaces in railway dynamical problems, the unstable equilibrium points are *saddle points* or simply *saddles*. The trajectories in the neighbourhood of a saddle approach the saddle asymptotically from certain directions in the state space and diverge from the saddle in at least one other direction. An example in the state plane of a saddle with two spirals is shown in Figure 1. Equation (1) may also have periodic, quasi-periodic or aperiodic solutions with the property that if the initial condition lies on the trajectory of such a solution in the state space, then the solution remains on the trajectory for all t. In other words, the solution is *invariant*. Since the trajectory of such a solution has stability properties that are very similar to the stability of equilibrium points, that is, the trajectories are *asymptotically orbitally stable or orbitally unstable*, they are called *equilibrium solutions* in this article. An equilibrium solution is *globally unique* if the asymptotic limit is independent of the initial values of the dynamic problem. An asymptotically stable equilibrium solution attracts the transient solutions in a certain domain in the state space, and it is therefore called an *attractor*, and the domain is called *its domain of attraction*. In Figure 1, for example, the domain of attraction of the right-hand spiral is shaded. The domain of attraction of a globally unique equilibrium solution is the entire state space

In vehicle dynamical problems, parameters such as the speed, V, or the applied brake force, P, vary, and then the solutions and their trajectories also change depending on the parameters. In this article, we only consider the changes caused by variation of the speed, V, which is called *the control parameter*. The number of equilibrium solutions of railway dynamical problems in the state space depends on the value of V. In most cases, the railway dynamical problems are formulated in an appropriate physical frame, where the desired solution in the state space is the stationary solution $\mathbf{x} = \mathbf{0}$.

The multiplicity of solutions arises through *bifurcations*, and the speeds at which the bifurcations occur in the parameter-state space are called *bifurcation points* or *branch points*. The parameter dependence of the number of solutions is plotted as a characteristic number for each and every solution versus the speed in a so-called *bifurcation diagram*. The plotted curve

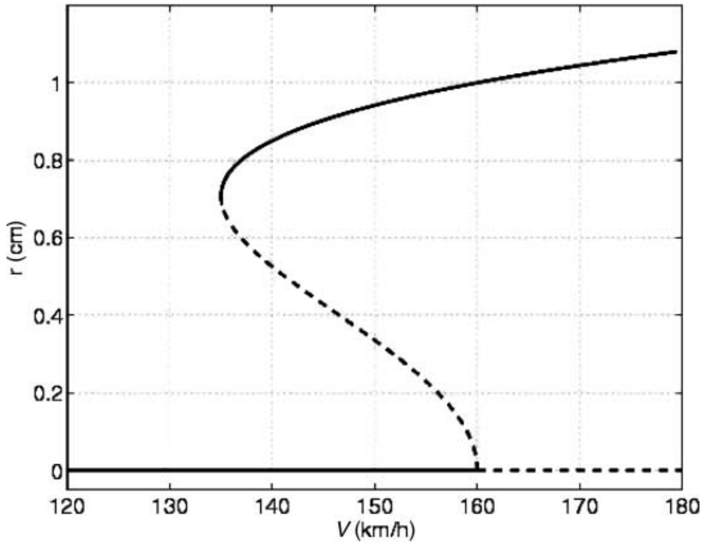

Figure 2. A bifurcation diagram with a subcritical bifurcation from the stationary solution and a tangent bifurcation between two periodic solutions, which are characterised by their amplitudes. $r \neq 0$ is the amplitude of a component of the periodic motion. The stationary solution is the axis $r = 0$. The stable solutions are drawn by a full line and the unstable solutions by a dotted line. The critical speed – the tangent bifurcation point – is 134 km/h, and there exists a subcritical Hopf bifurcation point at 160 km/h. Below 134 km/h, the stationary solution is globally unique.

is called a *path*. A typical and fairly simple example of a bifurcation diagram for a railway vehicle is shown in Figure 2. A bifurcation is called *subcritical* if the branching solution exists on the same side of the bifurcation point as the original stable solution, and the new branch is in most cases unstable. A bifurcation is called *supercritical* if the branching solution exists on the opposite side of the bifurcation point of the original stable solution, and in that case, the new branch is stable in most cases. The very often occurring bifurcation of a periodic solution from a stationary solution is called a *Hopf bifurcation*. An unstable solution may gain stability in a *saddle-node or tangent or fold bifurcation point*, where the path has a vertical tangent in the bifurcation diagram.

A brief guide to the nonlinear dynamics that apply to most railway dynamical problems can be found in the author's articles [1,2] and a thorough treatment of nonlinear dynamics in the numerous books about the topic.

In the case of the calculation of the *critical speed*, the upper limit of the vehicle speed for which an *equilibrium solution* of a nonlinear dynamical problem is *globally unique* must be found. It is *not* a stability problem. The lowest critical parameter needs not be the value at which the fundamental stationary solution loses its stability. This has been known now for decades, and the problem of finding the critical speed through the solution of a problem of existence has been described by the author in [1–6].

The mathematical problem of the calculation of the critical speed therefore leads to a calculation of the smallest bifurcation point. Bifurcations occur in railway dynamical problems often as a combination of a subcritical bifurcation of an unstable periodic solution from the stable stationary solution, in Figure 2 at $V = 160\,\text{km/h}$, and a fold or saddle-node bifurcation at a lower speed, where the unstable periodic solution meets a stable periodic solution. In Figure 2, it is at $V = 134\,\text{km/h}$. The stationary solution in Figure 2, $r = 0$, is asymptotically stable in the speed interval $0 < V < 160\,\text{km/h}$, in which it is an attractor. In this speed interval, the stationary solution is stable to infinitesimally small initial perturbations and it is therefore impossible to calculate the critical speed through a conventional stability analysis of the stationary solution. In the speed interval $0 < V < 134\,\mathrm{km/h}$, the stationary solution is a globally unique attractor, meaning that it is the only attractor in the entire state space. $V = 134\,\mathrm{km/h}$ is a bifurcation point in which an unstable and a stable periodic solution, *the hunting motion*, bifurcate from each other for increasing speed. The global uniqueness of the stationary solution is lost when the speed is larger than $134\,\mathrm{km/h}$, and in the interval $134 < V < 160\,\mathrm{km/h}$, three equilibrium solutions exist, out of which two are stable and one is unstable. In this speed interval, *multiple attractors* exist. From Figure 2, it can be easily seen that it is *the loss of uniqueness* of the stationary attractor that determines the critical speed, because above $V = 134\,\mathrm{km/h}$ finite disturbances of the stationary solution exist that will abruptly change the stationary solution into a periodic motion – hunting, if they are sufficiently large. The stationary solution loses its stability for growing speeds in the Hopf bifurcation point at $V = 160\,\mathrm{km/h}$.

2. The right way

The stationary solution can easily be found for sufficiently low speeds when it is the globally unique solution. In an inertial coordinate system, as in our example, it is the trivial solution – all state variables equal zero. In Figure 2, it is $r = 0$. It exists and is asymptotically stable for all speeds, V, up to a sufficiently high value. The first bifurcation point of the stationary solution can be calculated by a conventional stability analysis, that is, finding the eigenvalues of the linearised dynamical problem and calculating the smallest parameter value where the real part of an eigenvalue changes its sign from negative to positive. It may not be the smallest bifurcation point in the parameter-state space (Figure 2). The mathematically safe way to determine the smallest bifurcation point is to apply the method known as *path following* or *continuation* along the new bifurcating solution to the next bifurcation point. Path following is a computational procedure used to find an unknown path from a given point in the path. In Figure 2, for example, the path of the periodic unstable solution starting at the known value at $160\,\mathrm{km/h}$ and ending at $134\,\mathrm{km/h}$ is shown. In the railway dynamical problems, such as that shown in Figure 2, it will be the fold bifurcation that determines the critical speed. True and Kaas-Petersen [7] determined the critical speed by path following for a nonlinear vehicle dynamical system of ordinary differential equations applying Kaas-Petersen's routine PATH. Unfortunately until now, no numerical routine has been developed that can follow an unstable periodic solution of a differential-algebraic problem, which the railway dynamical problems usually are. Schupp [8] has, however, developed a routine, which can follow *stable periodic* solutions of railway vehicle dynamical differential-algebraic problems in the parameter-state space, but since the bifurcating solution from $V = 160\,\mathrm{km/h}$ in Figure 2 is unstable, we must use another strategy.

The 'True strategy' [1–6] is based on – first – a path following of the known stationary solution for growing speeds. The calculation stops when the computed equilibrium solution no longer is stationary. The speed at which this happens is the bifurcation point at which the stationary solution loses its stability. That speed is $160\,\mathrm{km/h}$ in Figure 2. Then, the full nonlinear dynamical problem must be solved with a small initial perturbation of the stationary solution for a slightly higher speed than the bifurcation point. The transient solution will tend to the nearest attractor in the state space since the stationary solution now is unstable. In most cases known, the attractor will be the periodic hunting motion (Figure 2). When the transient has disappeared, the attractor is found, and the end state of the calculation – a vector – must be stored. The method works as well in the case of a subcritical bifurcation, as shown in Figure 2, as in the case of a supercritical bifurcation as shown in Figure 3.

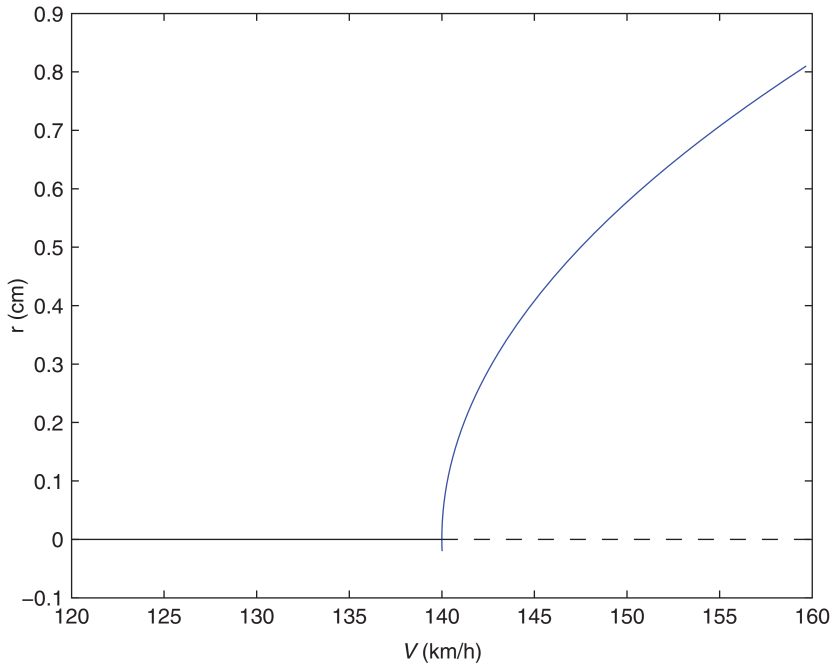

Figure 3. A bifurcation diagram with a supercritical bifurcation of a stable periodic solution from the stationary solution. $r \neq 0$ is the amplitude of a component of the periodic motion. The stationary solution is the axis $r = 0$. The stable solutions are drawn by a full line and the unstable solution by a broken line. The Hopf bifurcation point is 140 km/h and it is the critical speed if there are no other bifurcation points on the new path.

Second, the 'True strategy' is based on a path following of the hunting motion. Since the periodic solution is stable, Schupp's routine [8] with the end vector as the initial condition can be applied to a decreasing speed to find the next bifurcation point. In Figure 2, it is the fold bifurcation point 134 km/h. It is the critical speed! Alternatively, True's method [1–6] can be applied where the speed is changed manually in small steps. In every step, the end values of the preceding solution are used as the initial values for the subsequent calculation. The series of calculations end when the transient of the initial value problem again tends to the stationary solution, and the speed at which it happens is the next bifurcation point – the critical speed.

When the stationary solution – as in our case – is the trivial solution and an equilibrium point for all values of the control parameter, V, recall that zero is the only value a digital computer delivers with absolute accuracy, that is, zero error. Being an equilibrium point, zero can therefore be continued even with absolute accuracy although it has lost its stability through the parameter change. Therefore, in each and every new step in the path following of zero, a small disturbance must be added to the end vector, $\mathbf{0}$, before it is applied as the initial value in the next step. Since all the other possible solutions depend on the parameter value, it is not necessary to add a disturbance to the end vector of these solutions before it is used as the initial condition in the next step.

A fast estimate can be made by *ramping*. Instead of the stepwise discrete changes of the speed, a slowly changing speed is used in the solution of the dynamical problem. This quasi-stationary method can be used to calculate an approximation of the path and it yields a fast but inaccurate determination of the critical speed, because the method overshoots the bifurcation point. A more accurate value can subsequently be found by using discrete steps but now only in a much shorter speed interval. Note that these computations are a *mathematical method to* *find attractors* – to solve an existence problem – which in principle has nothing in common with stability.

3. The wrong way

The application of a conventional stability analysis, that is, to find the eigenvalues of the linearised dynamical problem and calculate the smallest parameter value where the real part of an eigenvalue changes its sign from negative to positive, is the wrong way to find the critical speed. The problem to be solved is *not a stability problem, it is a problem of existence of solutions*. A stability analysis will therefore in general *not* deliver the critical speed but only what it can deliver: a stability limit for the stationary motion to infinitesimal initial disturbances! In Figure 3, the stability limit may indeed be the critical speed, but it is not so in Figure 2, which illustrates the most often occurring case. It is never known in advance whether the bifurcation from the stationary solution is subcritical or supercritical. The difference between the bifurcation point on the stationary solution and the critical speed can be several hundred kilometres per hour for high-speed trains according to manufacturers! The details and the references are confidential. Jensen [9] calculated in his PhD thesis the Hopf bifurcation point and the critical speed for a 'half car' of the Danish IC3 train, Figure 4. For an adhesion coefficient of 0.30, the critical speed is $V = 317\,\mathrm{km/h}$ and the Hopf bifurcation point is $562\,\mathrm{km/h}$.

It is necessary to follow the bifurcating periodic solution in order to find the smallest bifurcation point. Also in the case of a supercritical bifurcation – when the bifurcating periodic solution is stable – it is necessary to follow the periodic solution to the next bifurcation point, because there may be a secondary bifurcation point in that branch from where a third solution will bifurcate to the left. This third solution might reach a smaller speed than the supercritical bifurcation point found on the stationary solution. An example can be found in [10,11], where a chaotic attractor bifurcates from the stable periodic solution and it disappears at a much lower speed in a so-called crisis. This speed is the critical speed. Another example was presented by Gasch *et al.* [12].

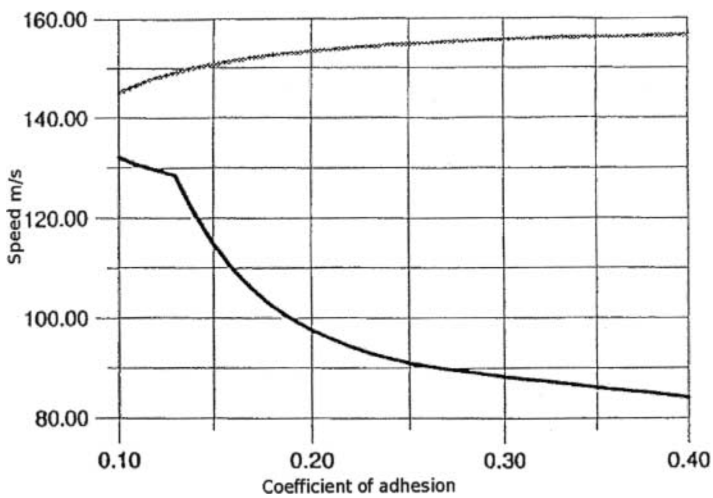

Figure 4. The Hopf bifurcation point (upper curve) and the critical speed (lower curve) versus the coefficient of adhesion for a 'half car' of the Danish IC3 train. The wheel profile is DSB 82-1 and the rails are UIC60 canted $\frac{1}{40}$ with the gauge 1435 mm.

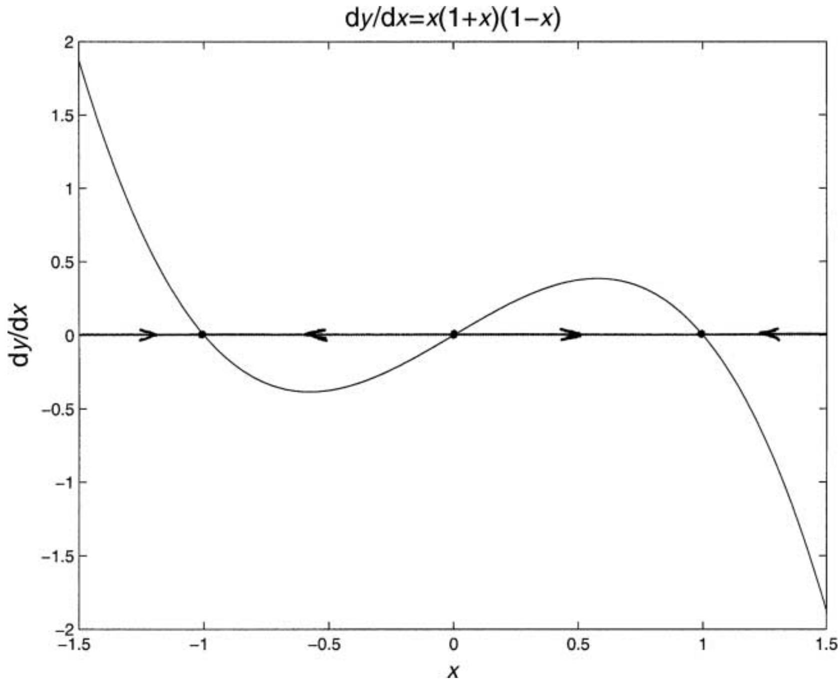

Figure 5. A one-dimensional nonlinear problem with three equilibrium points: the attractors -1 and 1, and 0, which is unstable. The abscissa is the state space, and the ordinate, $\mathrm{d}y/\mathrm{d}x$, determines the direction of the flow on the x-axis.

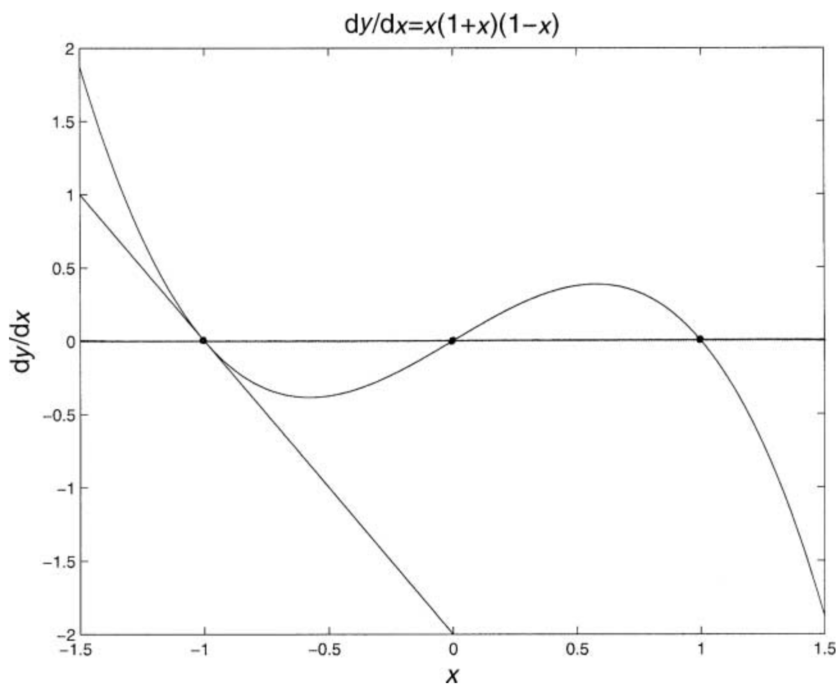

Figure 6. A problem the same as that shown in Figure 5, the inclined line through -1 is the linearisation of the nonlinear dynamical problem around the point -1.

Linearisations, such as those made in a conventional stability analysis of a nonlinear dynamical problem, must be treated with caution. First, we must choose an attractor in the nonlinear dynamical problem that we can linearise around. In Figure 5, a simple one-dimensional example of a nonlinear dynamical problem is illustrated. In this problem, we choose the attractor $x = -1$. Then, we linearise the problem around $x = -1$ and find the linearisation (Figure 6). Then, we isolate the problem linearised around $x = -1$, see Figure 7, and find that the linearised problem only has one attractor, namely $x = -1$, and all other equilibrium points have disappeared from the state space, which is the x-axis. Similarly, the vehicle dynamical problem that is linearised around the bifurcation point on the stationary solution contains no information about other equilibrium solutions in Figure 2 and, in particular, none about the existence of the hunting motion.

Linearisations are not always permitted. The first derivative of the function to be linearised must exist in the point of linearisation. For $\sqrt{x}$, it does not exist in $x = 0$. The railway dynamical problems are – practically all – *non-smooth*, and in the points of non-smoothness, the first derivative of the force and torque functions is not defined. The Jacobian is needed when implicit routines are applied for the numerical solution, but the Jacobian does not exist in the points of non-smoothness. Problems may also occur with the application of solvers of higher orders to problems with a discontinuity of the second derivatives, which occur in the description of a rail surface. The numerical procedure must therefore be modified in order to obtain reliable results. Appropriate modifications can be found in the works by Xia [13] and Hoffmann [14], and True has discussed the modification in [2].

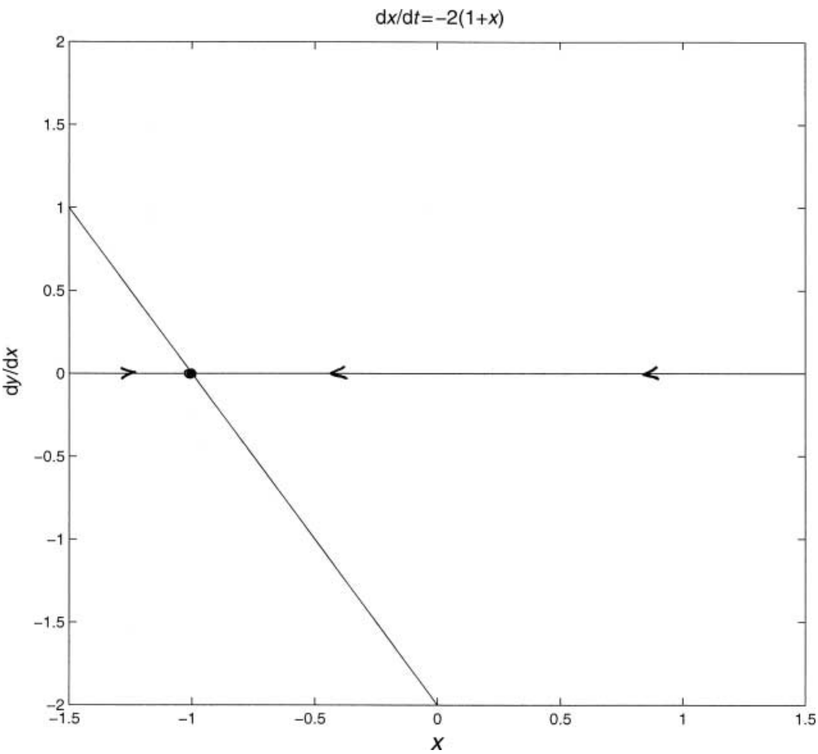

Figure 7. The problem shown in Figure 5 linearised around −1. Two of the equilibrium points have disappeared and with them every information about the stable equilibrium 1. When only the linearised problem is analysed is it impossible to find the point 1.

Two very recent papers on the numerical solution of vehicle dynamical problems are the state-of-the-art paper by Arnold *et al.* [15] and the survey paper by True *et al.* [16].

4. Gambling

Some railway dynamicists feel that 'the right method' for the calculation of the critical speed is either too complicated or too slow. They therefore present and argue for modifications of 'the right method'. Stichel [17] proposes the use of a finite disturbance of the stationary motion for growing speeds to find the periodic attractor. When it is found, the periodic attractor is followed backwards for a decreasing speed – using ramping – until the hunting motion stops. Thereby, Stichel saves computer time for the path following of the stationary solution up to the Hopf bifurcation (which is very fast) and back again from there along the periodic attractor to the speed at which he found the periodic attractor. This step can be time consuming. This method is also 'right', because Stichel *follows the hunting motion for a decreasing speed*. Stichel's method is not a stability investigation. He conducts a legitimate *search for other attractors in the state space*. Other authors, however, make the mistake of *not* following the hunting motion backwards in the parameter-state space and call the speed at which the hunting motion first occurs, when the speed is increased, the critical speed. It is pure luck if the found speed is anywhere near the critical speed of the vehicle, because the found value of the speed depends on the size and the kind of the initial disturbance – an initial condition, which is only *one of the many states that are excited* in a real situation on the railway line. This method is therefore a gamble. By disturbing only one component of the initial value vector, valuable computer time is wasted with excitation of all the other components until the transient disappears.

The railway dynamicists in the Southwest Jiaotong university in Chengdu have already realised this. They use the data from the measurements of a sufficiently long section of a railway line as the input to the dynamical problem. In that case, several components of the state of the vehicle are excited, and the critical speed found by growing speed will be based on a seemingly realistic calculation. The found hunting motion was, however, *not followed backwards in the parameter-state space*. During a visit to Chengdu in 2009, the author had the opportunity of testing the result of such a calculation together with one of its scientists. We started ramping, using commercial software, at the critical speed found by gambling in an earlier work. The hunting motion was now followed for a decreasing speed, but the calculations were stopped when the speed had decreased by 10%, and the vehicle model was still hunting! Obviously, the critical speed in the model was lower than the one that had been claimed to be the critical speed in the original report. In the test, a commercial routine was used, and it became obvious why 'the right way' is found cumbersome by dynamicists. The routine does not allow the download and storage of the end vector for a direct use as an initial value in the subsequent calculation. This is, however, a shortage of the routine and not of the method. On an earlier occasion, the author had the opportunity of using the same routine together with the head of the IT department in Bombardier. He was able to modify the routine in less than half an hour and enable its use in 'the right method'. The calculations with 'the right method' then ran fast and without any problems.

The conclusion is that *the published critical speeds that have been found by gambling are not reliable* – with the exception of Stichel's work [17], because he actually completed his calculations in 'the right way'.

It must be emphasised that the use of 'the right method' will not yield the correct result in all theoretically possible cases. The author has, however, never found any of these 'other possible cases' during 25 years of railway dynamical research.

5. The robustness of the hunting motion

More emphasis is often put on figures such as Figure 2 than what they are worth. For example, it is often stated – if we use Figure 2 as an example – that a lateral disturbance of the leading wheel set $\Delta r > 0.2$ cm at $V = 157$ km/h will initiate a hunting motion, and if $\Delta r < 0.2$ cm, then the transient will die out. It is tacitly assumed that the point $(157, 0.2)$ lies on the unstable limit cycle and that it separates the domain of attraction of the stationary motion, which is an attractor, from the domain of attraction of the hunting motion, which is also an attractor. *The statement is wrong*, because the state space is multi-dimensional, and it is only in the one-dimensional state space that a point can separate two domains of attraction. As an example, see the unstable point 0 in Figure 5. Furthermore, it is impossible to determine the position of the unstable limit cycle of differential-algebraic problems numerically in the state space by the methods available today. Daniele Bigoni, one of the students of the author, has even demonstrated that a disturbance of only the lateral motion of the leading axle, a little bit larger than the amplitude of *the stable limit cycle*, may start a transient *that tends asymptotically to the stationary motion*. This is certainly possible in a multi-dimensional state space. More details are given in the next section on multi-dimensional spaces.

A simple two-dimensional example will illustrate the point made. Consider Figure 8 with an unstable limit cycle – the ellipse – in a state space with the lateral displacement of the leading wheel set as the abscissa and the yaw angle as the ordinate. The point $(0, 0)$ is an attractor, and the interior of the ellipse is the domain of attraction of $(0, 0)$. The exterior of the ellipse is the domain of attraction of another attractor further away. The point $(157, 0.2)$ in Figure 2 corresponds to one of the points $(\pm 0.2, 0)$ in Figure 8, and it can be easily seen that for yaw $= 0$, all the trajectories through the points $|x| < 0.2$ will approach $(0, 0)$, because they lie in the domain of attraction of that point.

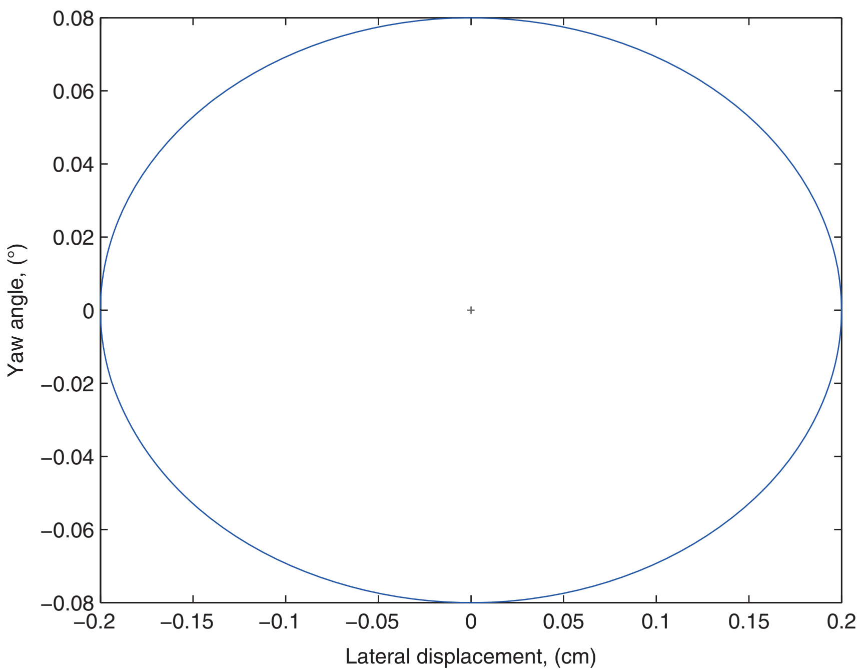

Figure 8. A stable equilibrium point $(0, 0)$ surrounded by an unstable limit cycle. The interior of the limit cycle is the domain of attraction of $(0, 0)$. The exterior of the limit cycle is the domain of attraction of another attractor further away.

If, on the other hand, the yaw $\neq 0$, then there are points with abscissae $|x| < 0.2$ – for example, also $x = 0$ – through which the transients will run away from $(0, 0)$, because the points lie outside the domain of attraction of $(0, 0)$. So, it is possible in Figure 2 – where Δr is the only component of the state of the system – that the disturbance of another mode *can start hunting of the vehicle even for* $\Delta r = 0$.

The conclusion is that *the existence of an unstable limit cycle should not be indicated by a curve on the bifurcation diagrams of dynamical systems in state spaces of dimension 2 or higher* because it may lead to misinterpretations.

We have seen that in railway vehicle dynamics there very often exists a speed interval in which multiple attractors exist – namely the stationary motion and the hunting motion. It is also a fact that the hunting motion is more robust in the sense that it is easy to start hunting, but virtually impossible to stop hunting without slowing down the vehicle. Due to the great complexity of the attractors in the high-dimensional state space, it is impossible to explain this fact exactly, but a simple picture may make it plausible. In Figure 9, the small ellipse, which is an unstable limit cycle, splits the domains of attraction of the stable equilibrium point $(0, 0)$, on the one side, and the domain of attraction of the stable limit cycle, the large ellipse, on the other side. The angle ϕ covers the domain of arguments of the vectors from P that can end inside the unstable limit cycle, the domain of attraction of the stable equilibrium point $(0, 0)$. Here – in two dimensions – it is easily seen that a sufficiently large initial disturbance of $(0, 0)$ will start the hunting independently of its direction in the plane. In contrast, an initial disturbance on the 'hunting ellipse' must not only be sufficiently large – and not too large – but also lie inside the angle ϕ in order to reach the domain of attraction of $(0, 0)$. Furthermore, $(0, 0)$ is an equilibrium point, that is, independent of time, while the point P moves around on the large ellipse all the time, whereby ϕ changes place and size.

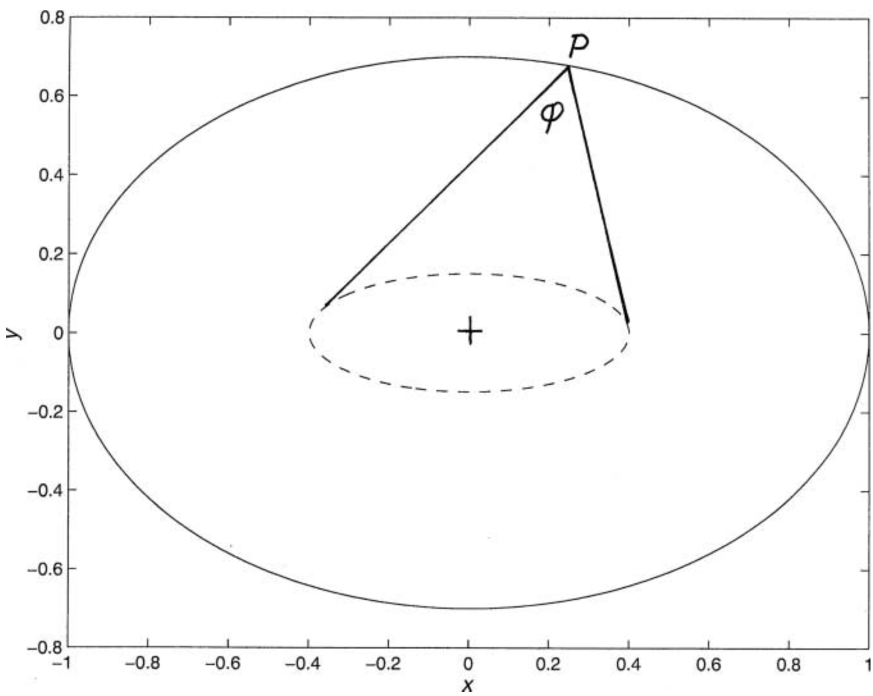

Figure 9. A picture of an unstable limit cycle – the small ellipse – with a stable equilibrium point in the centre, surrounded by a stable limit cycle – the large ellipse.

6. Multi-dimensional state spaces

The unstable limit cycle will lie on a surface in the N-dimensional state space. It can be mathematically proven that close to the subcritical bifurcation point, this surface has a tangent plane that is spanned by the eigenvectors belonging to the complex conjugated purely imaginary eigenvalues at the bifurcation point. The limit cycle is the base of the generators of an $(N-1)$-dimensional cylindrical subspace, which is *the inset of the unstable limit cycle*. It divides the N-dimensional state space into a domain of attraction of the origin and a domain of attraction of the stable limit cycle – the hunting motion. Without showing the actual geometry in the N-dimensional state space, Figure 10 might help the reader to understand the underlying dynamics. The cylindrical surface shown in Figure 10 is generated by a projection of an $(N-1)$-dimensional subspace onto a two-dimensional surface. All the trajectories on this cylindrical surface drift towards the unstable limit cycle while at the same time swirling around on the cylindrical surface. Outside the cylindrical surface, all the trajectories approach

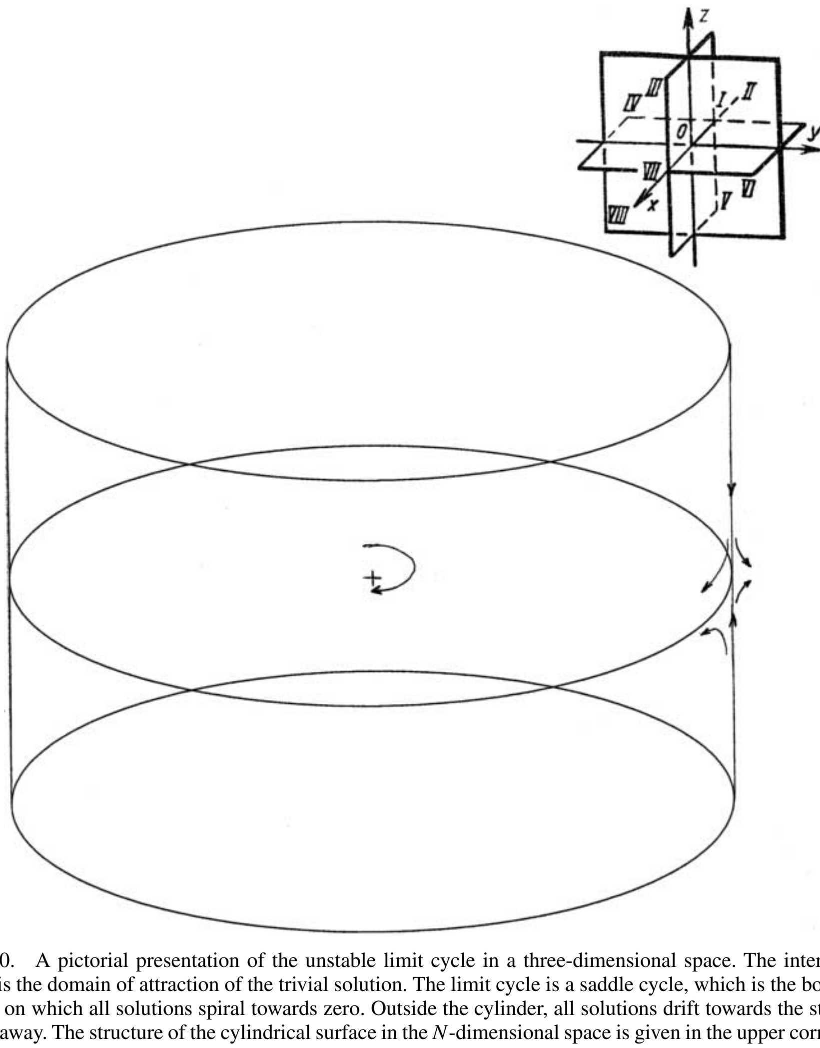

Figure 10. A pictorial presentation of the unstable limit cycle in a three-dimensional space. The interior of the cylinder is the domain of attraction of the trivial solution. The limit cycle is a saddle cycle, which is the boundary of a surface on which all solutions spiral towards zero. Outside the cylinder, all solutions drift towards the stable limit cycle far away. The structure of the cylindrical surface in the N-dimensional space is given in the upper corner, where each of the coordinate planes again has a similar three-dimensional structure and each of these coordinate planes again has a three-dimensional . . . and so on until the dimension $N-1$ has been reached in the full state space.

the stable limit cycle, and inside the surface, all the trajectories approach zero in a swirling fashion. In a state space with dimension 3 or higher, *it is the inset and NOT the unstable limit cycle* that splits the state space into the two domains of attraction of, respectively, zero and the hunting motion.

The tangent plane mentioned above lies obliquely in the state space with respect to some of the axes in the N-dimensional state space with projections of the limit cycle onto these axes of non-zero lengths. These axes represent the modes that are excited with the loss of the stability of the trivial solution in the subcritical bifurcation point. The tangent plane will be perpendicular to the remaining axes. With a decreasing speed, the unstable limit cycle will be deformed in such a way that its projections on all the axes in the N-dimensional state space will have a finite length, meaning that all modes of the dynamical system become excited. The shape of the deformed unstable limit cycle in the state space cannot be computed by the methods available today. An example of a closed curve in a three-dimensional space is shown in Figure 11.

Three different examples indicate that the domain of attraction of the trivial solution in the N-dimensional state space winds around the stable limit cycle in such a way that a very large disturbance may push the solution back onto the trivial one – but only for a short time. First, Jens Christian Jensen (personal communication) numerically demonstrated to the author in a realistic dynamical model of a bogie that a large disturbance may stop the hunting. Second, TTCI in Pueblo demonstrated to the author in a field test with a gondola wagon that the hunting motion of the wagon at a certain speed would abruptly end when the wheel sets are hit hard at the point in a frog. A few seconds later, the hunting starts again as could be expected. The

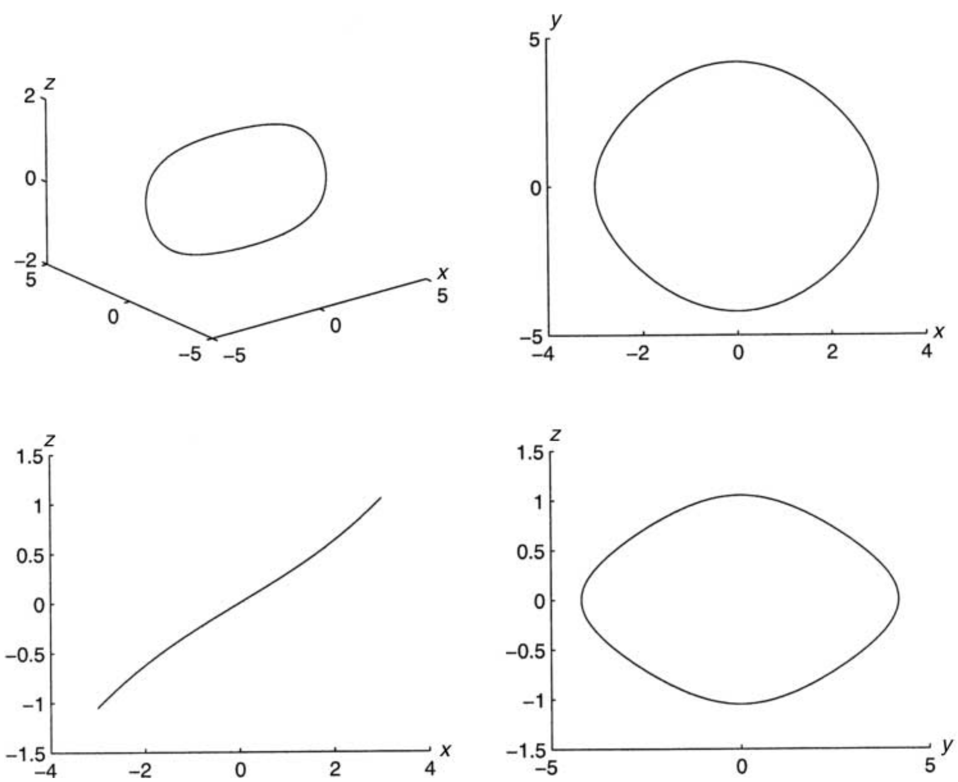

Figure 11. A closed curve in a three-dimensional xyz-space. Its projections on all the three coordinate planes are closed plane curves. The curve in the xz-plane is degenerate because the x- and z-components are in phase.

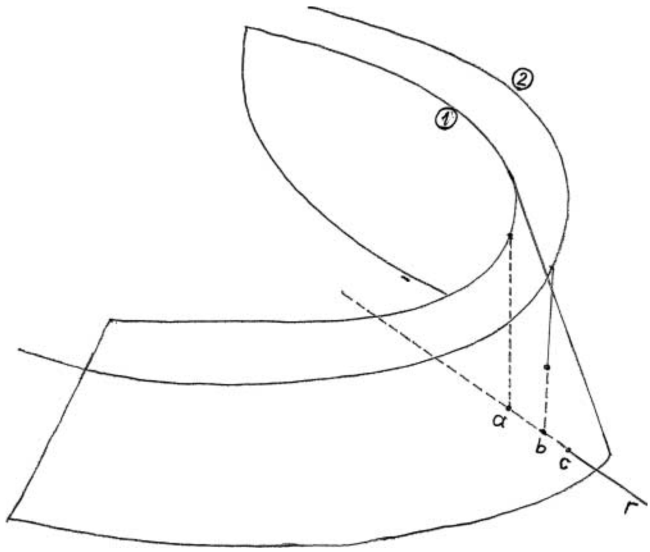

Figure 12. A demonstration in a three-dimensional state space as to how a disturbance of a single mode with V near by but larger than the fold bifurcation (the critical speed) can lie in the domain of attraction of the trivial solution, although the size of the disturbance is larger than the coordinate of the amplitude of the stable limit cycle. r is the ordinate shown in Figure 2. (1) is a part of the unstable limit cycle in the neighbourhood of its maximum value of r with a part of its inset on a conically shaped surface below. The domain of attraction of the trivial solution lies to the left of the surface. (2) is a part of the stable limit cycle (the hunting motion). a is the largest r-value of the unstable limit cycle, b is the largest r-value of the stable limit cycle and $c > b$ is the intersection of the r-axis with the inset of the unstable limit cycle.

test wagon had three-piece freight trucks, of which one had the snubbers removed, and it was this truck that would stop hunting by passing the point. Third, von Wagner [18] has calculated the domain of attraction of the trivial solution for a simple model of a wheel set. He showed some projections of the domain onto coordinate planes, where the projection of the domain of attraction was not connected.

Let us again consider the case of multiple attractors. In Figure 10, we can see that there exists an unstable limit cycle – a saddle cycle, which is the base of the inset – the subspace that separates the two domains of attraction of zero and the hunting motion, respectively. Figure 10 illustrates the geometry of the state space at a speed smaller than but very close to the subcritical bifurcation. The stable limit cycle is so far away that it cannot be seen in the figure. With a decreasing speed, the two limit cycles and the cylindrical surface are strongly deformed, and the limit cycles approach each other. Figure 12 illustrates in 3D how the situation may look locally, when the speed is larger than but close to the fold bifurcation point. The limit cycles are deformed but close to each other, and the inset may locally have a shape that is conical. We can see that an initial value vector with one component $r|b < r < c$, and all other components equal to zero, can fall inside the domain of attraction of the stationary solution although r is larger than the projection, b, of the amplitude of the stable limit cycle on the r-axis.

7. Driving with supercritical speed and chaos

It is often written in technical articles that ‘driving with supercritical speed’ is dangerous. I would warn against the use of the word ‘dangerous’ because many – if not most – freight wagons hunt at the normal service speeds. The public might get the impression that the highly praised safety on the railways is not guaranteed. Hunting is certainly undesirable, but it is only dangerous in extreme situations, for example, when the track or the vehicle has serious defects.

Self-excited chaotic oscillations were never found worse than hunting in any of the numerical investigations that the author and his co-workers have performed. The amplitudes of the chaotic motion were of the same order of magnitude as those of the hunting – *when the solutions in the state space lie on chaotic attractors*. The chaotic attractors attract the solutions in their neighbourhood in the state space so strongly that the external excitations are bounded and die out very fast.

In contrast, motions that are characterised as *chaotic transients* may be dangerous and must be avoided. The amplitudes of chaotic transients may be very large and in real life lead to dangerous situations. Chaotic transients may, for example, appear as a transient motion between two competing attractors. They have been found by the author in numerical investigations of low-dimensional dynamical systems such as the famous Lorenz system. At a certain parameter combination, a pair of stable equilibrium points exist in the Lorenz problem. Certain initial conditions can start a transient motion with large amplitudes. It has a chaotic nature, for example, sensitivity of the motion to infinitesimal changes of the initial conditions, and the motion may prevail for a fair amount of time until it dies out in one of the equilibrium points. Pascal [19] has simulated the occurrence of a derailment of a chaotically moving freight wagon. The amplitudes of the oscillations became so large that the wagon derailed. Due to the continuous excitation from the track, the chaos must be characterised as 'transient', because the wagon never reached an equilibrium state. The author recommends the book by Moon [20] for further studies on chaos, but it is only one of many.

8. Comments on the determination of the critical speed of real railway vehicles in road tests

This article has dealt with the theoretical aspects of defining and calculating the critical speed of vehicles. The method, however, carries over to the measurements of the critical speed in road tests. The critical speed of a real railway vehicle must be determined by 'Stichel's method'. It is mandatory that the critical speed is measured as *the speed at which the hunting stops during idle deceleration*. The method has been used at least in Germany for decades. The test vehicle running idle is pushed by an accelerating motorised vehicle until the test vehicle begins hunting. Then, the motorised vehicle slows down to release the test vehicle, which then runs idle and slows down due to friction and air resistance. When the test vehicle stops hunting, the critical speed is reached. The method used is the ramping method, which is described in the section 'The right way'. It is impossible to accelerate a railway vehicle on a real railway line all the way up to the speed of the subcritical bifurcation, because the track irregularities on all tracks in normal service are so large that the vehicle during acceleration will hunt before the speed of the subcritical bifurcation point is reached. So, the proposed method corresponds to 'Stichel's method' described in the section 'Gambling'. It is fortunate that it is impossible in the case with multiple attractors to reach the speed of the subcritical bifurcation, because it may be hundred or more kilometres per hour higher than the critical speed. In practice, therefore, the length of the track section needed for the idle slow down of the test vehicle is reasonable.

The results of an application of 'the right method' or 'Stichel's method' do not depend on the criteria chosen for the variable(s) that is/are calculated or measured. The physical laws and mathematical theorems guarantee that the disturbed motion in real life will not lead to wrong results, because all the equilibrium solutions have a domain of attraction. The results are therefore independent of the imposed disturbances within a realistically applicable range. Therefore, the criteria can be expressed as limits for displacements, rotations, velocities or accelerations of vehicle components. It is important to remember that *it is not the amplitude of a disturbance that defines the critical speed but the significant* change *of the amplitude, when the decelerating vehicle passes the critical speed*. The amplitude of a hunting motion is not well defined, but the decreasing speed, where the amplitude abruptly changes, is.

The standard of the selected railway line must, of course, be so high that it is possible to distinguish between the calculated or measured amplitudes caused by the deviations from the ideal track geometry and those caused by the errors plus hunting. The critical speed in curves may be lower than the critical speed on the tangent track. The effect is only relevant for curves with large radii, because the maximum speed in curves with radii below, say, 1000 m. usually is lower than the critical speed of the vehicle in these curves.

In the theoretical problem, the engineer can choose all the parameters in his vehicle/track model at will, but that is not possible in practice. The coefficient of adhesion, which is an important parameter in the wheel/rail stress relations, depends on the weather conditions and it may vary along the test track and with the time of the day. The suspension system in the real vehicle is manufactured with tolerances, and last but not least, a suitable section of a railway line must be available for the test. All this gives rise to practical problems that must be solved before 'Stichel's method' can be used in real-life tests for a reliable and repeatable determination of the critical speed of a railway vehicle.

9. Conclusion

The only safe computation of the theoretical critical speed of a railway vehicle uses *the right way* described in the section 'The right way', possibly using the modification proposed by Stichel [17] described in the section 'Gambling'. All other results are unreliable. In road tests of real vehicles, the same method applies by letting the vehicle roll out idling and measure the speed at which the hunting stops.

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