The Merino Welsh Conjecture

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Summary (English)

The goal of the thesis is to explain the Merino Welsh conjecture and related results to a degree understandable to a student of mathematics unfamiliar with the subject area. Specifically I deal with an article by my thesis advisor Carsten Thomassen titled "Spanning trees and orientations in graphs" published in 2010. I also explore some new results of my own related to the Merino Welsh conjecture.

Summary (Danish)

Målet for denne afhandling er at forklare Merino-Welsh formodningen og relaterede resultater, så de er forståelige for matematisk interesserede der ikke er hjemmevante i dette område af matematikken. Helt specifikt omhandler afhandlingen en artikel af Carsten Thomassen, min vejleder. Artiklen hedder "Spanning trees and orientations in graphs"og blev udgivet i 2010. Jeg udforsker også et par af mine egne resultater relateret til Merino-Welsh formodningen.d to the Merino Welsh conjecture.

Preface

The thesis deals with the Merino Welsh conjecture, which limits the number of spanning trees, $\tau(G)$, in a graph, G, in terms of the number of acyclic orientations, $a(G)$ and the number of totally cyclic orientations, $c(G)$. This is also a statement on the convexity of the so-called Tutte polynomial of a graph, which has a number of interesting applications.

This thesis is centered around an article by Carsten Thomassen: "Spanning trees and orientations of graphs" from 2010 ([Tho10]). In this article bounds are found on $\tau(G)$, $a(G)$ and $c(G)$, which in combination give the Merino-Welsh conjecture for graphs with either a small or large amount of edges in relation to the number of vertices (so there is only a narrow size gap where the conjecture is unsettled). Additionally, along with proofs of the Merino-Welsh conjecture for cubic graphs and for planar triangulations, the article also includes the statement of two open problems, one of which I use as a guide to find new results.

The work in this thesis comes in two parts: A presentation of the article and some original results. In the presentation of the article I rewrite all proofs from scratch such that they are clearly understandable to those unfamiliar with the topic. This is necessary, as the article is written for experts and leaves a lot of hidden assumptions and "obvious" shortcuts.

As for the original results, I am inspired by the result for planar triangulations and one of the open problems to try and prove the Merino Welsh conjecture for outerplanar near-triangulation. I succeed in this and from there also prove the Merino Welsh conjecture for any outerplanar graph. From there I start moving towards any planar near-triangulation. Apart from these results, this section also contains a few minor observations.

Acknowledgements

I would like to thank my thesis advisor, Carsten Thomassen, for his good advice, and for giving me the opportunity to do this project.

I would like to thank my friends, my parents, my brother and my sister for their boundless support and constructive criticism.

Contents

CHAPTER 1

Introduction

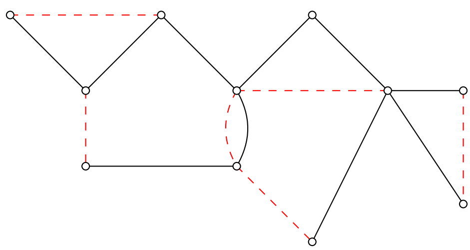
In this chapter I introduce the concepts needed to understand the Merino Welsh conjecture and why it is interesting. A few relevant proofs are included. I assume knowledge of graph theory equal to DTU course 01227 "Graph Theory". The contents of this chapter is based on knowledge from 01527, [Tho10] and other sources where mentioned

This chapter starts out with some facts about spanning trees, then moves on to acyclic and totally cyclic orientations. This leads to the Merino Welsh conjecture, followed by the chromatic and Tutte polynomials. I round off with a light touch on some applications.

In this work a graph has no multiple edges and no loops and a multigraph has multiple edges and no loops. So if I allow loops, I will mention it specifically. I use the term oriented path for a path where all edges are oriented in the same direction. In this work we use the concept of suspended paths. A suspended path is a path in a graph, where exactly the endvertices have degree > 2. I will often draw paths as zigzags ()

1.1 Spanning trees

The number of different[1] spanning trees of a graph is one of the three graph invariants the Merino Welsh conjecture is about. Just to refresh: a spanning tree of a graph G is a subgraph, with all the vertices of G and a subset of the edges to form a tree. Each valid subset is a different spanning tree, so in case of multiple edges, each edge counts for another spanning tree. We use $\tau(G)$ for the number of different spanning trees of a graph.



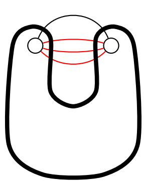
Any connected graph has at least one spanning tree, so $\tau(G) \geq 1$. Graphs that aren't connected have no spanning trees, $\tau(G) = 0$.

Importantly, the number of spanning trees obey the so-called Deletion-Contraction formula:

Theorem 1.1 *Let G be a multigraph and let e be an edge in G. Then* $\tau(G) = \tau(G/e) + \tau(G - e)$

This proof is adapted from my notes from DTU course 01227 "Graph Theory"

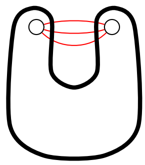
Proof. If we consider any edge e in G, here marked in black, with eventual multiple edges marked in red.



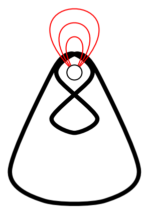
The spanning trees can be divided into those that include e and those that don't. If you look at the following drawing, it is easy to see that the spanning

[1]Two isomorphic spanning trees are not the same.

trees that don't include e are exactly the spanning trees of $G - e$. Note that the other edges can still be in a spanning tree.



The spanning trees of G that include e would be exactly spanning trees of G/e, if you contracted e in each of them. To illustrate, here is a drawing of G/e, the other edges are now loops and will never be in a spanning tree.



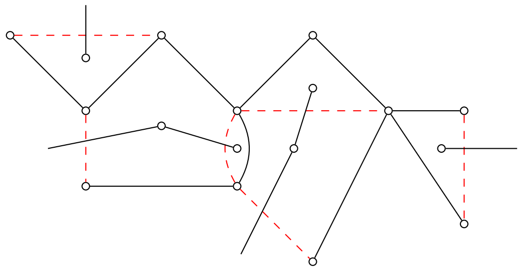
Since all spanning trees are accounted for, we have $\tau(G) = \tau(G/e) + \tau(G - e)$. □

Also of interest later on is the fact that planar duality preserves the number of spanning trees. As a quick reminder, edges in G correspond to edges in the dual graph G^*, while faces correspond to vertices and vice versa.

Theorem 1.2 *Let G be a planar multigraphs with loops allowed, and let G^* be the planar dual of this graph. Then $\tau(G) = \tau(G^*)$.*

I got a sketch of this proof from private communication with my advisor Carsten Thomassen.

Proof. I prove this by constructing a bijection between the spanning trees of G and G^*. Let G have n vertices, m edges and f faces. If T is a spanning tree in G, then let T^* consist of the edges of G^* that are not dual edges of edges of T. The situation is illustrated below with the spanning tree from earlier. The edges that "escape" should be seen as being connected to the outer face.



Since T has $n-1$ edges, T^* must have $m-(n-1)$ edges. Using Euler's polyhedron formula, $n-m+f=2$ $f-1=m-n+1$, T^* must have $f-1$ edges, the amount a spanning tree should have. Since T is a tree, there must be a path between any vertex in T^* and the vertex that corresponds to the outer face of G. [2] So since T^* is a connected subgraph with $f-1$ edges, it can only be a spanning tree. So each spanning tree in G corresponds to a unique spanning in G^*. Using the same argument in reverse (by starting with a tree in G^*), we have a bijection between the spanning trees of G and G^* and so there must be equally many. □

1.2 Acyclic orientations & Totally cyclic orientations

We need the concepts of acyclic orientations and totally cyclic orientations. Here are the definitions, an example of each is given in figure 1.1.

Definition 1.3 An acyclic orientation is an assignment of orientations such that there is no cycle with all edges oriented the same way. The number of different acyclic orientations in a graph is called $a(G)$

Definition 1.4 A totally cyclic orientation is an assignment of orientations such that there is no edges not in some cycle with all edges oriented the same way. The number of totally cyclic orientations is called $c(G)$.

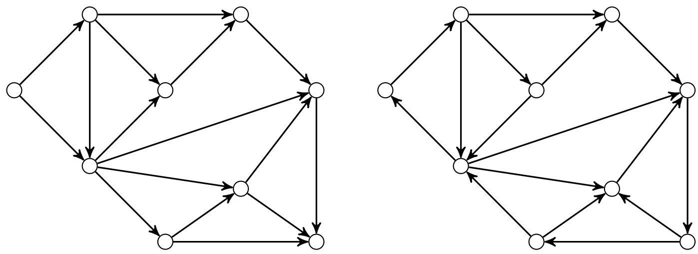
Obviously a orientation cannot be both acyclic and totally cyclic, except when the graph has no edges, then the only orientation is both acyclic and totally cyclic.

[2] If that wasn't the case, T would have had a cycle.

Figure 1.1: Two kinds of orientations of a graph

(a) An acyclic orientation

(b) A totally cyclic orientation



If a graph G has no edges $a(G) = c(G) = 1$, otherwise $a(G) \geq 2$ and if all components of G with edges is 2-edge-connected then $c(G) \geq 2$ otherwise $c(G) = 0$. If there is a loop $a(G) = 0$.

These two graph invariants[3] also obey the Deletion-Contraction formula:

Theorem 1.5 *Let G be a multigraph with an edge e. Then $a(G) = a(G - e) + a(G/e)$*

Theorem 1.6 *Let G be a multigraph with an edge e. If e is not part of a multiple edge or a bridge Let G be a multigraph with an edge $e = xy$. If the set of edges between x and y does not form a cut $c(G) = c(G - e) + c(G/e)$*

These proofs are from course 01527 "Graph Theory II" at DTU.

Proof. First we consider the case where e is not part of a multiple edge. We look at $G - e$, where $e = xy$. Let α_1 be the number of acyclic orientations of $G - e$ with a directed path from x to y. Let α_2 be the number of acyclic orientations of $G - e$ with a directed path from y to x. Let α_3 be the number of acyclic orientations of $G - e$ with neither a directed path from x to y nor one from y to x. If there were both kinds of path it would not be an acyclic orientation.

Then $a(G) = \alpha_1 + \alpha_2 + 2\alpha_3$ since the first two force a direction on e. $a(G - e) = \alpha_1 + \alpha_2 + \alpha_3$ and $a(G/e) = \alpha_3$, since the first two would create a directed cycle when contracting. It is now easy to see that $a(G) = a(G - e) + a(G/e)$.

If e is part of a multiple edge, G/e will have loops and so will have no acyclic orientations, that is $a(G/e) = 0$. $G - e$ will still have the other multiple edges,

[3]Invariants are $f(G)$, where $f(G) = f(G')$ when G and G' are isomorphic.

so if you add e back in, the rest of the multiple edge will force e to have a certain direction. So there is the same amount of acyclic orientations. Piecing it all together $a(G) = a(G - e) + 0 = a(G - e) + a(G/e)$. □

The proof for $c(G)$ goes similarly, except it is a bit more involving to handle the multiple edges.

PROOF. First we consider the case where e is not a multiple edge. We look at $G - e$, where $e = xy$.

Let β_1 be the number of orientations of $G - e$ where there is no directed path that corresponds to an y to x path in G.

Let β_2 be the number of orientations of $G - e$ where there is no directed path that corresponds to an x to y path in G.

Let β_3 be the number of orientations of $G - e$ where there is both a directed path that would correspond to one from x to y and one from y to x.

Then $c(G) = \beta_1 + \beta_2 + 2\beta_3$ since the first two force a direction on e, while the last allows e to be directed freely. $c(G - e) = \beta_3$, since the two first needs e to complete the cycle, and $c(G/e) = \beta_1 + \beta_2 + \beta_3$. It is now easy to see that $c(G) = c(G - e) + c(G/e)$.

If $e = xy$ is part of a multiple edge, let C be set of all edges between x and y. Now the βs from above are found on $G - C$ instead of $G - e$, but are otherwise the same.

The expressions for $c(G)$, $c(G - e)$ and $c(G/e)$ are a bit different:

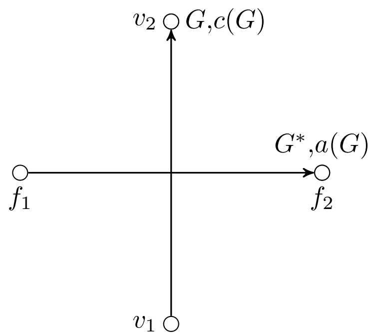
$c(G/e) = 2^{|C|-1}(\alpha_1 + \alpha_2 + \alpha_3)$, since the other edges between x and y have been turned into loops.

$c(G) = (2^{|}C| - 1)(\alpha_1 + \alpha_2) + 2^{|C|} * \alpha_3$, since the two first orientations force an orientation on one of the edges in C, but the last one doesn't

$c(G - e) = (2^{|C|-1} - 1)(\alpha_1 + \alpha_2) + 2^{|C|-1}\alpha_3$, since the two first orientations force an orientation on one of the edges in $C-$, but the last one doesn't

The expressions for $c(G - e)$ and $c(G/e)$ add up to the expression for $c(G)$, so Deletion-Contraction works in this case.

Figure 1.3: The relationship between the orientation of an edge and the orientation of the corresponding edge in the dual graph.



I

□

So both $\tau(G)$, $a(G)$ and $c(G)$ obey Deletion-Contraction, you would think there is a connection, wouldn't you? It turns out that there is, but it will have to wait until we get to the Tutte polynomial.

Theorem 1.7 *The number of acyclic orientations in a graph G is equal to the number of totally cyclic orientations in the dual G^*, i.e. $a(G) = c(G^*)$.*

This proof was first sketched in personal communication with my advisor, Carsten Thomassen.

Proof. In this proof I look at totally cyclic orientations in the original graph and acyclic orientations in the dual. If you want to go the opposite way, you can just pretend your graph was the dual graph all along.

I start by defining a relation between orientations of a graph G and orientations of its dual G^*. This relation is illustrated in figure 1.3. Imagine G and its dual G^* are drawn on top of each other. Then look at an edge $e = v_1v_2$ in G. If e is orientated from v_1 to v_2, start at v_1 and walk along the edge towards v_2. At some point you will meet the dual edge. Then the dual edge should be oriented to you right.

I now prove that each acyclic orientation corresponds to exactly one totally cyclic orientation, and vice versa, i.e. I find a bijection.

An oriented cycle in an totally cyclic orientation divides the dual graph into two components with all edges oriented from one to the other.

Since every edge in a totally cyclic orientation is in a directed cycle, the dual of that edge, xy is oriented from one such component to the other and there is no way to get from y to x. So no edge in the dual can be in a directed cycle, so the dual of a totally cyclic orientation will be an acyclic orientation. And since changing the orientation of one edge will change the orientation of exactly one edge in the dual (and no other change can change the orientation of that edge), each totally cyclic orientation corresponds to an unique acyclic orientation. So there must be $c(G) \leq a(G)$.

To go the other way: for any acyclic orientation in G^* look at any edge xy oriented from x to y. Since the orientation is acyclic, there can be no oriented path from y to x.

So there must exist an edge cut set C, dividing G^* into two components H_1 and H_2, with x in H_1 and y in H_2, and where all edges in C are directed from H_1 to H_2. The dual of those edges must form a directed cycle, and so the dual of xy must be in a directed cycle. So there must be $a(G^*) \leq c(G)$.

All in all tis gives $a(G^*) = c(G)$. □

1.3 Merino Welsh conjecture

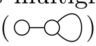
We have these three Deletion-Contraction formulas, but they split the numbers differently. Can we quantify how they split them in relation to each other? This is what the Merino Welsh conjecture is about.

Conjecture 1 (The Merino Welsh conjecture) *Let G be a bridgeless multigraph. Then* $\tau(G) \leq max\{a(G), c(G)\}$

First let's look at two simple cases, where we can verify the conjecture:

For a tree $\tau(G) = 1$, $a(G) = 2^n$ and $c(G) = 0$

For a cycle $\tau(G) = n$, $a(G) = 2^n - 2$ and $c(G) = 2$

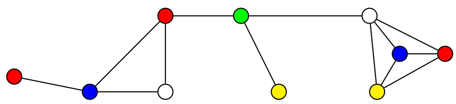
Note that the conjecture only works for bridgeless multigraphs: A connected graph G that has both a loop and a bridge, say (), will have $a(G) =$

$c(G) = 0$. But a connected graph will still have spanning trees, so the Merino Welsh conjecture doesn't hold for graphs with loops.

To put the Merino Welsh conjecture in a broader perspective, we need to look at two curious polynomials.

1.4 Chromatic & Tutte polynomials

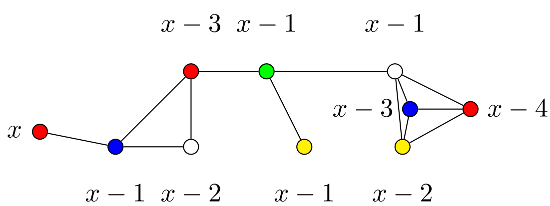
Graph colourings, an assignment of colours to vertices such that no two neighbouring vertices have the same colour:



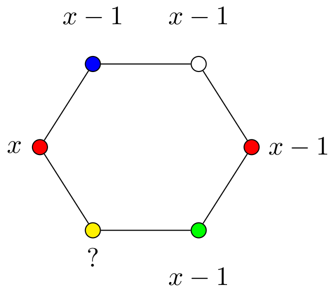
You could easily have used fewer colourings in the above example, leading to the famous question of how few you need in general. But here we are not (directly) interested in how many colours you need, but in how many different ways you can colour a graph if you are allowed to use a certain number of colours. This is what the chromatic polynomial expresses:

Definition 1.8 For a graph G, let $P(G, x)$ be the number of different ways to colour G with x different of colours.

It can be found by counting how many choices you have to colour each vertex, which is somewhat easy in simpler graphs where each vertex only depends on its neighbours. This includes the graph we coloured earlier, which has the chromatic polynomial $P(G, x) = x(x-1)^4(x-2)^2(x-3)^2(x-4)$. Starting from the left, each vertex has the number of choices marked next to it.



But in general, it is hard to find the chromatic polynomial. Here is a naive attempt at finding the chromatic polynomial of a cycle.



The number of choices at the yellow vertex depends on whether the neighbouring vertices have the same or different colours, which is hard to quantify. The chromatic polynomial of a cycle with n vertices is $P(C_n, x) = (x-1)^n + (-1)^n(x-1)$, by the way.[4]

The chromatic polynomial is related to one of our graph invariants

Theorem 1.9 *Let G be a graph. Then $a(G) = (-1)^n P(G, -1)$*

This proof is from DTU course 01527 "Graph Theory II" Proof. Proof by induction on number of edges.

If there is one edge, there are two acyclic orientations. The cromatic polynomial is $P(G, \lambda) = \lambda^{n-1}(\lambda - 1)$, so $(-1)^n P(G, -1) = (-1)^n(-1)^{n-1}(-2) = 2$. So in the base case, they are equal.

If we assume the result for fewer than m edges, then for any graph G with m edges: use deletion contraction! $a(G) = a(G-e) + a(G/e) = (-1)^n P(G - e, -1) + (-1)^n P(G/e, -1) = (-1)^n P(G, -1)$. $\square$

The chromatic polynomial can be generalized[5] as the Tutte polynomial: $P(G, x) = (-1)^n T(G, 1-x, 0)$. [6]

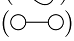
Both the chromatic and the Tutte polynomial obey deletion contraction, except the formula is slightly different in the case of the chromatic polynomial. For the chromatic polynomial the formula is $P(G, x) = P(G-e, x) - P(G/e, x)$. For

[4] I know from DTU course 01527 "Graph Theory II".
[5] The Tutte polynomial is defined in general on matroids, other examples of matroids are the columns of matrices. Matroids could be said to generalize the notion of independence.
[6] For more about this see [Wel10], [Tut54] or DTU course 01257 "Graph Theory II"

the Tutte polynomial the formula is $T(G,x,y) = T(G-e,x,y) + T(G/e,x,y)$. I will now prove the formula for the chromatic polynomial.

PROOF. For an edge $e = xy$, $P(G/e,x)$ is the number of colourings where x and y have the same colour; $P(G-e,x)$ is the number of colourings where you ignore that x and y are neighbours; If you take all the colourings where you ignore that x and y are neighbours and subtract the colourings where x and y have been given the same colour, you get the valid colourings of G. So $P(G,x) = P(G-e,x) - P(G/e,x))$ □

Now that we see that these two polynomials have been shown to also obey Deletion-Contraction, the question again arises of how these things are related. It turns out that they are something called Tutte-Grothendieck invariants. They are exactly the graph invariants that obey Deletion-Contraction, and all of them are evaluations of the Tutte polynomial.[7]

To see how our invariants are related to the Tutte polynomial, we need to look at to small graphs. Let's call the graph that only consists of a vertex and a loop L () and the graph that consists of two vertices with an edge between them I ().

$a(G)$, $\tau(G)$ and $c(G)$ are these evaluations of the Tutte polynomial.

THEOREM 1.10 *For a multigraph with loops G*

$$a(G) = T(G,2,0)$$

$$c(G) = T(G,0,2)$$

If G is connected

$$\tau(G) = T(G,1,1)$$

The exception for the number of spanning trees, is because $T(G,1,1)$ actually gives the number of maximal cycle-free subgraphs. In connected graphs the maximal cycle-free subgraphs are exactly the spanning trees.

The first has a very simple proof: $a(G) = (-1)^n P(G,-1) = (-1)^n((-1)^n T(G,1-(-1),0)) = T(P,2,0)$

All of them can be proved like this (This is very much a work in progress)
PROOF. Proof by induction on number of edges. Base cases are with one edge, I and L.

[7] See [Wel10], DJA Welsh

We use these identities to prove the base cases: $T(L,x,y) = y$ $T(I,x,y) = x$

$a(L) = 0 = T(L,2,0)$ and $a(I) = 2 = T(I,2,0)$.

$\tau(L) = 1 = T(L,1,1)$ and $\tau(I) = 1 = T(I,1,1)$.

$c(L) = 2 = T(L,0,2)$ and $c(I) = 0 = T(I,0,2)$.

As for the induction step, assume the theorem for all graphs with less than m vertices. Then look at any edge e.

If e is a loop, $T(G,x,y) = yT(G-e,x,y)$. So $T(G,2,0) = 0 = a(G)$, since there can be no acyclic orientations in a graph with loops. $T(G,1,1) = T(G-e,1,1) = \tau(G-e) = \tau(G)$, since there never are loops in spanning trees. Finally $T(G,0,2) = 2T(G-e,0,2) = 2c(G-e) = c(G)$, since each loop makes an acyclic orientation by itself and can be directed two ways.

If e is a bridge, $T(G,x,y) = xT(G-e,x,y)$. So $T(G,2,0) = 2T(G-e,2,0) = 2a(G-e) = a(G)$, which makes sense since the bridge never could be part of a oriented cycle. $T(G,1,1) = T(G-e,1,1) = \tau(G)$, this takes a bit of explaining. Since $T(G,1,1)$ is actually the number of maximal cycle-free subgraphs, $T(G-e,1,1)$ will be the product of the number of spanning trees in the two components. The bridge will be in all spanning trees of G, so $\tau(G)$ is also equal to the product of the number of spanning trees in the two components. Finally $T(G,0,2) = 0 = c(G)$, since the bridge could never be in an oriented cycle.

For all other edges, both the Tutte polynomial and the graph invariants obey Deletion-Contraction, so look at $G-e$ and G/e and use the induction hypothesis.
$\square$

You can see the Merino Welsh conjecture as statement on the convexity of the Tutte polynomial, in the sense $T(G,1,1) \leq max\{T(G,2,0), T(G,0,2)\}$. Information about what happens between them would be an even stronger result, according the original Merino Welsh article [MW99].

The Tutte polynomial has many applications outside graph theory. According to Merino and Welsh [MW99] it is used for, among other things, the Jones polynomial of alternating knots, the partition function in the Ising and Potts models of statistical physics, the flow polynomial and the terminal reliability polynomial.

CHAPTER 2

On "Spanning trees and orientations of graphs" by Carsten Thomassen

This chapter is a presentation of the article "Spanning trees and orientations" by Carsten Thomassen ([Tho10])

The theorems and corollaries from the article are referred to as "Article Theorems" and "Article Corollaries", respectively. For these I have kept their statement verbatim. This chapter largely follows the structure of the article, at least in terms of the corollaries and theorems.

The article discusses some previous/known results. I will mention these when they fit with my exposition, not in the order they appear in the article. I do this to make these fit better in the context.

First we look at graphs with a high number of edges compared to the number of vertices. We find a bound on the number of trees, and then a bound on the number of totally cyclic orientations; combined these prove the Merino Welsh conjecture for graphs G with more than $4|V(G)|-4$ edges.

Then the Merino Welsh conjecture is proved directly for graphs with a low number of edges compared to vertices, less than $\frac{16}{15}|V(G)|$ to be exact.

Then we look at cubic graphs[1] and planar triangulations: We first find a bound on the number of acyclic orientations, which we use to prove the Merino Welsh conjecture on cubic graphs. This leads to a simple proof of the Merino Welsh conjecture on planar triangulations.

This section rounds off with a look at some related open problems.

Apart from adapting, expanding and reorganizing the proofs all illustrations are also original work, though some were first sketched during meetings with my advisor.

2.1 Many edges

The first two results in the article are Theorem 1 and Corollary 1. These are bounds on the number of spanning trees in a graph, which is the left hand side of the Merino Welsh conjecture.

Article Theorem 1 *Let G be a multigraph with n vertices and vertex-degrees $d_1, d_2, \ldots, d_n$. Then*

$$\tau(G) \leq d_1 d_2 \ldots d_{n-1}$$

with equality if and only if either some d_i is zero or the vertex of degree d_n is incident with all edges.

Moreover, if M is any matching in G, then for each edge in M joining the vertices of degrees d_i, d_j, say, (both distinct from the vertex of degree d_n), the term $d_i d_j$ may be replaced by $d_i d_j - 1$ in the above product.

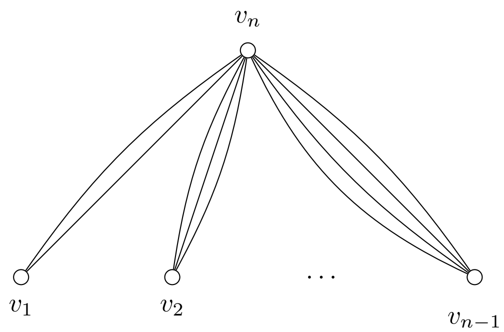
In my proof I first consider two special cases where the theorem can be proven outright. I then prove the inequality with one induction proof, and then the0 matching part with another induction proof. In the article this is proven as one big induction with everything else inside. Other than that, the proof is the same, just much more detailed.

[1] Just to remind, a cubic graph is a graph where all vertices have degree 3.

Proof. Let v_i be the vertex of degree d_i for all i. In two cases we can prove the theorem outright: If there is a vertex of degree 0, or if there is a vertex incident with all edges.

If there is a vertex of degree zero, there can be no spanning trees, which is less than or equal to any product with non-negative terms, so $\tau(G) = 0 \leq d_1 d_2 \ldots d_{n-1}$. If there is a degree zero vertex that isn't v_n, there is equality, $\tau(G) = 0 = d_1 d_2 \ldots 0 \ldots d_{n-1}$. Since a matching can't include a vertex of degree 0, if the edge $v_i v_j$ is an a matching, we can safely replace $d_i d_j$ with $(d_i d_j - 1)$ and the product will still be 0. Remember that no vertex is matched twice in a matching; this means we can replace all pairs without having to use the same d_i twice.

If there is a vertex incident with all edges, first lets assume it's v_n. This case looks like this:



In this case $\tau(G) = d_1 d_2 \ldots d_{n-1}$, since each of the $d_1 d_2 \ldots d_{n-1}$ vertices will be incident to exactly one edge in a spanning tree. Any matching in this multigraph must be a single edge. That edge must be incident with v_n, and so the matching part of the theorem doesn't apply.

If it was not v_n that was incident with all edges, but instead v_i, then

$$d_1 d_2 \ldots d_{i-1} d_i d_{i+1} \ldots d_{n-1} \geq d_1 d_2 \ldots d_{i-1} d_{i+1} \ldots d_{n-1} d_n = \tau(G)$$

If we have a matching, assume the vertex v_i is matched with is v_j. Since $d_n \leq d_i - 1$, we have that $d_n d_j \leq d_i d_j - d_j \leq d_i d_j - 1$ and so

$$\begin{aligned}\tau(G) &\leq d_1 d_2 \ldots d_{i-1} d_{i+1} \ldots d_{j-1} d_j d_{j+1} \ldots d_{n-1} d_n \\ &\leq d_1 d_2 \ldots d_{i-1} d_{i+1} \ldots d_{j-1} d_{j+1} \ldots d_{n-1} (d_i d_j - 1)\end{aligned}$$

The rest of this proof goes by induction, twice. The first part of the theorem I will now prove by induction on the number of vertices, n, with the base cases of one or two vertices. In a multigraph with one vertex, that vertex will have degree 0, which was covered above. In a multigraph with two vertices, both vertices are incident with all edges, which was also covered above.

Now to the actual induction step. First we assume the first part of the theorem for all multigraphs with less than n vertices. If there is no vertex incident with all edges, there must be an edge e not incident with v_n. Assume this edge is between v_{i_1} and v_{j_1} (where $i_1, j_1 < n$). The deletion-contraction formula for spanning trees is

$$\tau(G) = \tau(G - e) + \tau(G/e)$$

Using the induction hypothesis

$$\tau(G - e) \leq (d_{i_1} - 1)(d_{j_1} - 1)d_1 d_2 \dots d_{n-1} \tag{2.1}$$

$$\tau(G \setminus e) \leq (d_{i_1} - 1 + d_{i_1} - 1)d_1 d_2 \dots d_{n-1} \tag{2.2}$$

so

$$\tau(G) \leq (d_{i_1} - 1)(d_{j_1} - 1)d_1 d_2 \dots d_{n-1} + (d_{i_1} - 1 + d_{i_1} - 1)d_1 d_2 \dots d_{n-1} \tag{2.3}$$

and if you expand that expression out you get

$$\tau(G) \leq (d_{i_1} d_{j_1} - 1)d_1 d_2 \dots d_{n-1}$$

which means that

$$\tau(G) < d_1 d_2 \dots d_{n-1}$$

and so the first part of the theorem have been proved – note the lack of equality.

Now for the matching part. We do an induction proof on the size of the matching, m'. As the base case we have a matching with a single edge. In a multigraph with two vertices, $\tau(G) = d_1 = d_2$, and so the matching part of the theorem doesn't apply. In larger graphs, if $v_{i_1} v_{j_1}$ is the edge in the matching, this is exactly what we did for equation (2.3).

For the induction step, we assume the theorem for all matchings of size less than m' on all graphs. If $e = v_{i_m} v_{j_m}$ is an edge in the matching M, then $M - e$ must be a matching on both $G - e$ and G/e. To see that this is the case, remember that none of the other edges in the matching touch v_{i_m} or v_{j_m}. Using the induction hypothesis we get

$$\tau(G{-}e) \leq d_1 d_2 \dots (d_{i_1} d_{j_1} {-} 1)(d_{i_2} d_{j_2} {-} 1) \dots (d_{i_{m-1}} d_{j_{m-1}} {-} 1)(d_{i_m} {-} 1)(d_{j_m} {-} 1) \dots d_{n-1}$$

$\tau(G\backslash e) \leq d_1 d_2 \ldots (d_{i_1} d_{j_1} - 1)(d_{i_2} d_{j_2} - 1) \ldots (d_{i_{m-1}} d_{j_{m-1}} - 1)(d_{i_m} - 1 + d_{i_m} - 1) d_{n-1}$

Due to Deletion-Contraction

$$\tau(G) = \tau(G - e) + \tau(G/e)$$

and so

$\tau(G) \leq d_1 d_2 \ldots (d_{i_1} d_{j_1} - 1)(d_{i_2} d_{j_2} - 1) \ldots (d_{i_{m-1}} d_{j_{m-1}} - 1)(d_{i_m} d_{j_m} - 1) \ldots d_{n-1}$

Note that there are no restrictions on equality when matchings are involved.□

From this you can find a bound that requires less information about the graph:

Article Corollary 1 *Let G be a multigraph with n vertices and m edges. Then*

$$\tau(G) \leq \left(2\frac{m}{n}\right)^{n-1}$$

This doesn't have an explicit proof in the article—just the facts that are necessary to prove it. Even then my proof is barely any longer.

Proof. The sum of degrees of G is $\sum_{i=1}^{n} d_i = 2m$. For a fixed m the product

$$\tau(G) \leq d_1 d_2 \ldots d_{n-1}$$

is highest when d_n is smallest. d_n is smallest when all vertices have the same degree, namely the average degree. So

$$\tau(G) \leq \left(2\frac{m}{n}\right)^{n-1}$$

□

The article mentions a similar result by Kostochka [Kos95]: $\tau(G) \leq d_1 d_2 \ldots d_n/(n-1)$. It then goes on to compare these two bounds with some classes of graphs where the number of trees is known exactly. These exact formulas are from [Ber71].

Cayleys formula states that a complete graph K_n has n^{n-2} spanning trees. Since all vertices have degree $n-1$ in a spanning tree, Theorem 1 and Kostochka's formula gives $(n-1)^{n-1}$ as a bound.

Scoin's formula states that a complete bipartite graph $K_{p,q}$ has $p^{q-1}q^{p-1}$ spanning trees. In such a bipartite graph, there will be p vertices of degree q and q of degree p. If we assume, without loss of generality, that $p \geq q$, Theorem 1 gives $q^p p^{q-1}$, since we have to leave out a p or a q. Kostochka's formula gives $p^q q^p/(p+q-1)$, which is slightly tighter than the bound from Theorem 1.

It is proposed that for graphs with $\Omega(n^2)$ edges, the ratio of the product of vertex degrees and $\tau(G)$ is bounded by a polynomial of n. Kostochka has proved something weaker than this.

The next result is Theorem 2, which is a bound on the number of totally cyclic orientations, used on the right hand side of the Merino Welsh conjecture.

Theorem 2 in combination with Corollary 1 will give us our first proof of the Merino Welsh conjecture for some class multigraphs.

Article Theorem 2 *Let G be a connected, bridgeless multigraph with n vertices and m edges. Then*

$$c\,(G) \geq 2^{m-n+1}$$

This proof goes by induction on how many more edges than vertices there are. Most of the proof is explaining how the induction step works. The original proof in the article has the same structure, only shorter and much less explicit.

Proof. We prove by induction on k the difference between m and n: $k = m - n$. A connected bridgeless subgraph has minimum degree 2,[2] so a minimal connected bridgeless multigraph is a cycle, so this will be our base case. In this case $m - n = 0$ and a cycle has two totally cyclic orientations, so

$$c(G) = 2 \geq 2^{0+1}$$

In the induction case, we assume that $c(H) \geq 2^{m'-n'+1}$ for all graphs H with n' vertices, m' edges and $m' - n' < k$.

Then for any graph G with $m - n = k$ let $H \subset G$ be a maximal connected bridgeless proper subgraph of G. Let n' be the number of vertices and m' the number of edges in H.

[2] If there were degree 0 vertices, it wouldn't be connected, and if there were degree 1 vertices, it wouldn't be bridgeless.

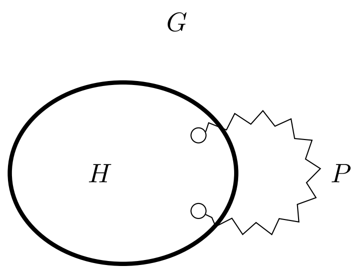
Since H is a proper subgraph, there must be some part P' of G that isn't in H. Since G is connected, some of the edges in P' must be incident with vertices in H. Let P be P' with these vertices in H included.

P must be connected, otherwise you could add one of the components of P to H without getting G, which contradicts the maximality of H.

P must also have at most two vertices in H, otherwise you could take two vertices in $H \cap P$ and find a path Q between them in P. There must be an edge in $P - Q$ for the third vertex, so $H \cup Q \subsetneq G$, which goes against the maximality of H.

Which means P must be a path or a cycle, since otherwise you could find a path as subset of P with endvertices in $P \cap H$ or a cycle as a subset of P with a vertex in $P \cap H$, both contradicting the maximality of H.

If P is a path, it must have one more vertex than it has edges. That means P' must have fewer vertices than edges, since there are two in $P \cap H$. If P is a cycle, it must have as many vertices as edges. That means P' must have fewer vertices than edges, since there is a vertex in $P \cap H$. Eiter way that means $|m' - n'| < |m - n|$ and H is covered by our induction hypothesis.



By the induction hypothesis we know that

$$c(H) \geq 2^{m'-n'+1}$$

In addition, no matter how we orient[3] P it will give a totally cyclic orientation of G in combination with a totally cyclic orientations of H[4], so

$$c(G) \geq 2c(H) \geq 2 \cdot 2^{m'-n'+1} = 2^{(m'-n'+1)+1} \tag{2.4}$$

Since there is one more edge than vertex in P, we have

$$m - n = m' - n' + 1$$

and so, in combination with (2.4), we have

$$c(G) \geq 2^{m-n+1}$$

□

All the results so far can be combined to get Corollary 2: that the Merino Welsh conjecture is true for multigraphs with many edges. Interestingly enough, we don't need to consider $a(G)$ at all in this case, where we have plenty of edges. This is intuitively plausible, as many edges would lead to more ways of making cycles in the graph.

Article Corollary 2 *Let G be a bridgeless multigraph with n vertices, m edges. If $m \geq 4n - 4$, then*

$$\tau(G) < c(G)$$

Again with this corollary, there was no proof given in the article, as it is a pretty straight forward piece of formula manipulation.

Proof. Corollary 1 states that

$$\tau(G) \leq \left(2\frac{m}{n}\right)^{n-1}$$

if we assume that $m \geq 4n - 4$, we get

[3] In this case orienting a path means orienting all the edges the same way

[4] Note that there might be orientations that become totally cyclic with the addition of the orientations of P

$$\tau(G) \leq \left(2\frac{m}{\frac{m}{4}+1}\right)^{n-1}$$

which can be rewritten as

$$\tau(G) \leq \left(8\frac{m}{m+4}\right)^{n-1} \tag{2.5}$$

Theorem 2 states that

$$c(G) \geq 2^{m-n+1}$$

if we again assume that $m \geq 4n-4$, we get

$$c(G) \geq 2^{3n-3} = 8^{n-1} \tag{2.6}$$

since $\frac{m}{m+4} < 1$, we can combine (2.5) and (2.6) to get

$$\tau(G) < 8^{3n} \leq c(G)$$

□

2.2 Few edges

Now we look at the situation where there is a large number of edges in relation to vertices. The first step is this proposition, which is used in Theorem 3, but no proof or source was given in the article.

Proposition 2.1 *For all real $x \geq 3q > 0$*

$$2^q\left(\frac{x}{q}+1\right)^q \leq 2^x$$

Proof. For $y \geq 3$ it holds that

$$2y+2 \leq 2^y$$

Since $\frac{x}{q} \geq 3$ replace y by $\frac{x}{q}$ to get

$$2\frac{x}{q} + 2 \leq 2^{\frac{x}{q}}$$

Take the qth power on both sides:

$$\left(2\left(\frac{x}{q}+1\right)\right)^q \leq 2^{\frac{x}{q}q}$$

which is

$$2^q\left(\frac{x}{q}+1\right)^q \leq 2^x$$

$\square$

Now we can prove Theorem 3 from the article, a direct verification of the Merino Welsh conjecture for graphs with a small number of edges compared to vertices. And again we only need one of the invariants; this time just the number of acyclic orientations. Intuitively this still makes sense, as fewer edges would mean that there are fewer ways you could make a cycle.

Article Theorem 3 *Let G be a graph with n vertices and m edges. If $m \leq \frac{16}{15}n$ then*

$$\tau(G) < a(G)$$

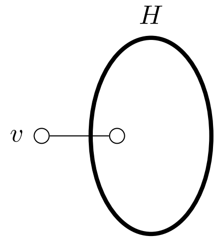
This theorem is proven by induction on the number of vertices, but the case where the minimum degree is 2 takes up most of the proof. In this case, we prove that the graph G is a subdivision of a unique graph H with minimum degree 3 except for vertices incident with a single double edge. This H might have a special structure, where the theorem is easy to prove. If not, the edges and vertices of H are subject to an inequality, which allows us to prove the theorem. The proof in the article follows the same structure, but is again much more concise and implicit.

Proof. We will prove this by induction on the number of vertices, n. Our base case is the 2-path (○—○) which has 1 spanning tree and 2 acyclic orientations.

Now for the induction step, assume the theorem holds for all graphs with less than n vertices.

If G is not connected, there are no spanning trees and at least 1 acyclic orientation. If there are vertices of degree 0, the graph is not connected.

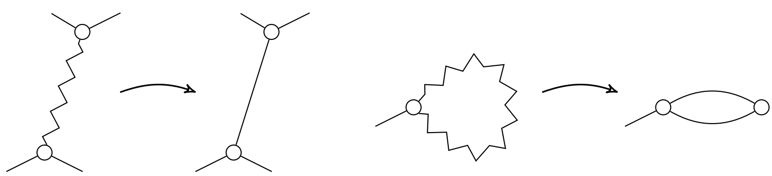
If there are vertices of degree 1, let v be a vertex of degree 1 and let H be G without v.



Since there are fewer than n vertices in H, the induction hypothesis tells us that $\tau(H) < a(H)$. The edge from v to H must be in all spanning trees of G, so $\tau(G) = \tau(H)$, but no matter how we orient that edge, it will never be part of a oriented cycle, so $a(G) = 2a(H)$. Combine these three equations and we get $\tau(G) < a(G)$.

Then G must have minimum degree 2, which has two subcases: If the graph is a cycle, we already proved this in chapter 1. If G is not a cycle, then the situation is a bit more complicated.

Then we need to look at a multigraph H related to G. We can construct H from G, like so: For every vertex in G of degree at least 3 we make a corresponding vertex in H. Any edges between these vertices we keep. The remaining vertices in G now all have degree 2. That means they must be on a suspended path in G. For any such path, say between vertices x and y in G, if $x \neq y$, add an edge between the vertices corresponding to x and y in H. If $x = y$, add a new vertex in H with a double edge to the vertex corresponding to x. Note that H needs to be a multigraph, since there could be many different paths between x and y.

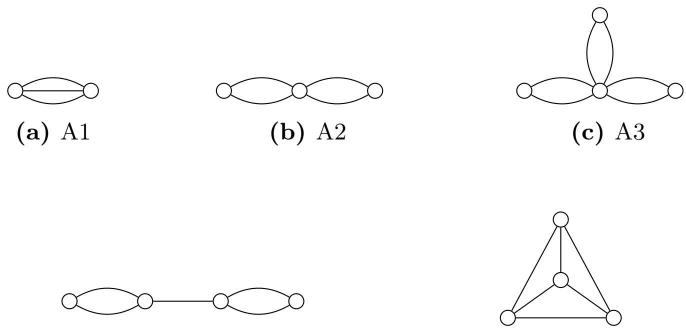


Now the only vertices in H of degree 2 are incident with a double edge, all the other vertices have degree at least 3. Let p be the number of vertices in H and let q be the number of edges. I now claim that $q \geq \frac{4}{3}p - \frac{1}{3}$ and that H has a special structure when there is equality. I will examine this structure after I have proven the formula.

If all vertices have degree at least 3, we must have $q = \frac{3}{2}p > \frac{4}{3}p - \frac{1}{3}$ [5]. If there are vertices of degree 2, I prove the claim by induction on the number of vertices.

As a base case, we have graphs with at most 4 vertices. There are five archetypical graphs with at most 4 vertices and minimum degree 2, excluding the 2-cycle, since we already took care of cycles. The archetypical graphs are shown in figure 2.1. All other graphs on at most four with minimum degree 2 can be made by adding edges to these graphs:

Figure 2.1: The five archetypical graphs



(a) A1 **(b)** A2 **(c)** A3

(d) A4 **(e)** A5

	A1	A2	A3	A4	A5
p	2	3	4	4	4
$4p/3 - 1/3$	7/3	11/3	5	5	5
q	3	4	6	5	6

Table 2.1

In table 2.1 you can see p, q and the right hand side of the inequality. We see that they all are good and that there only is equality in the case of A4. If you added edges to these graphs, you would not change p and so you would just make the left hand side of the inequality bigger.

So for the induction step, assume that $q \geq \frac{4}{3}p - \frac{1}{3}$ for all graphs of this sort with fewer than q vertices. Then look at a vertex x of degree 2.[6] This vertex must be incident with a double edge to, say, a vertex y. There are six different such cases.

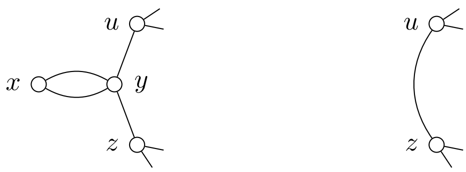
[5] since we have a minimum degree, $p \geq 1$

[6] we have already considered the case where all vertices have degree 3 or more.

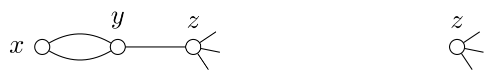
If y has degree 5 or more, use the induction hypothesis on $H-x$ with p', q'. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $1 = q' + 2$ and $\frac{4}{3}(p'+1) - \frac{1}{3} = q' + \frac{4}{3} < q$.

If y has degree 4 and is incident with another double edge to some vertex z, use the induction hypothesis on $H - x - y$. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $q = q' + 4$ and $\frac{4}{3}(p'+2) - \frac{1}{3} = q' + 2\frac{4}{3} < q$.

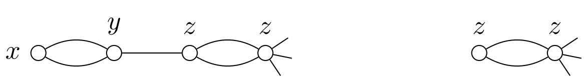
If y has degree 4 and is incident with 2 single edges to vertices z and u, use the induction hypothesis on $H - x - y$ with an edge added between z and u. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $q = q' + 3$ and $\frac{4}{3}(p'+2) - \frac{1}{3} = q' + 2\frac{4}{3} < q$.



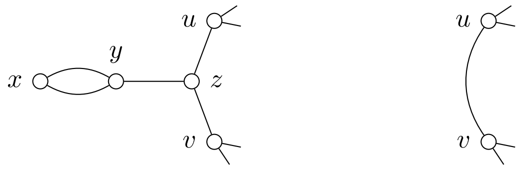
If y has degree 3 and the other neighbour z of y has degree 4 or more, use the induction hypothesis on $H - x - y$ with p', q'. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $q = q' + 3$ and $\frac{4}{3}(p'+2) - \frac{1}{3} = q' + 2\frac{4}{3} < q$.



If y has degree 3 and the other neighbour z of y has degree 3, and z has a double edge to say u, use the induction hypothesis on $H - x - y$ with p', q'. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $q = q' + 3$ and $\frac{4}{3}(p'+2) - \frac{1}{3} = q' + 2\frac{4}{3} < q$.

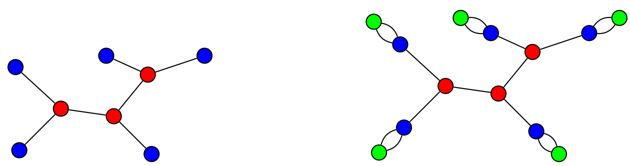


If y has degree 3 and the other neighbour z of y has degree 3, and z has 2 neighbours say u and v, use the induction hypothesis on $H - x - y - z$ adding an edge between u and v with p', q' vertices, edges. Then $q' \geq \frac{4}{3}p' - \frac{1}{3}$, adding the part we removed we get $q = q' + 4$ and $\frac{4}{3}(p' + 3) - \frac{1}{3} = q' + 4\frac{4}{3} = q$. So this is the only place we have equality.



That is all the induction steps, now for the special structure. The structure is partly a tree with only degree 1 and 3 vertices, with a vertex with a double edge added to each degree 1 vertex. That way all vertices have degree 2 or 3. This is the case for the base case with equality; it consists of the tree on 2 vertices plus degree 2 vertices.

The structure is illustrated below. The red vertices are degree 3 vertices in the tree, the blue vertices are degree 1 in the tree, green vertices are added to the blue ones.



The induction step with equality adds a degree 3 vertex in the middle of any edge, connected to a degree 1 vertex with a degree 2 vertex on. This actually permits adding the degree 3 vertex to one of the double edges, but that would leave a degree 2 vertex without a double edge. Such a H could not have been constructed.

If H is a graph with such a structure, call the tree part T. In G this corresponds to a tree T where all vertices have degree 1, 2 or 3. Let l be the number of vertices in T, then T has 1 spanning tree and 2^{l-1} acyclic orientations.

The rest of H is 2-cycles attached to the degree 1 vertices of T', which corresponds cycles in G. For each of these cycles, if the cycle is of length k_i, it has k_i spanning trees and $2^{k_i} - 2$ acyclic orientations. The number of acyclic orientations in G is the product of the number of acyclic orientations in T and all the cycles, and likewise for the spanning trees. So if there are I degree 2 vertices in H,

$$\tau(G) = 1 \cdot \prod_{i=0}^{I} k_i$$

and

$$a(G) = 2^l \prod_{i=0}^{I} \left(2^{k_i} - 2\right) s$$

So when H has this special structure, $\tau(G) \leq a(G)$.

Now we look at the remaining versions of H where $q > \frac{4}{3}p - \frac{1}{3}$. In this case we will prove that if s is a nonnegative real number and $m \leq \frac{16}{15}n + \frac{s}{15}$ then

$$\tau(G) \leq 2^s a(G)$$

which is slightly stronger than the original theorem. By the definition of H, G can be formed by inserting p_i vertices into each edge e_i of H. We introduce $r = p_1 + p_2 + \cdots + p_q = n - p$. A very rough bound on the number of spanning trees of H is 2^q, since each edge can either be or not be in a tree. If we have a spanning tree of H, we can make a spanning tree of G by, for each edge e_i not in the tree, adding all but one of the $p_i + 1$ edges to the tree. This can be done in $p_i + 1$ ways per edge not in the tree, and so a very rough bound on number of spanning trees of G is

$$\tau(G) < 2^q (p_1 + 1)(p_2 + 1) \ldots (p_q + 1)$$

and, in likeness with Corollary 2, since a product is maximized when all the terms are around the same size and the average is $\frac{r}{q}$, we have

$$\tau(G) < 2^q \left(\frac{r}{q} + 1\right)^q$$

We can create acyclic orientations of G by, for each $e_i \in N$, orienting all but one of the edges of a e_i as we want, so each $e_i \in H$ gives rise to at least 2^{p_i} acyclic orientations, and so we have

$$a(G) \geq 2^{p_1} 2^{p_2} \ldots 2^{p_q} = 2^r$$

where the right part follows from the definition of r.

To go on, we need that

$$q + r = m$$

and using the expanded assumption of the theorem that $m \leq \frac{16}{15}n + \frac{s}{15}$ for any real $s \geq 0$ we get

$$q + r \leq \frac{16}{15}n + \frac{s}{15}$$

Using that $n = p + r$

$$q + r \leq \frac{16}{15}(p + r) + \frac{s}{15}$$

and since $q \geq p\frac{4}{3}$

$$q + r \leq \frac{16}{15}\left(q\frac{3}{4} + r\right) + \frac{s}{15}$$

which expanded gives

$$q + r \leq q\frac{4}{5} + r\frac{16}{15} + \frac{s}{15}$$

and reduces to

$$s + r \geq 3q$$

Now we can add s to the inequality on the number of trees like this

$$\tau(G) < 2^q\left(\frac{r}{q} + 1\right)^q \leq 2^q\left(\frac{r + s}{q} + 1\right)^q$$

If we use proposition 2.1 with $x = r + s$ we get

$$\tau(G) \leq 2^{r+s}$$

and since $2^r < a(G)$

$$\tau(G) < 2^s a(G)$$

which is Theorem 3. □

2.3 Cubic graphs and triangulations

This time the approach is more like the first part; we prove a bound on $a(G)$ and combine it with a bound on the number of trees to get the conjecture. That means we will also have independent bounds on all the invariants

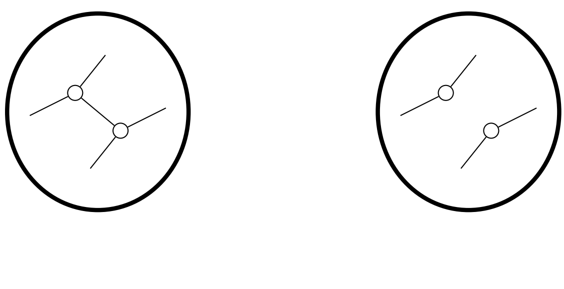
So we will now look at Theorem 4, a bound on number of acyclic orientations.

Article Theorem 4 *If G is a 3-connected cubic graph with n vertices, then for all n,*

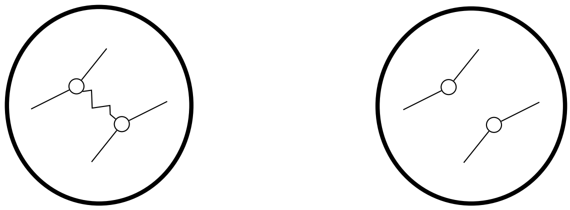
$$a(G) \geq \frac{2}{3} \cdot 6^{n/2} - 2 \cdot 3^{n/2-1}$$

To prove this, we first prove that we can decompose the graph into suspended paths and a cycle. Then we calculate a bound on $a(G)$ with that decomposition. There is no structural difference from the article.

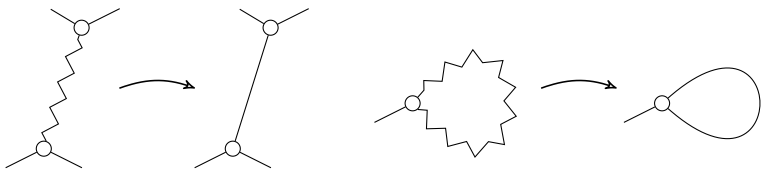
Proof. Since the graph is cubic, we start by removing an edge to have suspended paths. Then we continue to remove suspended paths until we get a cycle. From now on there will always be at least 2 vertices of degree 2, and so at least 1 suspended path. This can be seen in the following illustrations. First when removing an edge



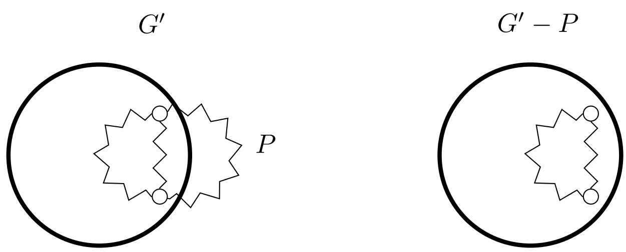
And then when we remove a suspended path from a formerly cubic graph.



To see that this is possible we will prove that whenever you delete a suspended path in this process, you leave a graph that has only 1 component with edges and this component is bridgeless. Look at a step in the process and call the component with edges G'. If G' is a cycle, we are done. If not, there must be at least 2 vertices left of degree 3.[7] Then G' is the subdivision of a unique cubic multigraph H. You can construct H from G' by taking a copy of all vertices in G' of degree 3. Then add an edge between 2 vertices in H if there is a suspended path between the corresponding vertices in G'. The question of suspended paths with the same vertex v in both ends does not come up, as that would make the remaining edge incident with v a bridge.

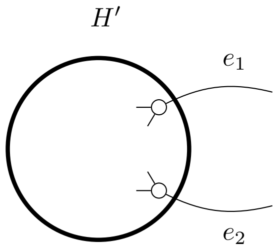


If H is 3-connected, there are 3 disjoint paths between any pair of points. So if P is a suspended path in G' there are two other paths between the endvertices of P. That means $G' - P$ is bridgeless, and if you remove the edges from P, there will be only one component with edges.So $G' - E(P)$ has 1 component with edges and this component is bridgeless.

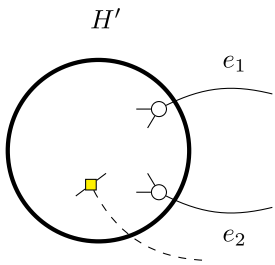


[7]There must be an even number of odd vertices, and 3 is the only allowed odd degree.

So let's look at the case when H is not 3-connected. Then there must exist edges e_1 and e_2 such that $H - e_1 - e_2$ is disconnected. Now choose e_1 and e_2 such that the smallest component of $H - e_1 - e_2$ is smallest possible. Call this component H'. Since H is cubic there must be other edges in H'.



Since G was originally 3-connected, there must at some earlier point in the process have been another path from H' to the rest of H. This path must have had an endvertex somewhere in the H' part of G'. This endvertex is now a part of a suspended path that is an edge in H'. This is illustrated below, the endvertex as a yellow square and the removed path out of H' as a dashed line.



Now since we chose H' to be minimal, e cannot be a bridge in H', and so $H - e$ will still be 2-connected, and so $G' - E(P)$ will have no bridge and there will only remain 1 component with edges.

So since you can keep removing edges until you get a cycle, we do that. Let r_1 be the number of edges in this cycle, it will have $2^{r_1} - 2$ acyclic orientations. We then add the suspended paths back in, one by one, in the right order. If the number of edges in each of the suspended paths is $r_2, r_3, \ldots, r_k$, each of them can be oriented in $2^{r_i} - 1$ ways without causing cycles.[8]

$$a(G) \geq (2^{r_1} - 2)(2^{r_2} - 1)(2^{r_3} - 1) \ldots (2^{r_k} - 1) \tag{2.7}$$

[8]If both the fully oriented ways gave totally cyclic orientations, there would already be a cycle

Since each suspended path takes 2 vertices from degree 2 to degree 3, there must be $\frac{n}{2} - 1$ suspended paths, since the edge from the start also needs two endvertices. Including the cycle, that means $k = \frac{n}{2}$

Furthermore

$$r_1 + r_2 + \cdots + r_k = m - 1 = \frac{3n}{2} - 1$$

The right hand side of equation (2.7) will be minimized when most of the terms are as small as possible, and since the suspended paths can be of length 2 and the cycle needs to be of length at least 3, the suspended paths will be made small and all in all absorb $2 \cdot (\frac{n}{2} - 1)$ edges. Then there are $\frac{3n}{2} - 1 - (n-2)$ edges left for the cycle, i.e. $r_1 = \frac{n}{2} + 1$.

$$a(G) \geq (2^{\frac{n}{2}+1} - 2)(2^2 - 1)^{\frac{n}{2}-1}$$

which expanded out give

$$a(G) \geq \frac{2}{3} 6^{\frac{n}{2}} - 2 \cdot 3^{\frac{n}{2}-1}$$

which is Theorem 4. □

With this in hand we continue with Theorem 5, verifying the Merino Welsh conjecture for cubic graphs. This time it is a different kind of class, maximum degree instead of number of edges. But maximum degree is also a limit on the number of edges.

As stated in the article, Theorem 5 is

Article Theorem 5 *If G is a multigraph of maximum degree 3, then $\tau(G) \leq a(G)$.*

But there is an exception: 2 vertices with a triple edge between them ( ), where $a(G) = 2$ and $\tau(G) = 3$. In this case $c(G) = 6$, so the Merino Welsh conjecture holds, but this still makes the induction go wrong. This problem turns out to be solvable.

I have found another problem, with the induction step. In the cases where replace a degree two vertex ( ) with an edge, and in the case where we replace ( ) with an edge, the formulas in the article were wrong. Take the case where H is a 4-cycle as a counterexample. The graphs for this counterexample are shown in figure 2.2.

The 4-cycle has 14 acyclic orientations, while the two expansions have 30 and 62, respectively. In the first case, the number of acyclic orientations should have tripled, in the second case it should have grown sevenfold. The good news is that this problem disappears when stop adding an edge to H. In the example above H would have been the 3-path instead, which has 8 acyclic orientations, which makes the mulitpliier work. As an added bonus this turns the problem with (○⊜○) into a non-issue, since it will never be the H graph in an induction step.

Figure 2.2: The bad decompositions

(a) H

(b) H + degree 2 vertex

(c) H + path with double edge

(d) What H should be

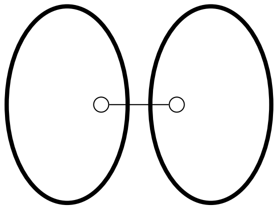
The theorem I prove is

Theorem 2.2 *If G is a multigraph of maximum degree 3, then $\tau(G) \leq a(G)$, unless G consists of a pair of vertices with 3 edges between them.*

The proof is an induction proof on the number of vertices. The first part of the proof takes care of multiple edges and vertices with degree different from 3. Now that the problem is reduced to cubic graphs, we consider the case where the graph is not 3-connected. The last part of the proof is for 3-connected cubic graphs. In the article this proof is by induction on the number of edges, but I think this method is clearer. As explained above, the induction step is also different.

Proof. If the graph consists of exactly 2 vertices with a triple edge between them, the situation is as described above. In all other cases we do induction on the number of vertices, n. Base case are multigraphs with 1 or 2 vertices and max degree 2. With 1 vertex, there can be no edges, so $\tau(G) = a(G) = 1$. If there are 2 vertices, there can be either 0, 1 or 2 edges between them. Then they have 0, 1 or 2 spanning trees, respectively, and 1, 2 or 2 acyclic orientations, respectively.

Now for the induction step, we assume the theorem for graphs with less that n. If G is disconnected, then $\tau(G) = 0$ and $a(G) \geq 1$. This covers the case where there is a component with a triple edge. If G has a bridge, then we remove it and consider the 2 connected components, say H_1 and H_2.



Then the number of spanning trees in G is $\tau(G) = \tau(H_1)\,\tau(H_2)$, since each spanning tree in G is one of the spanning trees from each connected component, plus the bridge. By the induction hypothesis $\tau(H_1) \leq a(H_1)$ and $\tau(H_2) \leq a(H_2)$, and so $\tau(H_1)\,\tau(H_2) \leq a(H_1)\,a(H_2)$[9]. Since any combination of acyclic orientations of H_1 and H_2, along with any orientation of the bridge, will result in an acyclic orientation of G, $2a(H_1)\,a(H_2) = a(G)$. All in all

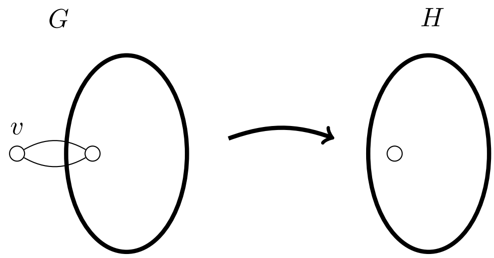
$$\tau(G) = \tau(H_1)\,\tau(H_2) \leq a(H_1)\,a(H_2) = \frac{1}{2}a(G)$$

This also takes care of vertices with degree 1, since the incident edge will be a bridge.

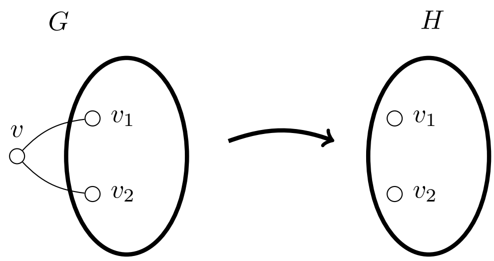
Consider the case when $v \in G$ is a point of degree 2. Then there are two cases:

If v is incident with a double edge, call the rest of the graph H. $\tau(G) = 2\tau(H)$, since a spanning tree in H plus exactly 1 of the 2 edges is a spanning tree in G. Likewise $a(G) = 2a(H)$, since an acyclic orientation of H plus any of the 2 parallel orientations (illustration of parallel) of the 2 edges gives an acyclic orientation of G. By the induction hypothesis, $\tau(H) \leq a(H)$, so $\tau(G) \leq a(G)$.

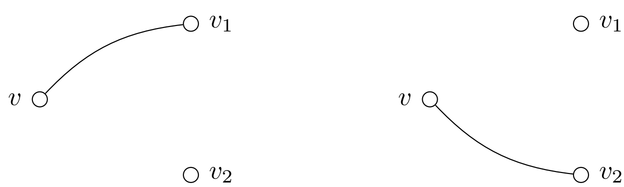
[9]We use that $\tau \geq 0$



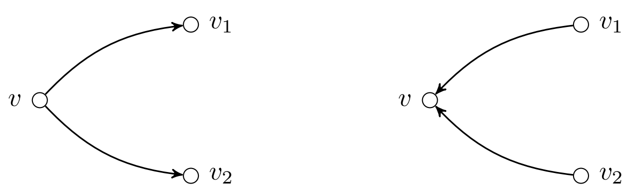
Otherwise v is incident with say v_1 and v_2. Consider now the graph H where we remove v.



Looking at the spanning trees of H the worst case is when v_1v_2 isn't in the spanning tree of H, since then we need to connect v. This can be done in exactly two ways, so $\tau(G) \leq 2\tau(H)$.



Orientation-wise, no matter how v_1v_2 is oriented, if the orientations of v_1vv_2 are not aligned. there will be no cycles, that were not originally there.

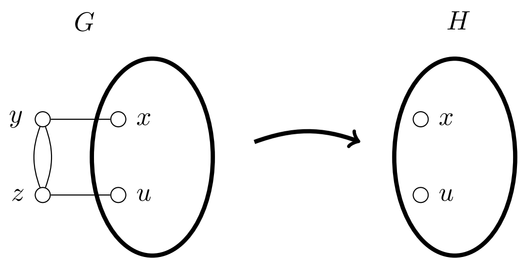


so each orientation in H corresponds to at least 2 orientations in G, i.e. $2a\,(H) \leq a\,(G)$, so

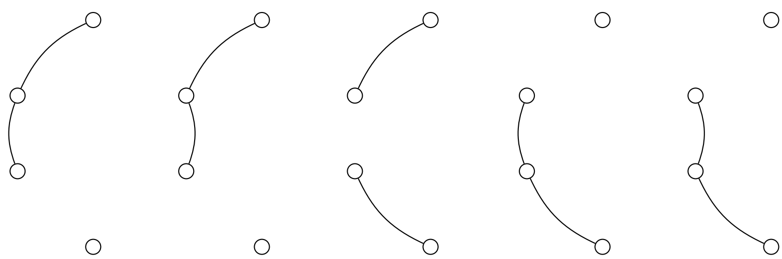
$$\tau(G) \leq 2\tau(H) \leq 2a(H) \leq a(G)$$

and the theorem holds for vertices of degree 2.

Consider the case where G contains a path $xyzu$, where yz is joined by a double edge. In case $x \neq u$, remove y and z to form H.

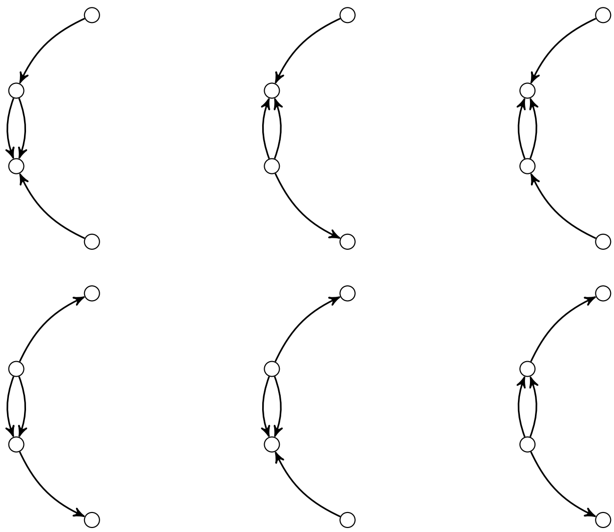


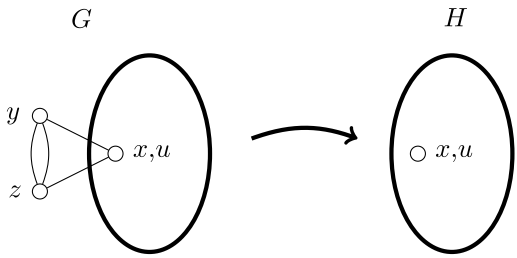
Worst case is when the tree in H does not contain xu, since then we need to connect y and z in G. This can be done in five ways, so $\tau\,(G) \leq 5\tau\,(H)$.



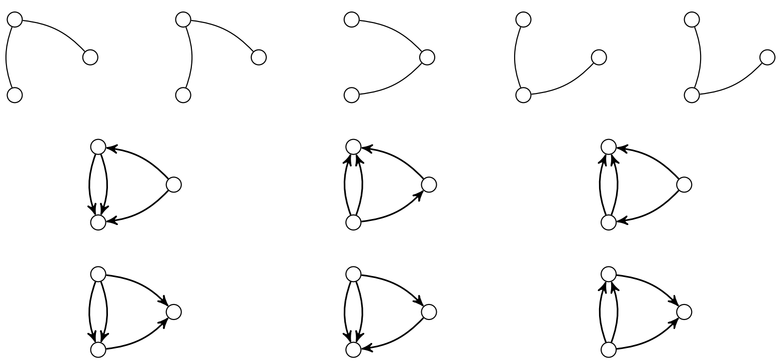
In case of acyclic orientations, there are six ways to orient the edges so they aren't aligned, and since we can always agree with xu, $7a\,(H) \leq a\,(G)$, i.e.

$$\tau\,(G) \leq 5\tau(H) \leq 5a(H) \leq 7a(H) \leq a\,(G)$$





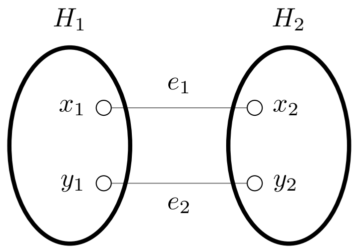
In case $x = u$, just remove y and z, leaving no edge joining x and u. Then the argument is the same, except there is no xu-edge to agree with, so $6a\,(G) \leq a\,(G)$.



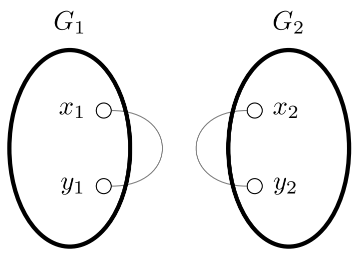
So we may assume that G is 2-connected and cubic. Assume now that G is not

3-connected. Then there exists a pair of edges $e_1 = x_1x_2$ and $e_2 = y_1y_2$ such that $G - e_1 - e_2$ is disconnected.

Let H_1, H_2 be the resulting connected components.



Now we form G_1 by adding the edge x_1y_1 to H_1 and G_2 by adding x_2y_2 to H_2, whether or not those edges existed to begin with. Remember, that since G is 2-connected, $x_1 \neq y_1$ and $x_2 \neq y_2$, so G_1 and G_2 will be loopless.



A spanning tree in G can contain either both of e_1 and e_2 or only 1 of them. A spanning tree that contains both e_1 and e_2 must contain a path between them; assume this path is in H_1. Then the part of the spanning tree that is in H_2 must be a spanning forest, which becomes a spanning tree when we add x_2y_2.

If we introduce $\tau'(G_2)$ as the number of spanning trees in G_2 that includes the edge x_2y_2, then the number of spanning trees in G containing both e_1 and e_2 and a path between them in H_1 must be $\tau(H_1)\tau'(G_2)$. Similarly the number of spanning trees in G containing both e_1 and e_2 and a path between them in H_2 must be $\tau(H_2)\tau'(G_1)$.

A spanning tree in G that contains exactly one of e_1 and e_2 must contain a spanning tree in both H_1 and H_2, and so there are $2\tau(H_1)\tau(H_2)$ such trees. All in all

$$\tau(G) = \tau(H_1)\,\tau'(G_2) + \tau(H_2)\,\tau'(G_1) + 2\tau(H_1)\,\tau(H_2)$$

and since the spanning trees of H_1 are exactly the spanning trees of G_1 not containing x_1y_1, we have $\tau(G_1) = \tau(H_1) + \tau'(G_1)$, and similarly $\tau(G_2) = \tau(H_2) + \tau'(G_2)$. This means that

$$\tau(G) = \tau(H_1)\,\tau(G_2) + \tau(H_2)\,\tau(G_1)$$

Now we introduce $a_1(H_i)$ as the number of acyclic orientations of H_i that contains a directed path between x_i, y_i , and $a_0(H_i)$ as the number of acyclic orientations that doesn't contain a directed path between x_i, y_i. Since an acyclic orientation either does or doesn't contain a directed path between x_i, y_i, we have

$$a(H_i) = a_0(H_i) + a_1(H_i) \tag{2.8}$$

and

$$a(G_i) = 2a_0(H_i) + a_1(H_i) \tag{2.9}$$

since a directed path between x_i and y_i enforces[10] a direction on x_iy_i.

If we combine orientations of H_1 and H_2 and orient e_1 and e_2 willy-nilly, we get

$$a(G) \leq 4a(H_1)\,a(H_2)$$

If e_1 and e_2 both are oriented from H_1 to H_2 (or the other way), they cannot cause cycles, so let's look at the case where e_1 is oriented from H_1 to H_2 and e_2 from H_2 to H_1. Then a cycle would only appear when there is a path in H_1 directed from e_2 to e_1 and a path in H_2 directed from e_1 to e_2. The number of such orientations is $\frac{a_1(H_1)}{2}$ and $\frac{a_1(H_2)}{2}$, respectively, since there is only one edge-disjoint path and you can just flip the edges. That makes $\frac{a_1(H_1)}{2}\frac{a_1(H_2)}{2}$ cyclic orientations with e_1 oriented from H_1 to H_2, and by symmetry we get the same amount with e_1 oriented from H_2 to H_1. So all in all

$$a(G) = 4a(H_1)\,a(H_2) - 2\frac{a_1(H_1)}{2}\frac{a_1(H_2)}{2}$$

[10] in order to preserve acyclicness

If we use formula (2.8) to expand we get

$$a(G) = 4a_0(H_1)\,a_0(H_2) + 4a_0(H_1)\,a_1(H_2) + 4a_1(H_1)\,a_0(H_2) + \frac{7}{2}a_1(H_1)\,a_1(H_2) \tag{2.10}$$

Putting it all together we have

$$\tau(G) = \tau(H_1)\,\tau(G_2) + \tau(H_2)\,\tau(G_1)$$

if we use the induction hypothesis on H_1, H_2, G_1 and G_2, we get

$$\tau(G) \leq a(H_1)\,a(G_2) + a(H_2)\,a(G_1)$$

expanding these with equation (2.8) and equation (2.9), we get

$$\begin{aligned}\tau(G) \quad \leq \quad & (a_0(H_1) + a_1(H_1))\,(2a_0(H_2) + a_1(H_2)) \\ & + (a_0(H_2) + a_1(H_2))\,(2a_0(H_1) + a_1(H_1))\end{aligned}$$

expanding this we get

$$\tau(G) \leq 4a_0(H_1)\,a_0(H_2) + 3a_0(H_1)\,a_1(H_2) + 3a_1(H_1)\,a_0(H_2) + 2a_1(H_1)\,a_1(H_2)$$

which is less than the right hand side of equation (2.10), so we get

$$\tau(G) \leq a(G)$$

So we may assume G is 3-connected and cubic. If G has more than 26 vertices then Theorem 4 combined with

$$\tau(G) \leq 2^{\frac{5}{3}}\left(\frac{16}{3}\right)^{\frac{n}{2}}$$

from [McK83] gives $\tau(G) \leq a(G)$.

Since G is cubic and every graph must have an even number of odd vertices, G must have an even number of vertices. So we only need to check graphs with an even number of vertices. In the article the following bounds are presented

n	4	6	8	10	12	14	16
max $\tau(G)$	248832	16	81	392	2000	9800	50421
min $a(G)$	$1.1 \cdot 10^6$	18	126	810	5000	$3 \cdot 10^4$	$1.8 \cdot 10^5$

n	18	20	22	24	26
max $\tau(G)$	1265625	6422000	32710656	$\leq 8.6 \cdot 10^8$	$\leq 4.5 \cdot 10^9$
min $a(G)$	$6.7 \cdot 10^6$	$4 \cdot 10^7$	$2.4 \cdot 10^8$	$1.4 \cdot 10^9$	$8.7 \cdot 10^9$

The bounds on $a(G)$ are from Theorem 4. The bounds on $\tau(G)$ originate from private communication between Carsten Thomassen and Gordon Royle for $n = 4, 6, \ldots 22$, and with Brendan McKay for $n = 24, 26$. $\square$

Now that we have proved Theorem 5, we can do another Merino Welsh verification straight away! This time it is Theorem 6, verifying the Merino Welsh conjecture for planar triangulations.

Article Theorem 6 *If G is a planar triangulation, then*

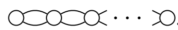
$$\tau(G) \leq c(G)$$

This proof is by direct usage of some identities for planar dual graphs.

Proof. Let H denote the geometric dual graph of G. Then $\tau(G) = \tau(H)$. Since G is a planar triangulation, H must be a cubic multigraph, and so by theorem 5 $\tau(H) \leq a(H)$ and $\tau(G) \leq a(H)$. Finally since $a(H) = c(G)$, we have that $\tau(G) \leq c(G)$. $\square$

2.4 Two open problems

It is interesting that, in our results, $\tau(G)$ is bounded by $a(G)$ when there are few edges, and $c(G)$ when there are many edges. You could ask yourself if that is the general situation, and that there might be some place they meet. This question is the first of 2 open problems posed in the article, except it is stated in even stronger terms: Is the dividing line $m = 2n - 2$?

This dividing line is motivated by the "double-path" on n vertices: . This multigraph has $\tau(G) = a(G) = c(G) = 2^{n-1}$. Add an edge between any pair of consecutive vertices and you get a graph with $2n-1$ vertices and $a(G) = 2^{n-1} < 3 \cdot 2^{n-2} = \tau(G)$. The dual multigraph has $2n-3$ edges and $c(G) < \tau(G)$. So if there is a dividing line, it must be $2n-2$. There are further considerations of this problem in chapter 3.

Do likewise about the best possible bound for the number of acyclic orientations in a 2- or 3-connected graph. $6^{smt} \rightarrow 7^{smt}$

Another open problem pertains to the number of acyclic orientations in a cubic graph. A modification of Theorem 4 gives $a(G) \geq 6^{\frac{n}{2}}/2$ for 2-connected cubic graphs. This is best possible within a factor 2. The question is then: How good can you do for 3-connected graphs? Does there exist a 3-connected cubic graph with less that $7^{n/2}n^{-1000}$ acyclic orientations.

CHAPTER 3

New results

The first proof of the Merino Welsh conjecture in the article is corollary 2, where we prove the conjecture for graphs with a large number of edges compared to vertices. Here we find that $\tau(G) \leq c(G)$, in agreement with problem 1. The second Merino Welsh result is theorem 3, where we find that a small number of edges compared to vertices leads to $\tau(G) \leq a(G)$, also in agreement with problem 1. The next Merino Welsh result is theorem 5, where we find that cubic graphs have $\tau(G) \leq a(G)$ and since cubic graphs have $m = \frac{3}{2}n \leq 2n - 2$ for $n \geq 3$[1], this is also in agreement with problem 1.

To go on we need something called Euler's polyhedron formula. It states that, if a planar graph G has n vertices, m edges and f faces $n - m + f = 2$. In more general terms, you could also say that the Euler characteristic of the plane is 2.

As for planar triangulations, we need to use Euler's polyhedron formula. If we look at the dual graph of a planar triangulation, we have a cubic graph with f vertices, n faces and m edges. So cubic means that $f = \frac{2}{3}m$ and using the Euler characteristic $m = 3n - 6$, so a planar triangulation has more than $2n - 2$ vertices. This fits with the fact that $\tau(G) \leq c(G)$. While this is nowhere near a proof of problem 1, it still suggests a bound on a class of graphs, if that class is all on one side of $2n - 2$.

[1] Which is good enough, since the smallest cubic graph has four vertices

It is on this background I decided to look at outerplanar near-triangulations.

3.1 The Merino Welsh conjecture for outerplanar near-triangulations

Planar near-triangulations are planar graphs where the outer face is the only face that has degree different from 3. Outerplanar graphs are planar graphs where all vertices are lie on the outer face. I look at the graphs that fulfill both those conditions. Note that outerplanar near-triangulations are edge-maximal outerplanar graphs.

A way to look at an outerplanar near-triangulation is to consider it as a cycle with a number of chords. The cycle provides n edges and there must be $n-3$ chords to triangulate the inner face. So an outerplanar near-triangulation has $2(n-2)+1$ edges, so problem 1 would suggest $\tau(G) \leq a(G)$.

What I acually prove is that for any outerplanar near-triangulation $2\tau(G) \leq a(G)$.

Proof. The proof goes by induction on the number of vertices, and 2-path is my base case. The 2-path has one spanning tree and two acyclic orientations, so it's good. The 2-path isn't actually a near-triangulation, but due to the way my proof works, it is the fundamental building block of outer-planar near-triangulations.

For the induction step, look at any face/triangle in the graph. Call the edges of the triangle a, b and c, and the three (edge-)disjoint subgraphs that contain each of the edges A, B and C. Exactly half of the acyclic orientations of A will orient an edge in the triangle each way, since flipping all edges in an acyclic orientation is still an acyclic orientation. Likewise for B and C. A triangle has 6 acyclic orientations, and we can just combine the acyclic orientations of A, B and C based on the edges of the triangle, so $a(G) = \frac{6}{8}a(A)a(B)a(C)$.

We now introduce the number of spanning trees in A that contains the edge in the triangle as $\tau'(A)$. Now we combine the spanning trees of A, B and C, based on the spanning trees of the triangle. If a and b are in the spanning tree of the triangle, we can use any spanning tree of A and B, but when c isn't in the spanning tree, we need to use a tree from C where c is in, but remove c when we add it. For the three spanning trees of the triangle, that gives

$\tau(G) = \tau(A)\tau(B)\tau'(C) + \tau'(A)\tau(B)\tau(C) + \tau(A)\tau'(B)\tau(C)$. This is less that $3\tau(A)\tau(B)\tau(C)$, and using the induction hypothesis $\tau(G) \leq \frac{3}{8}a(A)a(B)a(C) = \frac{1}{2}a(G)$ □

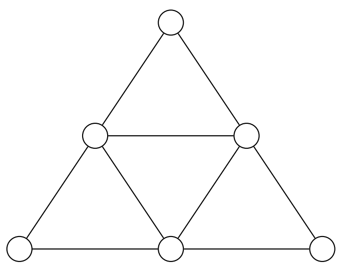
3.2 Towards outerplanar graphs

The next step from outerplanar near-triangulations is to prove the Merino Welsh conjecture for all outerplanar graphs. Since outerplanar near-triangulations are edge maximal outerplanar graphs, outerplanar graphs will have at most $2n - 3$ edges. So we are still looking for $\tau(G) \leq a(G)$.

I found out that, if you remove a chord from an outerplanar graph, you will at worst halve the number of acyclic orientations. So if all chords in outerplanar graphs are in at least half of all spanning trees, we would be able to prove the Merino Welsh conjecture for outerplanar graphs. This would happen by induction with the outerplanar near triangulations as base cases.

I did some work towards this, but it didn't bear any fruit. Chords incident with vertices of degree 3 have this property on their own, but they interact strangely.

Unfortunately, I found a counterexample:



An internal edge in this graph is in 24 different spanning trees, but there are 54 in total.

I did then find that a small reformulation of the induction step for outerplanar near-triangulations works for any outerplanar graph

Take any face. A cycle has $2^n - 2$ acyclic orientations, so $a(G) = \frac{2^n-2}{2^n}a(G_1)a(G_2)\ldots a(G_n)$. A cycle has n spanning trees, so $\tau(G) \leq n\tau(G_1)\tau(G_2)\ldots\tau(G_n)$. Using the induction hypothesis, $\tau(G) \leq \frac{n}{2^n} a(G_1)a(G_2)\ldots a(G_n) \leq a(G)$ □

3.3 Near-triangulations

Since near-triangulations have only degree 3 faces, except the outer face, the dual of a near-triangulation is a graph where all vertices except one has degree 3.

Outerplanar near-triangulations are already covered above, so we are now interested in near-triangulations with vertices in the middle of the graph. $m-2n+3$ seems to be the number of internal vertices, but I have not managed to prove it. If it is we can combine this with Euler's polyhedron formula to get a bound on the degree of the only vertex of degree ≥ 3 in the dual graph.

If you split the large vertex of the dual graph repeatedly, you can get a cubic graph, but it won't help, since all the other graphs of the Deletion-Contraction formula "go the wrong way" Merino Welsh-wise.

I have found an alternative proof of Theorem 1 from the article, but it only works for planar graphs:

PROOF. Look at the dual graph. Let the vertex of degree d_n correspond to the outer face of the dual. Then for each of the "inner faces" in the dual graph we need to remove an edge from each to get rid of the cycles of the dual graph. We can do this in $d_1d_2\ldots d_{n-1}$ ways (Argument for why this includes all spanning trees of the dual graph. Possibly using the Euler characteristic.) This gives $\tau(G^*) \leq d_1d_2\ldots d_{n-1}$. Since the number of spanning trees is the same in the two graphs, we have $\tau(G) \leq d_1d_2\ldots d_{n-1}$

For the second half: Each edge in the matching is a pair of adjecent faces in the dual. Assume these faces have degrees d_i and d_j. The case where both faces remove the same edge is useless, so for those two faces we can use $d_id_j - 1$ instead. □

The dual of a near-triangulation is 2-vertex-connected and 3-edge-connected. A sketch of the proof:

PROOF. There is one path between any two inner faces using only inner faces.

Each of those faces have two vertex-disjoint paths to the outer face, otherwise they have an additional inner path. If they have two inner paths the graph is 3-vertex-connected, otherwise it is 2-vertex connected. □

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