Partial differential equations related to multivariate phase type distributions

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Preface

This thesis was conducted in the department of informatics and mathematical modelling (IMM), Technical University of Denmark (DTU). It is submitted in partial fulfilment of the requirement for the degree of Master of Science (Mathematical Modelling and Computation).

This is a Master exam project, which is the result of work performed in the period from March 2011 to October 2011 with the amount of 30 ECTS points workload.

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I would also give my gratitude to my whole family for their sustained assistance and encouragement. In fact I have every reason to say that my family is a very important part in this half-year-long endeavour.

Last but not least, I would like to thank my friends for giving me advices and inspiring my life.

Abstract

Phase-type distributions can be understood as the distributions of absorption times of certain Markov jump processes. Phase type distributions constitute a class of distributions on the nonnegative real axis. The concept can be expanded into higher dimensions, thus multivariate phase type distributions (MPH) are obtained.

We could associate the total accumulated reward until absorption in a finite state with phase type distributions. Under this background V.G. Kulkarni introduced a new class of multivariate phase type distributions (denoted by MPH*). Usually it is difficult to compute this distribution directly. There are several computation techniques for the distributions in MPH*, of which we have a particular interest in the PDE method.

Since it is not an easy task to solve the partial differential equations directly, we have introduced power series method to see the possibilities of obtaining the distributions or survival functions. Several concrete examples have shown that the recursive equations generated from the PDEs are valid.

Finally we use some numerical tests to confirm the feasibility of the PDE-Power series method to obtain the approximate survival functions of multivariate phase type distributed random variables.

Content

Chapter 1 Introduction

A phase-type distribution is the distribution of the first passage time in a finite-state Markov chain. This distribution plays an important role in a number of applied probability problems, such as risk theory and queuing theory. The univariate case is the simplest case since univariate PH distributions can be written in a closed form including their densities, Laplace transforms and their moments. Due to these nice properties, it is relatively easy to evaluate their probabilistic quantities. The concept can also be expanded into bivariate and multivariate cases. Several techniques are possibly helpful in computing their distributions. These include partial differential equations (PDE) method and Laplace Stieltjes Transforms. The former draws a particular interest since it can be easily associated with other known analysis tools, i.e. power series method. Thus the goal of this thesis is to see the possibilities of obtaining distributions/survival functions of phase type distributed random vectors by the help of partial differential equations.

For a better understanding of the project, we present some necessary background. This means a review on the basic theory will come first because the initial idea of phase type distribution arises from the framework of continuous Markov processes. We then present the concept and the corresponding derivation of phase type distributions in both the univariate case and the multivariate case. After that the motivation of using partial differential equation method is presented; the motivation is also supported by some concrete examples. By looking at these examples, the reader will get the feeling about how real examples could work in the framework of partial differential equations. Afterwards, power series method is introduced to supplement the PDE method. Similarly, there will be concrete examples to illustrate the ideas. Finally we use some numerical tests to confirm the feasibility of PDE-Power series method to obtain the approximate survival functions of phase type distributed random variables.

A brief summary of the thesis is as follows:

Chapter 2 contains basic definitions and a brief introduction to computation techniques for obtaining survival functions of phase-type distributed variables. The importance of the partial differential equations method is highlighted.

Chapter 3 focuses on the partial differential method. We present the motivation and then show the detailed computation processes in several examples, which include the case using the identity reward matrix and a more complicated reward structure. These examples indicate the potential applications of PDE method.

Chapter 4 shows the application of PDE-power series method to obtain the survival functions of phase type distributed variables. The idea is supported by numerical test, where we have used finite power series coefficients to obtain the approximate values of the survival functions.

Chapter 5 presents the conclusion of the thesis. Some limitation of the method is pointed out. We also give some suggestions about the future work in this area.

Chapter 2 Basic theory

2.1 Continuous-Time Markov chains

In probability theory, a stochastic process is a mathematical model for the evolution of systems that either exhibits inherent randomness, or operates in an unpredictable environment. A continuous-time Markov process is a stochastic process which can describe physical systems in continuous time.

Definition 2.1.1 (Stochastic process). [1]

Let I be either $\mathbb{N}$ or $[0,\infty)$. A stochastic process on I with state space E is a collection of random variables $\{X_t : t \in I\}$ on E.

Definition 2.1.2 (Continuous-time stochastic process) [2]

A continuous-time stochastic Process $(X(t)_{t\geq 0})$, with state space E is a collection of random variables X(t) with values in E.

One fundamental example is the Poisson process, which is defined as follows:

Definition 2.1.3 (Homogeneous Poisson Process) [2]

A point process N(t) on the positive half-line is called a homogeneous Poisson process (HPP) with intensity $\lambda > 0$ if

a) For all times $t_i, i \in [1,k]$, such that $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, the random variables $N(t_i, t_{i+1}], i \in [1, k-1]$, are independent.
b) For any interval $(a,b] \subset \mathbb{R}_+, N(a,b]$ is a Poisson random variable with mean $\lambda(b-a)$.

The Poisson process describes the number of “events” taking place until time t while the waiting time from one event to the next is exponentially distributed and independent of all other waiting times.

In the following work, we are only concerned about the continuous-time phase type distributions, therefore what is needed is only the concept of continuous time Markov chains.

Definition 2.1.4 (Finite Continuous Time Markov Chains) [2]

A stochastic process $(X(t)_{t\geq 0})$ with finite state space E is a finite continuous-time Markov chain (or Markov jump process) if for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < t$ and all $j_1, ..., j_n, j \in E$

$$P(X(t) = j \mid X(t_1) = j_1, ..., X(t_n) = j_n) = P(X(t) = j \mid X(t_n) = j_n)$$

Whenever $P(X(t_1) = j_1, ..., X(t_n) = j_n) > 0$

This property is the Markov property of X(t).

From this property, we can see that the distribution of a continuous-time Markov chain is determined by the initial distribution

$$\alpha(i) = P(X(0) = i) \qquad i \in E$$

and the transition probability $P(t) = \{p_{ij}(t)\}$ $i, j \in E$

Where $p_{ij}(t) = p(X(t+s) = j \mid X(s) = i)$

Obviously P(0)=I,

Where I is the identity matrix.

In other words, $\{P(t)\}_{t\geq 0}$ is a transition semigroup on E, and we have the well-known Chapman-Kolmogorov equation:

P(t+s)=P(t)P(s) for each $t, s \geq 0$

For any state i, there exists $q_i = \lim_{h\to 0} \frac{1 - p_{ii}(h)}{h} \in [0, \infty)$,

We denote $q_{ii} = -q_i$

And for any state $i, j (i \neq j)$, there exists $q_{ij} = \lim_{h\to 0} \frac{p_{ij}(h)}{h} \in [0, \infty)$

The matrix of $Q = (q_{ij})_{i,j\in E}$ is critical and will lead to the following concept.

Definition 2.1.5 (Infinitesimal Generator) [2]

The numbers q_{ij} are called the local characteristics of the continuous-time Markov chain[3]. The matrix

$$Q = \{q_{ij}\}_{i,j\in E}$$

Is called the infinitesimal generator of the continuous-time Markov chain.

For the purpose of showing an important property of the infinitesimal generator, the following concept of inter-arrival times and embedded process is also necessary.

There exists a sequence of transition times $(\tau_n)_{n\in N}$ when X(t) jumps i.e.

$$\tau_n = \inf\{t \geq \tau_{n-1} : X(t) \neq X(\tau_{n-1})\}$$

Clearly if $\tau_n = \infty$ then $\tau_{n+k} = \infty$ for all $k \in N$.
The inter-arrival times are defined as $\tau_{n+1} - \tau_n$

Definition 2.1.6 (Embedded process) [3]

Let $(\tau_n)_{n\in N}$ be the nondecreasing sequence of transition times of the regular jump process $(X(t)_{t\geq 0})$, where $\tau_0 = 0$ and $\tau_n = \infty$ if there are strictly fewer than n transitions in $(0,\infty)$. The process $\{X_n\}_{n\geq 0}$ with values in $E_\Delta = E \cup \{\Delta\}$, where Δ is an arbitrary element not in E, is defined by

$$X_n = X(\tau_n),$$

with the convention $X(\infty) = \Delta$, and it is called the embedded process of the jump process.

Theorem 2.1.1 (Regenerative structure) [2]

Let $(X(t)_{t\geq 0})$ be a continuous-time Markov chain with state space E and infinitesimal generator $Q = \{q_{ij}\}_{i,j\in E}$
Then given the embedded process $(X_n)_{n\in \mathbb{N}_0}$ the sequence of inter-arrival times, $(\tau_{n+1} - \tau_n)_{n\in N_0}$, are independent with distribution given by

$$P(\tau_{n+1} - \tau_n > s \mid (X_n)_{n\in\mathbb{N}_0}) = \begin{cases} \exp(-q_{X_n} s) & \text{if } X_n \neq \Delta \\ 1 & \text{if } X_n = \Delta \end{cases} \quad s > 0$$

Where $q_i = \sum_{j\in E\setminus\{i\}} q_{ij}$ for all $i \in E$

2.2 Phase type distribution

With the basic theory of continuous-time Markov chains described above, the concept of phase type distributions can be now presented.

First we look at the univariate case. Consider a continuous time Markov chain (CTMC) with finite state-space E: (1, 2, … , m+1) where the first m states are transient and state m+1 absorbing. Then this Markov process has an intensity matrix

$$Q = \begin{bmatrix} T & t \\ 0 & 0 \end{bmatrix}$$

Where T is a non-singular $n \times n$ sub-intensity matrix and $t = -Te$

Here e is the m-dimensional column vector of 1's.

Let the initial distribution of $(X(t)_{t \geq 0})$ be α_i , and $\tau = \min\{t > 0 \mid X(t) = m+1\}$ be the absorbing time of the Markov process. It is clear that the distribution of τ is only dependent on α and T.

Definition 2.2.1 (Phase type distribution) [4]

The distribution of τ is called a phase-type distribution with initial distribution α and the sub-intensity matrix T. We write $\tau \sim PH(\alpha, T)$.

We can show that the probability that absorbing state is not visited at time t is $\alpha e^{tT} e$.

Proof.

$$P(\tau > t) = \sum_{j=1}^{m} P(X(t) = j)$$

$$= \sum_{i,j=1}^{m} P(X(t) = j \mid X(0) = i) P(X(0) = i)$$

$$= \sum_{i,j=1}^{m} \alpha(i) P_{ij}^{t} = \alpha e^{tT} e$$

Similarly the following theorem is obtained.

Theorem 2.2.2 If $\tau \sim PH(\alpha,T)$, then τ has the following density:

$$f(x) = \alpha e^{Tx} t$$

Proof.

$$f(x)dx = P(\tau \in (x, x+dx))$$

$$= \sum_{i,j=1}^{m} P(\tau \in (x, x+dx) \mid X(x) = j, X(0) = i) P(X(x) = j \mid X(0) = i) P(X(0) = i)$$

$$= \sum_{i,j=1}^{m} P(\tau \in (x, x+dx) \mid X(x) = j) P_{ij}^{x} \alpha(i)$$

By the definition of $t = -Te$, we get $P(\tau \in (x, x+dx)) = t_j dx$, where t_j is the jth element of t.

Thus $f(x)dx = \sum_{i,j=1}^{m} \alpha \exp(Tx)_{ij} t_j dx = \alpha e^{Tx} t dx$

In quest of moments of τ, we rely on the method of Laplace transform.

Theorem 2.2.3 If $\tau \sim PH(\alpha,T)$, then the laplace transform of τ is given by

$$L_\tau(s) = E(e^{-s\tau}) = \alpha(sI - T)^{-1} t$$

Proof. We use the density function $f(x) = \alpha e^{Tx} t$ to calculate $L_\tau(s)$, then we get

$$L_\tau(s) = \int_0^\infty e^{-sx} \alpha e^{Tx} t dx$$

$$= \int_0^\infty \alpha e^{-(sI-T)x} t dx$$

$$= \alpha(sI - T)^{-1} t$$

By successive differentiation of the Laplace transform, then the kth moment of τ is :

$$E(\tau^k) = k! \alpha(-T^{-1})^k e$$

We will give one simple example of a phase type distributions.

Example: Generalized Erlang distribution

Let $X_1, X_2, ..., X_n$ be independent and $X_i \sim \exp(\lambda_i)$, then $Y = X_1 + X_2 + ... + X_n$ has a phase type distribution with $\alpha = (1,0,0,...,0)$ and

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n \end{bmatrix}$$

Think of Y as the absorption time of a Markov chain, which starts from state 1 and can only jump to the “adjacent” state. Only from the last transient state n is the Markov process ready to be absorbed.

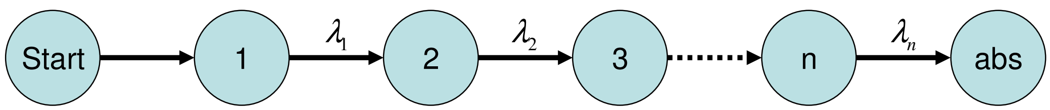


Fig.1: Markov process of the Generalized Erlang distribution.

With the knowledge gained from the univariate case, the model of the bivariate case can be introduced.

Take a continuous-time Markov chain $(X(t)_{t\geq 0})$ with state space E and sub-intensity matrix T and initial distribution α into consideration. Let A_1 and A_2 be two nonempty closed subsets of E. Also assume that the probability of eventually being absorbed into $A_1 \cap A_2$ is 1. Without loss of generality, E=(1, 2, … , m, m+1). As before, the first m states are transient while the state m+1 is the absorbing state. Then the infinitesimal generator is

$$Q = \begin{bmatrix} T & t \\ 0 & 0 \end{bmatrix}$$

We define

$$\tau_k = \inf\{t \geq 0 : X(t) \in A_k\},\ k = 1,2$$

The vector (τ_1, τ_2) then has a bivariate phase type (BPH) distribution. Mathematically, $(\tau_1, \tau_2) \sim BPH(\alpha, T)$, where α is the initial distribution. [6]

We will give one simple example of a bivariate phase type distribution.

Example [6]:

Given two independent random variables T_1 and T_2. If $T_1 \sim \exp(\lambda_1)$, $T_2 \sim \exp(\lambda_2)$, then the joint distribution of (T_1, T_2) is BPH. This could be understood by constructing a CTMC with state space $E = \{1, 2, 3, \Delta\}$, $A_1 = \{2, \Delta\}$, $A_2 = \{3, \Delta\}$ and $\alpha = (1, 0, 0)$, and

$$T = \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_2 & \lambda_1 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & -\lambda_2 \end{bmatrix}$$

In this case, the survival function $\overline{F}$ of (T_1, T_2) is $\overline{F}(t_1, t_2) = e^{-\lambda_1 t_1 - \lambda_2 t_2}$

Likewise, the above bivariate phase-type distribution could be expanded to multidimensional phase-type distribution.

Consider a continuous-time Markov chain $(X(t)_{t \geq 0})$ with state space E and sub-intensity matrix T and initial distribution α. Given k closed subsets of E: $A_1, A_2, \ldots A_k$, then it could be defined

$$\tau_i = \min\{t > 0 \mid X(t) = A_i\}\ \ (1 \leq i \leq k)$$

The random vector ($\tau_1, \tau_2, ..., \tau_k$) is then said to have a multivariate phase type distribution. Mathematically, that is $(\tau_1, \tau_2, ..., \tau_k) \sim MPH(\alpha, T; A_1, A_2, ..., A_k)$

2.3 Accumulated reward system and The Class MPH*

One of the essential applications of phase type distribution can be found by the combination of a continuous-time Markov chain and reward system. If a process has already been modelled by a CTMC, and during the sojourn in each state before being absorbed, there might be a reward gained dependent on the time spent in each state and the corresponding reward rate. The goal is thus to find the distribution or survival function of the accumulated reward. In the following work, we focus on deriving the survival function due to the convenience of calculating.

First the model is to be illustrated mathematically.

Consider again a one dimensional phase type distribution.
Consider a continuous time Markov chain with state-space (1, 2,…, m+1) with corresponding generator matrix

$$Q = \begin{bmatrix} T & t \\ 0 & 0 \end{bmatrix}$$

The absorption time is $\tau = \min\{t > 0 \mid X(t) = m+1\}$

It is assumed that before the random variable X enters the absorbing state, there is reward gained. Mathematically, there exists a vector r=(r(1),r(2),…,r(m)), where r(i) is the reward rate at state i. $(1 \le i \le m)$. Throughout our project work, we assume all reward rates are non-negative.

Therefore the total reward until absorption in the state m+1 is $Y = \int_0^{\tau} r(X(t))dt$

It can be derived that Y has a phase type distribution. The basic idea is to find a CTMC such that the distribution of the absorbing time is the same as the distribution of Y. [4]

Likewise, this one dimensional Y could be extended to k-dimensional $(Y_1, Y_2, ..., Y_k)$ using the same Markov chain mentioned above.

Suppose there are k reward systems: $r_i = (r_i(1), r_i(2), ..., r_i(m)), \quad 1 \le i \le k$.

Denote

$$R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{bmatrix}$$

as the reward matrix.

Similarly, it could be defined $Y_i = \int_0^\tau r_i(X(t))dt, \quad 1 \le i \le k.$

The new random vector $(Y_1, Y_2, ..., Y_k)$ is said to have a multivariate phase* type distribution. We write $(Y_1, Y_2, ..., Y_k) \sim MPH^*(\alpha, T; R)$ [4]

Theorem 1.3.1 MPH* is a strict superset of MPH [4]

Proof omitted.

Since phase type distributions play an important role in applied probability models, it is important to study their distributions. Usually it is difficult to compute the distributions directly. There are several computation techniques for the distributions in MPH*, of which we have a particular interest in the PDE method. In the next section an introduction will be given to these techniques including Laplace transforms and partial differential equations.

2.4 Computation techniques related to MPH*

2.4.1 Laplace Stieltjes Transforms

Consider a vector $Y=(Y_1,Y_2,...,Y_k) \sim MPH^*(\alpha,T;R)$. The conditional Laplace Stieltjes transforms are defined by

$$\begin{aligned}&\phi_i(s_1,s_2,...,s_k)\\&=E(\exp(-s_1Y_1-s_2Y_2-...-s_kY_k)\mid X(0)=i),\ 1\le i\le m\end{aligned}$$

Therefore the Laplace transform of Y is given by

$$\begin{aligned}&\phi(s_1,s_2,...,s_k)\\&=E(\exp(-s_1Y_1-s_2Y_2-...-s_kY_k))\\&=\sum_{i=1}^{m}\alpha_i\phi_i(s_1,s_2,...,s_k)+\alpha_{m+1}\end{aligned}$$

The properties of the conditional Laplace transforms will be analyzed by introducing new random variables H : the time spent in state i. [4]

By the properties of conditional expectation and Markov process, what is obtained is

$$\begin{aligned}&\phi_i(s_1,s_2,...,s_k)\\&=E(\exp(-s_1Y_1-s_2Y_2-...-s_kY_k)\mid X(0)=i)\\&=E(\exp(-s_1r_1(i)H-s_2r_2(i)H-...-s_kr_k(i)H)\mid X(0)=i)\\&\quad\cdot[\sum_{\substack{j=1\\j\ne i}}^{m}(\frac{q_{ij}}{q_i})\phi_j(s_1,s_2,...,s_k)+(\frac{q_{i,m+1}}{q_i})]\\&=\frac{q_i}{q_i+s_1r_1(i)+s_2r_2(i)+...+s_kr_k(i)}\cdot[\sum_{\substack{j=1\\j\ne i}}^{m}(\frac{q_{ij}}{q_i})\phi_j(s_1,s_2,...,s_k)+(\frac{q_{i,m+1}}{q_i})]\\&=\frac{\sum_{j=1,j\ne i}^{m}q_{ij}\phi_j(s_1,s_2,...,s_k)+q_{i,m+1}}{q_i+s_1r_1(i)+s_2r_2(i)+...+s_kr_k(i)}\end{aligned}$$

Keep in mind that $q_{i,m+1}=-\sum_{j=1}^{m}q_{ij}$, and the following equation can be finally derived:

$$\begin{aligned}&(s_1r_1(i)+s_2r_2(i)+...+s_kr_k(i))\phi_i(s_1,s_2,...,s_k)\\&=\sum_{j=1}^{m}q_{ij}\phi_j(s_1,s_2,...,s_k)+q_{i,m+1}\end{aligned}$$

This is a good result, since if written in the matrix form, it becomes

$(D-T)\phi = -Te$

Where D is a diagonal matrix with $D_{ii} = \sum_{j=1}^{k} s_j r_j(i)$, T is the sub intensity matrix of the CTMC, $\phi = (\phi_1, \phi_2, ..., \phi_k)'$, e is the column of 1.

Since T is invertible, $(D-T)$ is also invertible in a nonempty neighbourhood of $s = (s_1, s_2, ..., s_k) = 0$. Therefore the equation $(D-T)\phi = -Te$ has a unique solution, that is $\phi = -(D-T)^{-1}Te$.

However, it is difficult to invert a multidimensional Laplace transform to get the joint distribution. Nevertheless, this formula can be used to get the joint moments by taking proper derivatives. We will investigate the possibility of this computation. **[4]**

If we define $m_i(a_1, a_2, ..., a_k) = E(Y_1^{a_1} Y_2^{a_2} ... Y_k^{a_k} \mid X(0) = i),\ 1 \le i \le m$

where a_i are nonnegative integers for $1 \le i \le m$

Then using the result above, the following equations are satisfied:

$m_i(0,0,...,0) = 1,\ 1 \le i \le m$

$\sum_{j=1}^{m} q_{ij} m_j(a_1, a_2, ..., a_k) = -\sum_{j=1}^{m} a_j r_j(i) m_i(a_1 ..., a_{j-1}, a_j - 1, a_{j+1}, ..., a_k)$

One immediate outcome of these equations is the feasibility of computing the variance covariance matrix of $(Y_1, Y_2, ..., Y_k)$

A more explicit expression of the joint moments can be derived in the following theorem.**[7]**

Theorem 2.4.1. The joint moments $E(\prod_{i=1}^{n} Y_i^{a_i})$, where $a_i \in \mathbb{N}$, is given by

$$\alpha \sum_{l=1}^{a!} \prod_{i=1}^{a} (-T)^{-1} \Delta(r_{\sigma_l(i)}) e.$$

Here $a = \sum_{i=1}^{n} a_i$, r_j is the jth column of R and σ_l is one of the $a!$ possible ordered permutations of the derivatives, with $\sigma_l(i)$ being the value among 1…n at the ith position of that permutation.

2.4.2 Partial Differential Equations

Consider a vector $Y = (Y_1, Y_2, ..., Y_k) \sim MPH^*(\alpha, T; R)$. Let $\overline{F_i}$ be the survival function of Y given the initial state is i, that is

$$\overline{F_i}(y_1, y_2, ..., y_k) = P(Y_1 > y_1, Y_2 > y_2, ..., Y_k > y_k \mid X(0) = i)$$
$$1 \le i \le m$$

To obtain the important result of PDE, it is better to start with the derivation from the well-known properties of conditional probability. In the following work, we assume $y_i > 0$ for $1 \le i \le k$.

$$\overline{F_i}(y_1, y_2, ..., y_k) = P(Y_1 > y_1, Y_2 > y_2, ..., Y_k > y_k \mid X(0) = i)$$
$$= \sum_{j=1}^{m} P(Y_1 > y_1, Y_2 > y_2, ..., Y_k > y_k \mid X(0) = i, X(h) = j)$$
$$\cdot P(X(h) = j \mid X(0) = i)$$
$$= \sum_{j=1}^{m} P(Y_1 > y_1 - r_1(i)h, ..., Y_k > y_k - r_k(i)h \mid X(0) = j) \cdot P(X(h) = j \mid X(0) = i)$$

As $h \to 0$, we know $P(X(h) = j \mid X(0) = i) = q_{ij}h + o(h)$, so finally we get

$$\overline{F_i}(y_1, y_2, ..., y_k) = P(Y_1 > y_1, Y_2 > y_2, ..., Y_k > y_k \mid X(0) = i)$$
$$= \overline{F_i}(y_1 - r_1(i)h, ..., y_k - r_k(i)h)(1 - q_i h) + \sum_{\substack{j=1 \\ j \ne i}}^{m} \overline{F_j}(y_1 - r_1(i)h, ..., y_k - r_k(i)h) q_{ij} h + o(h)$$

After rearranging terms, we obtain

$$\overline{F_i}(y_1, ..., y_k) - \overline{F_i}(y_1 - r_1(i)h, ..., y_k - r_k(i)h)$$
$$= \sum_{j=1}^{m} q_{ij} \overline{F_j}(y_1 - r_1(i)h, ..., y_k - r_k(i)h)h + o(h)$$

By the definition of partial differential equations, by dividing both sides by h and letting $h \to 0$, the following important result comes out:

$$\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{m} q_{ij} \overline{F_j} \ , 1 \le i \le m$$

Although the above derivation is based on the assumption that $y_i > 0$ for $1 \le i \le k$, the final PDEs could be also valid for any case involving marginal survival functions. We can not give a thorough proof like we did in the above derivation. In the following work, we simply assume the marginal survival functions also satisfy the above PDEs.

The PDEs are very important in the following work. Some “derivative” result could also be generated with the help of this theorem. To see whether this PDE works for a real model, we will look at a simple example below.

Example: Kibble model

The model of the bivariate case is illustrated below.

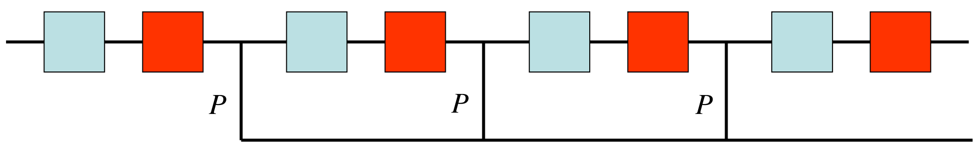


Fig 2: Markov process of the Kibble model

For the purpose of easy illustration, it is considered that the Markov chain consists of 2 transient states and one absorbing state. The time spent in each transient state (state 1 and 2) is exponentially distributed with parameter λ . When it comes to state 2, the probability that the next jump is to absorbing state is P, and that the next jump is to state 1 is (1-p).

The sub-intensity matrix of this CTMC becomes

$$T=\begin{bmatrix} -\lambda & \lambda \\ (1-p)\lambda & -\lambda \end{bmatrix}$$

In this model the reward matrix is $R=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In order to obtain two conditional survival functions $\overline{F_1}(y_1, y_2)$ and $\overline{F_2}(y_1, y_2)$, we would use the properties of the mixture of Erlang distributions.

Recall that if $X \sim Erlang(i, \lambda)$, then $\overline{F_x}(x, i, \lambda) = P(X > x) = \sum_{k=0}^{i-1} e^{-\lambda x} (\lambda x)^k / k!$

Under this condition, using the properties of 2-dimensional Erlang distributions we get the two conditional survival functions in probabilistic forms as follows:

$$\overline{F_1}(y_1, y_2) = p(Y_1 > y_1, Y_2 > y_2 \mid X(0) = 1)$$
$$= \sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$

And

$$\overline{F_2}(y_1, y_2) = p(Y_1 > y_1, Y_2 > y_2 \mid X(0) = 2)$$
$$= p\cdot 0 + \sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})$$
$$= \sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$

We would like to see how these two survival functions work in the framework of PDEs.

$$\sum_{j=1}^{2} r_j(1)\frac{\partial \overline{F_1}}{\partial y_j} = r_1(1)\frac{\partial \overline{F_1}}{y_1} = \sum_{i=1}^{\infty}[p(1-p)^{i-1}e^{-\lambda y_1}\sum_{k=0}^{i-1}(-\lambda\frac{(\lambda y_1)^k}{k!} + \frac{\lambda^k y_1^{k-1}}{(k-1)!}\cdot 1(k \geq 1))\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2}]$$
$$= -\lambda\sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})] + \lambda\sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})$$
$$= \sum_{j=1}^{2} q_{1j}\overline{F_j}$$

$$\sum_{j=1}^{2} r_j(2)\frac{\partial \overline{F_2}}{\partial y_j} = r_2(2)\frac{\partial \overline{F_2}}{\partial y_2}$$
$$= \sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}(-\lambda e^{-\lambda y_2}) + e^{-\lambda y_2}\sum_{k=1}^{i-1}\frac{\lambda^k y_2^{k-1}}{(k-1)!})]$$
$$= (1-p)\lambda\sum_{i=2}^{\infty}[p(1-p)^{i-2}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-2}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$
$$-\lambda\sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2}]$$
$$= (1-p)\lambda\sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$
$$-\lambda\sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2}]$$
$$= \sum_{j=1}^{2} q_{2j}\overline{F_j}$$

Therefore we get

$$\sum_{j=1}^{2} r_j(1)\frac{\partial \overline{F_1}}{\partial y_j} = \sum_{j=1}^{2} q_{1j}\overline{F_j}$$

$$\sum_{j=1}^{2} r_j(2)\frac{\partial \overline{F_2}}{\partial y_j} = \sum_{j=1}^{2} q_{2j}\overline{F_j}$$

The result is consistent with the PDEs. This is a good example to show how the partial differential equations work for phase type distributions. We have more interest in how to use the PDEs to obtain the distribution or survival functions. To be more specific in this bivariate model, given the reward matrix R and the sub-intensity matrix T, is it possible for us to solve the above partial differential equations to get $\overline{F_1}$ and $\overline{F_2}$? To see such possibilities, we will introduce the power series method in the next chapter.

Chapter 3 Power series method

Power series are used to solve differential equations, including many important differential equations with nonconstant coefficients such as the Legendre and Bessel equations. [8] In many cases, the solutions of differential equations have power series expansions. It might be possible to find the power series solution (if it exists) directly from the differential equation. Therefore the power series method plays an important role in PDE studies. It is worth making efforts to see whether this method is useful in phase type distribution studies.

As is known in mathematics, a power series (in one dimension) is an infinite series of the form:

$$f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + (x-c)^3 + ...$$

Where a_n is called the coefficient of the nth term and c is a constant.

Similarly, we could extend the power series from one dimension to multi-dimensions:

$$f(x_1, x_2, ...x_n) = \sum_{j_1,..j_n=0}^{\infty} a_{j_1,..j_n} \prod_{k=1}^{n} (x_k - c_k)^{j_k}$$

In our following work, it is assumed that $c_k = 0$ for $k \forall \mathbb{R}$.

As explained in the previous chapter, if $Y = (Y_1, Y_2, ..., Y_k) \sim MPH^*(\alpha, T; R)$, then

$$\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{m} q_{ij} \overline{F_j} \text{ ,} 1 \le i \le m$$

Where $\overline{F_i}$ is the survival function of Y given the initial state is i.

Since

$$\overline{F_i}(y_1, y_2, ..., y_k) = P(Y_1 > y_1, Y_2 > y_2, ..., Y_k > y_k \mid X(0) = i) \quad 1 \le i \le m,$$

$\overline{F_i}$ is a k-variate function. With the theory of multivariate power series, it is interesting to find out whether there are some interrelations among power series coefficients and if any, how these relations could be used to find some solutions of the PDEs.

Let's first consider the bivariate case. Suppose $Y = (Y_1, Y_2) \sim MPH^*(\alpha, T; R)$, and throughout the thesis we use the following notation in the bivariate case:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

By the analysis in the previous sections, there is

$$\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{2} q_{ij} \overline{F_j} \ , 1 \le i \le 2$$

By the expansion of power series, come the representations of $\overline{F_1}$ and $\overline{F_2}$:

$$\overline{F_1}(y_1, y_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij} y_1{}^i y_2{}^j$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij} y_1{}^i y_2{}^j$$

Here the analysis is going to be made step by step, the simplest reward system-identity matrix therefore being the first. After that a more "general" reward system will be taken to verify the results.

3.1 Using the identity matrix reward system

As mentioned above, the first analysis is made with the simplest reward system

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After plugging the value of this matrix into

$$\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{2} q_{ij} \overline{F_j} \ ,1 \le i \le 2$$

It gives

$$\frac{\partial \overline{F_1}}{y_1} = T_{11}\overline{F_1} + T_{12}\overline{F_2} \qquad (1)$$

$$\frac{\partial \overline{F_2}}{y_2} = T_{21}\overline{F_1} + T_{22}\overline{F_2} \qquad (2)$$

Using the power series theory, the left hand side of (1) can be rewritten in more details as below:

$$\frac{\partial \overline{F_1}}{y_1} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij} \cdot i \cdot y_1^{i-1} y_2^{\ j} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{i+1\,j} \cdot (i+1) \cdot y_1^{\ i} y_2^{\ j}$$

And the right hand side of (1) is

$$T_{11}\overline{F_1} + T_{12}\overline{F_2} = T_{11}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij} y_1^{\ i} y_2^{\ j} + T_{12}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{ij} y_1^{\ i} y_2^{\ j} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (T_{11}\alpha_{ij} + T_{12}\beta_{ij}) y_1^{\ i} y_2^{\ j}$$

Thus there comes an equation

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{i+1\,j} \cdot (i+1) \cdot y_1^{\ i} y_2^{\ j} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (T_{11}\alpha_{ij} + T_{12}\beta_{ij}) y_1^{\ i} y_2^{\ j}$$

Recall that if two power series are equal, then the coefficients corresponding to the term with the same power should be equal. Therefore we get

$$\alpha_{i+1,\,j} \cdot (i+1) = (T_{11}\alpha_{ij} + T_{12}\beta_{ij}) \qquad (3)$$

Similarly (2) can be rewritten in more detail as below:

$$\frac{\partial \overline{F_2}}{y_2} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \beta_{ij} \cdot j \cdot y_1^{\ i} y_2^{\ j-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{i,j+1} \cdot (j+1) \cdot y_1^{\ i} y_2^{\ j}$$

$$= T_{21}\overline{F_1} + T_{22}\overline{F_2} = T_{21}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij} y_1^{\ i} y_2^{\ j} + T_{22}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{ij} y_1^{\ i} y_2^{\ j}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (T_{21}\alpha_{ij} + T_{22}\beta_{ij}) y_1^{\ i} y_2^{\ j}$$

That is $\beta_{i,j+1} \cdot (j+1) = (T_{21}\alpha_{ij} + T_{22}\beta_{ij})$ (4)

(3) and (4) are therefore the two recursive equations which could be used to obtain all coefficients if only given some boundary values.

In fact, (3) and (4) contain a lot of information about the PDE. No matter how complicated the representations of the power series coefficient might look, they all satisfy (3) and (4). A simple example is illustrated below.

Example: Kibble model.

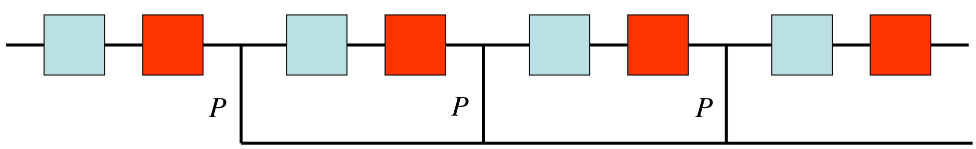


Fig 3: Markov process of the Kibble model

To see the function of (3) and (4), we still use the example of the Kibble model mentioned in the previous chapter.

The simplest case of the sub-intensity matrix would be as follows:

$$T = \begin{bmatrix} -\lambda & \lambda \\ (1-p)\lambda & -\lambda \end{bmatrix}$$

Recall that in this case the two conditional survival functions are as follows:

$$\overline{F_1}(y_1, y_2) = \sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}e^{-\lambda y_1})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}e^{-\lambda y_2})]$$

The two conditional survival functions can be written in the form of power series:

$$\overline{F_1}(y_1, y_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{ij}{y_1}^i{y_2}^j$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\beta_{ij}{y_1}^i{y_2}^j$$

However, α_{ij} and β_{ij} are not directly obtainable by only looking at the original survival functions. Instead, we insert the power series expressions of the exponential functions, by this we get

$$\overline{F_1}(y_1, y_2) = \sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!}) \quad (5)$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!}) \quad (6)$$

In order to get explicit representations of α_{ij} and β_{ij}, the following method has to be resorted to.

First we work with α_{ij}.

For the ease of explanations, we will introduce notation G such that

$G = \overline{F_1}(y_1, y_2) = \sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!})$, and then divide G into infinite sum such that

$$G = \sum_{i=1}^{\infty}G_i \qquad (7)$$

where

$$G_i = p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!}) \qquad (8)$$

It is clear that $G_i = P(Y_1 > y_1, Y_2 > y_2, U = i \mid X(0) = 1)$, where the random variable U is the times the CTMC repeats in state 1 and 2.

Obviously G_i contributes to α_{nm}, $\quad 0 \le n, m < \infty$
Let's write $G_i(n,m)$ as the term which contributes to the series coefficient α_{nm}, e.g.
$G_i(0,0) = p(1-p)^{i-1}$

What interests us more is $\alpha_{nm}(n,m>0)$
3 cases need discussing:

a) If $i-1 \le n$ and $i-1 \le m$ $(1 \le i \le (m,n)^-)$

$$G_i(n,m) = p(1-p)^{i-1}\left(\sum_{k=0}^{i-1} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{n-k}}{(n-k)!}\right)\left(\sum_{k=0}^{i-1} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{m-k}}{(m-k)!}\right) \qquad (9)$$

b) If $i-1 > n$ and $i-1 > m$ $(i \ge (m,n)^+ + 2)$

$$G_i(n,m) = p(1-p)^{i-1}\left(\sum_{k=0}^{n} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{n-k}}{(n-k)!}\right)\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{m-k}}{(m-k)!}\right) \qquad (10)$$

c) If $(m,n)^- + 1 \le i \le (m,n)^+ + 1$, then

$$G_i(n,m) = p(1-p)^{i-1}\left(\sum_{k=0}^{i-1} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{(m,n)^+-k}}{(n-k)!}\right)\left(\sum_{k=0}^{(m,n)^-} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{(m,n)^--k}}{((m,n)^- - k)!}\right) \qquad (11)$$

Recall the binomial formula:

$$(a+b)^m = \left(\sum_{k=0}^{m} \frac{a^k}{k!} \cdot \frac{b^{m-k}}{(m-k)!}\right)$$

Therefore $\sum_{k=0}^{m} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{m-k}}{(m-k)!}) = (\lambda - \lambda)^m = 0$

$$\sum_{k=0}^{(m,n)^-} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{(m,n)^- - k}}{((m,n)^- - k)!} = (\lambda - \lambda)^{(m,n)^-} = 0$$

This indicates that both (10) and (11) are equal to 0.

Without loss of generality, we assume n>m, then by using (7), (9)-(11), we get

$$\alpha_{nm} = \sum_{i=1}^{\infty} G_i(n,m) = \sum_{i=1}^{m} p(1-p)^{i-1}\left(\sum_{k=0}^{i-1} \frac{\lambda^n}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!}\right)\left(\sum_{k=0}^{i-1} \frac{\lambda^m}{k!} \cdot \frac{(-1)^{m-k}}{(m-k)!}\right) \quad (n > m > 0) \qquad (12)$$

With the same method, now we work with β_{ij}

We write
$\overline{F_2}(y_1,y_2)=\sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!})=H$, and then divide H into an infinite sum such that

$$H=\sum_{i=2}^{\infty}H_i \qquad (12)$$

where $H_i=p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!})$ (13)

Obviously H_i contributes to β_{nm}, $0\le n,m<\infty$
Let's write $H_i(n,m)$ as the term which contributes to the series coefficient β_{nm}

What also interests us is $\beta_{nm}(n,m>0)$

Similarly 4 cases need discussing:

a) If $i-2\le n$ and $i-1\le m$ ($2\le i\le(m+1,n+2)^-$), then

$$H_i(n,m)=p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!}) \qquad (14)$$

b) If $i-2>n$ and $i-1>m$ ($i\ge(m+1,n+2)^++1$) then

$$H_i(n,m)=p(1-p)^{i-1}(\sum_{k=0}^{n}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{m}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!}) \qquad (15)$$

c) If $i-2\le n$ and $i-1>m$ then

$$H_i(n,m)=p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{m}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!}) \qquad (16)$$

d) If $i-2>n$ and $i-1\le m$ then

$$H_i(n,m)=p(1-p)^{i-1}(\sum_{k=0}^{n}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!}) \qquad (17)$$

By the binomial formula, we realize that equations (15)-(17) are equal to 0.
Without loss of generality, assume n>m, then by using (12), (14)-(17), what is obtained is

$$\beta_{nm}=\sum_{i=2}^{\infty}H_i(n,m)=\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!}) \quad (n>m>0) \quad (18)$$

Even though this model is simple, the representations of α_{nm} and β_{nm} are quite complicated. Therefore out come the advantages of the recursive equations (3)-(4), since it is more difficult to get a concise representation of power series coefficients when the model becomes more complicated. Nevertheless, (3)-(4) are always satisfied, so they can be used to retrieve coefficients with some known boundary values.

We have now found explicit expressions for α_{nm} and β_{nm} in the model and will continue to verify that these indeed satisfy (3)-(4). Without loss of generality, again we assume n>m.

Proof of $\alpha_{n+1,m}\cdot(n+1)=(T_{11}\alpha_{nm}+T_{12}\beta_{nm})$ in the Kibble model

In this model, $T_{11}=-\lambda$ and $T_{12}=\lambda$

Using (12), the task is to see whether

$$\alpha(n+1,m)=\sum_{i=1}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{\lambda^{n+1}}{k\,!}\cdot\frac{(-1)^{n+1-k}}{(n+1-k)\,!})(\sum_{k=0}^{i-1}\frac{\lambda^{m}}{k\,!}\cdot\frac{(-1)^{m-k}}{(m-k)\,!})$$

is equal to

$$\frac{1}{n+1}[-\lambda\cdot\sum_{i=1}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{\lambda^{n}}{k\,!}\cdot\frac{(-1)^{n-k}}{(n-k)\,!})(\sum_{k=0}^{i-1}\frac{\lambda^{m}}{k\,!}\cdot\frac{(-1)^{m-k}}{(m-k)\,!})$$
$$+\lambda\cdot\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^{k}}{k\,!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)\,!})(\sum_{k=0}^{i-1}\frac{\lambda^{k}}{k\,!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)\,!})]$$

Which leave us to prove

$$\sum_{i=1}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k\,!\cdot(n+1-k)\,!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k\,!\cdot(m-k)\,!})$$
$$=\frac{1}{n+1}[-\sum_{i=1}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(-1)^{n-k}}{k\,!\cdot(n-k)\,!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k\,!\cdot(m-k)\,!}) \qquad (19)$$
$$+\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(-1)^{n-k}}{k\,!\cdot(n-k)\,!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{(m-k)\,!})]$$

To prove (19), we have to divide the sum into two parts: i=1 and i>1.

When i=1, the part on the left side is

$$p(\frac{(-1)^{n+1-0}}{0!\cdot(n+1-0)\,!})(\frac{(-1)^{m-0}}{0!\cdot(m-0)\,!})$$

and the part of the right side is

$$\frac{1}{n+1}[-p(\frac{(-1)^{n-0}}{0!\cdot(n-0)\,!})(\frac{(-1)^{m-0}}{0!\cdot(m-0)\,!})]$$

Obviously $p(\frac{(-1)^{n+1-0}}{0!\cdot(n+1-0)!})(\frac{(-1)^{m-0}}{0!\cdot(m-0)!})=\frac{1}{n+1}[-p(\frac{(-1)^{n-0}}{0!\cdot(n-0)!})(\frac{(-1)^{m-0}}{0!\cdot(m-0)!})]$

When i>1, the part of the left side is $\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k!\cdot(m-k)!})$

and the part of the right side is

$$\frac{1}{n+1}[-\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(-1)^{n-k}}{k!\cdot(n-k)!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k!\cdot(m-k)!})$$
$$+\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(-1)^{n-k}}{k!\cdot(n-k)!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{(m-k)!})]$$

After rearranging terms, this is to prove

$$\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k!\cdot(m-k)!})$$
$$=\frac{1}{n+1}\sum_{i=2}^{m}p(1-p)^{i-1}(\frac{(-1)^{n-i+2}}{(i-1)!\cdot(n-i+1)!})(\sum_{k=0}^{i-1}\frac{(-1)^{m-k}}{k!\cdot(m-k)!})$$

Which leaves us to prove

$$\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!}=\frac{(-1)^{n-i+2}}{(i-1)!\cdot(n-i+1)!\cdot(n+1)} \qquad (20)$$

We will prove equation (20) by induction.

When i=2, the left side of (20) is $\frac{(-1)^{n+1-0}}{0!\cdot(n+1-0)!}+\frac{(-1)^{n+1-1}}{1!\cdot(n+1-1)!}$

and the right side of (20) is $\frac{1}{n+1}\cdot\frac{(-1)^{n-2+2}}{(2-1)!\cdot(n-2+1)!}$

Obviously $\frac{(-1)^{n+1-0}}{0!\cdot(n+1-0)!}+\frac{(-1)^{n+1-1}}{1!\cdot(n+1-1)!}=\frac{1}{n+1}\cdot\frac{(-1)^{n-2+2}}{(2-1)!\cdot(n-2+1)!}$

So when i=2, (20) is satisfied.

Suppose $\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!}=\frac{(-1)^{n-i+2}}{(i-1)!\cdot(n-i+1)!\cdot(n+1)}$ when $i\geq 2$

Then

$$\sum_{k=0}^{i}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!}=\sum_{k=0}^{i-1}\frac{(-1)^{n+1-k}}{k!\cdot(n+1-k)!}+\frac{(-1)^{n+1-i}}{i!\cdot(n+1-i)!}$$
$$=\frac{(-1)^{n-i+2}}{(i-1)!\cdot(n-i+1)!\cdot(n+1)}+\frac{(-1)^{n+1-i}}{i!\cdot(n+1-i)!}=\frac{i\cdot(-1)^{n-i+2}}{i!\cdot(n-i+1)!\cdot(n+1)}+\frac{(n+1)\cdot(-1)^{n+1-i}}{i!\cdot(n-i+1)!\cdot(n+1)}$$
$$=\frac{(n+1-i)\cdot(-1)^{n+1-i}}{i!\cdot(n-i+1)!\cdot(n+1)}=\frac{(-1)^{n+1-i}}{(n+1)\cdot i!\cdot(n-i)!}$$

This is the end of the induction proof.
Thus $\alpha_{n+1,m}\cdot(n+1)=(T_{11}\alpha_{nm}+T_{12}\beta_{nm})$

Likewise, there is also the need to prove

$$\beta_{n,m+1}\cdot(m+1)=(T_{21}\alpha_{nm}+T_{22}\beta_{nm})=(1-p)\lambda\alpha_{nm}-\lambda\beta_{nm}$$

Once again assume n>m, and by the result in (18), this is going to test whether

$$\beta_{n,m+1}=\sum_{i=2}^{m+1}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m+1-k}}{(m+1-k)!})$$

equal to

$$\frac{1}{m+1}[(1-p)\lambda\sum_{i=1}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{\lambda^n}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^m}{k!}\cdot\frac{(-1)^{m-k}}{(m-k)!})$$
$$-\lambda\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{\lambda^k}{k!}\cdot\frac{(-\lambda)^{m-k}}{(m-k)!})]$$

This indicates testing whether

$$\sum_{i=2}^{m+1}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{m+1-k}}{(m+1-k)!})$$

equal to

$$\frac{1}{m+1}[\sum_{i=1}^{m}p(1-p)^{i}(\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{m-k}}{(m-k)!})$$
$$-\sum_{i=2}^{m}p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{m-k}}{(m-k)!})]$$

This is equal to prove

$$\sum_{i=1}^{m} p(1-p)^i (\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i}\frac{1}{k!}\cdot\frac{(-1)^{m+1-k}}{(m+1-k)!})$$
$$=\frac{1}{m+1}[\sum_{i=1}^{m} p(1-p)^i (\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{m-k}}{(m-k)!})$$
$$-\sum_{i=1}^{m-1} p(1-p)^i (\sum_{k=0}^{i-1}\frac{1}{k!}\cdot\frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i}\frac{1}{k!}\cdot\frac{(-1)^{m-k}}{(m-k)!})]$$

which leaves us to prove

$$\sum_{k=0}^{i}\frac{1}{k!}\cdot\frac{(-1)^{m+1-k}}{(m+1-k)!}=\frac{1}{m+1}\cdot\frac{(-1)^{m-i+1}}{i!\cdot(m-i)!} \tag{21}$$

Obviously when i=1 the (21) is satisfied.

With the same method of induction, assume $\sum_{k=0}^{i}\frac{1}{k!}\cdot\frac{(-1)^{m+1-k}}{(m+1-k)!}=\frac{1}{m+1}\cdot\frac{(-1)^{m-i+1}}{i!\cdot(m-i)!}$

$$\rightarrow\sum_{k=0}^{i+1}\frac{1}{k!}\cdot\frac{(-1)^{m+1-k}}{(m+1-k)!}=\frac{1}{m+1}\cdot\frac{(-1)^{m-i+1}}{i!\cdot(m-i)!}+\frac{(-1)^{m-i}}{(i+1)!\cdot(m-i)!}$$
$$=\frac{(i+1)\cdot(-1)^{m-i+1}+(m+1)\cdot(-1)^{m-i}}{(m+1)\cdot(i+1)!(m-i)!}$$
$$=\frac{(m-i)\cdot(-1)^{m-i}}{(m+1)\cdot(i+1)!(m-i)!}=\frac{(-1)^{m-i}}{(m+1)\cdot(i+1)!(m-i-1)!}$$

In sum, equation (21) is valid.

All of the above is based on the assumption that n>m>0, the proof is exactly the same if the case is tested when $0<n\le m$.

It is also necessary to highlight the results in this section since they are very important: the following recursive power series equations are valid in the simple Kibble model.

$$\alpha_{n+1,m}\cdot(n+1)=(T_{11}\alpha_{nm}+T_{12}\beta_{nm})$$
$$\beta_{n,m+1}\cdot(m+1)=(T_{21}\alpha_{nm}+T_{22}\beta_{nm})$$

Typically, when n>m, by using the induction method, the following representations of α_{nm} and β_{nm} can be obtained:

$$\alpha_{nm} = \sum_{i=1}^{\infty} G_i(n,m) = \sum_{i=1}^{m} p(1-p)^{i-1} (\sum_{k=0}^{i-1} \frac{\lambda^n}{k!} \cdot \frac{(-1)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1} \frac{\lambda^m}{k!} \cdot \frac{(-1)^{m-k}}{(m-k)!}) \quad (n > m > 0) \quad (12)$$

$$\beta_{nm} = \sum_{i=2}^{\infty} H_i(n,m) = \sum_{i=2}^{m} p(1-p)^{i-1} (\sum_{k=0}^{i-2} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{n-k}}{(n-k)!})(\sum_{k=0}^{i-1} \frac{\lambda^k}{k!} \cdot \frac{(-\lambda)^{m-k}}{(m-k)!}) \quad (n > m > 0) \quad (18)$$

If other reward system other than identity matrix is chosen, (3) and (4) have to be modified. Another example is taken below to show the functionality of other recursive equations.

3.2 Using a more complicated reward system

Recall that $\sum_{j=1}^{k} r_j(i)\frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{m} q_{ij}\overline{F_j}$

The PDEs for the general bivariate case can be written as:

$$R_{11}\frac{\partial \overline{F_1}}{y_1} + R_{21}\frac{\partial \overline{F_1}}{y_2} = T_{11}\overline{F_1} + T_{12}\overline{F_2} \qquad (22)$$
$$R_{12}\frac{\partial \overline{F_2}}{y_1} + R_{22}\frac{\partial \overline{F_2}}{y_2} = T_{21}\overline{F_1} + T_{22}\overline{F_2} \qquad (23)$$

Similarly with the method of deriving (3)-(4), the following power recursive equations is obtained:

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = T_{11}\alpha_{ij} + T_{12}\beta_{ij} \qquad (24)$$
$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = T_{21}\alpha_{ij} + T_{22}\beta_{ij} \qquad (25)$$

Thus (24) and (25) contain the information of (22) and (23) in our model, and these two concise equations could be expected to retrieve power series coefficients with given boundary values.

As what has been done in the previous section, now is the turn to see how (24)-(25) work in a concrete example.

Example: Erlang (2) distribution

Consider $Y = (Y_1, Y_2) \sim MPH^*(\alpha, T; R)$, with

$$\text{T=}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \text{ and R=}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Y can be represented in another way:

$$Y=\begin{bmatrix}Y_1\\Y_2\end{bmatrix}=\begin{bmatrix}2/3 & 1/3\\1/3 & 2/3\end{bmatrix}\begin{bmatrix}Z_1\\Z_2\end{bmatrix}$$

Where (Z_1, Z_2) are independent exp(1) distributed random variables.
The two conditional survival functions of Y will be derived.

For the purpose of getting $\overline{F_1}(y_1, y_2) = p(Y_1 > y_1, Y_2 > y_2 \mid X(0) = 1)$, 3 cases have to be discussed:

1. If $y_1 \geq 2y_2$, then

$$\begin{aligned}\overline{F_1}(y_1, y_2) &= p(Y_1 > y_1) = p(\tfrac{2}{3}Z_1 + \tfrac{1}{3}Z_2 > y_1) = p(Z_2 > 3y_1 - 2Z_1)\\&= 1\cdot p(2Z_1 > 3y_1) + \int_0^{\frac{3}{2}y_1} \exp(2Z_1 - 3y_1)\exp(-Z_1)dZ_1\\&= \exp(-\tfrac{3}{2}y_1) + \exp(-3y_1)\int_0^{\frac{3}{2}y_1}\exp(Z_1)dZ_1\\&= \exp(-\tfrac{3}{2}y_1) + \exp(-3y_1)(\exp(-\tfrac{3}{2}y_1) - 1) = 2\exp(-\tfrac{3}{2}y_1) - \exp(-3y_1)\end{aligned}$$

2. If $y_1 \leq \frac{1}{2}y_2$, then $\overline{F_1}(y_1, y_2) = p(Y_2 > y_2) = 2\exp(-\frac{3}{2}y_2) - \exp(-3y_2)$
3. If $\frac{1}{2}y_2 < y_1 < 2y_2$, then

$$\begin{aligned}\overline{F_1}(y_1, y_2) &= \int_{y_1}^{\infty}\int_{y_2}^{\infty} f(u,v)dudv\\&= \int_{y_1}^{2y_2}\int_{y_2}^{2u} 3\exp(-u-v)dudv + \int_{2y_2}^{\infty}\int_{\frac{1}{2}u}^{2u} 3\exp(-u-v)dudv\\&= 3\exp(-y_1 - y_2) - \exp(-3y_1) - \exp(-3y_2)\end{aligned}$$

It is much easier to get

$$\overline{F_2}(y_1, y_2) = p(Y_1 > y_1, Y_2 > y_2 \mid X(0) = 2) = \exp(-\max(3y_1, \tfrac{3}{2}y_2))$$

Likewise, we will see how the survival functions work in the PDE framework in (24) and (25). To do this, 3 cases have to be discussed:

1. $y_1 \leq \frac{1}{2}y_2$

$$\overline{F_1}(y_1, y_2) = 2\exp(-\tfrac{3}{2}y_2) - \exp(-3y_2) = 2\sum_{k=0}^{\infty}\frac{(-\frac{3}{2}y_2)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-3y_2)^k}{k!}$$

$$\overline{F_2}(y_1, y_2) = \exp(-\tfrac{3}{2}y_2) = \sum_{k=0}^{\infty}\frac{(-\frac{3}{2}y_2)^k}{k!}$$

In this case, $\alpha_{0m} = 2\frac{(-\frac{3}{2})^m}{m!} - \frac{(-3)^m}{m!}$, $\beta_{0m} = \frac{(-\frac{3}{2})^m}{m!}$, and all other coefficients are 0.

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$

$$= \tfrac{2}{3}\cdot 0 + \tfrac{1}{3}[2\frac{(-\frac{3}{2})^{j+1}}{(j+1)!} - \frac{(-3)^{j+1}}{(j+1)!}](j+1) = -2\frac{(-\frac{3}{2})^j}{j!} + \frac{(-3)^j}{j!} + \frac{(-\frac{3}{2})^j}{j!}$$

$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$

$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$

$$= \tfrac{1}{3}\cdot 0 + \tfrac{2}{3}(j+1)\frac{(-\frac{3}{2})^{j+1}}{(j+1)!} = 0 - \frac{(-\frac{3}{2})^j}{j!}$$

$$= T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

Therefore (24)-(25) are satisfied in this case.

2. $y_1 \geq 2y_2$

$$\overline{F_1}(y_1, y_2) = 2\exp(-\tfrac{3}{2}y_1) - \exp(-3y_1) = 2\sum_{k=0}^{\infty}\frac{(-\frac{3}{2}y_1)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!}$$

$$\overline{F_2}(y_1, y_2) = \exp(-3y_1) = \sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!}$$

In this case, $\alpha_{m0} = 2\frac{(-\frac{3}{2})^m}{m!} - \frac{(-3)^m}{m!}$ $\quad \beta_{m0} = \frac{(-3)^m}{m!}$

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$

$$= \tfrac{2}{3}[2\frac{(-\frac{3}{2})^{i+1}}{(i+1)!} - \frac{(-3)^{i+1}}{(i+1)!}](i+1) + \tfrac{1}{3}\cdot 0 = -[2\frac{(-\frac{3}{2})^i}{i!} - \frac{(-3)^i}{i!}] + \frac{(-3)^i}{i!}$$

$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$

$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$

$$= \tfrac{1}{3}\frac{(-3)^{i+1}}{(i+1)!}(i+1) + \tfrac{2}{3}\cdot 0 = 0 - \frac{(-3)^i}{i!}$$

$$= T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

Therefore (24)-(25) are satisfied in this case.

3. $\frac{1}{2}y_2 < y_1 < 2y_2$

$$\overline{F_1}(y_1, y_2) = 3\exp(-y_1 - y_2) - \exp(-3y_1) - \exp(-3y_2)$$

$$= 3\sum_{k=0}^{\infty}\frac{(-y_1)^k}{k!}\sum_{k=0}^{\infty}\frac{(-y_2)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-3y_2)^k}{k!}$$

$$\overline{F_2}(y_1, y_2) = \exp(-3y_1) = \sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!}$$

This case is more complicated so the following 3 subcategories need to be considered:

a) If m>0, n>0, $\alpha_{mn} = 3\frac{(-1)^{m+n}}{m!n!}$ $\beta_{mn} = 0$

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$

$$= \tfrac{2}{3}\cdot 3\frac{(-1)^{i+j+1}}{(i+1)!\,j!}(i+1) + \tfrac{1}{3}\cdot 3\frac{(-1)^{i+j+1}}{i!(j+1)!}(j+1) = -3\frac{(-1)^{i+j}}{i!\,j!} + 0$$

$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$

$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$

$$= 0 + 0 = T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

Therefore (24)-(25) are satisfied in this case.

b) If m>0, n=0, $\alpha_{m0} = 3\frac{(-1)^m}{m!} - \frac{(-3)^m}{m!}$, $\beta_{m0} = \frac{(-3)^m}{m!}$

$$R_{11}\alpha_{i+1,0}(i+1) + R_{21}\alpha_{i,0+1}(0+1) = \frac{2}{3}\alpha_{i+1,0}(i+1) + \frac{1}{3}\alpha_{i,0+1}(0+1)$$

$$= \tfrac{2}{3}\cdot(3\frac{(-1)^{i+1}}{(i+1)!} - \frac{(-3)^{i+1}}{(i+1)!})(i+1) + \tfrac{1}{3}\cdot 3\frac{(-1)^{i+1}}{i!1!}(0+1) = -(3\frac{(-1)^i}{i!} - \frac{(-3)^i}{i!}) + \frac{(-3)^i}{i!}$$

$$= T_{11}\alpha_{i0} + T_{12}\beta_{i0}$$

$$R_{21}\beta_{i+1,0}(i+1) + R_{22}\beta_{i,0+1}(0+1) = \frac{1}{3}\beta_{i+1,0}(i+1) + \frac{2}{3}\beta_{i,0+1}(0+1)$$

$$= \frac{1}{3}\frac{(-3)^{i+1}}{(i+1)!}(i+1) + \frac{2}{3}\cdot 0$$

$$= 0 - \frac{(-3)^i}{i!} = T_{21}\alpha_{i0} + T_{22}\beta_{i0}$$

Therefore (24)-(25) are satisfied in this case.

c) If m=0, n>0, $\alpha(0,n) = 3\frac{(-1)^n}{n!} - \frac{(-3)^n}{n!}$, $\beta(0,n) = 0$

In the case that i>0,j=0

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$
$$= \tfrac{2}{3}\cdot[3\frac{(-1)^{i+1}}{(i+1)!} - \frac{(-3)^{i+1}}{(i+1)!}](i+1) + \tfrac{1}{3}\cdot 3\frac{(-1)^{i+0+1}}{i!(0+1)!}(0+1) = -(3\frac{(-1)^i}{i!} - \frac{(-3)^i}{i!}) + \frac{(-3)^i}{i!}$$
$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$
$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$
$$= \tfrac{1}{3}\frac{(-3)^{i+1}}{(i+1)!}(i+1) + 0 = 0 - \frac{(-3)^i}{i!} = T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

In the case that i=0, j>0

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$
$$= \tfrac{2}{3}\cdot 3\frac{(-1)^{j+1}}{j!} + \tfrac{1}{3}\cdot[3\frac{(-1)^{j+1}}{(j+1)!} - \frac{(-3)^{j+1}}{(j+1)!}](j+1) = -[3\frac{(-1)^j}{j!} - \frac{(-3)^j}{j!}] + 0$$
$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$
$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$
$$= 0 + 0 = T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

In the case that i=0, j=0

$$R_{11}\alpha_{i+1,j}(i+1) + R_{21}\alpha_{i,j+1}(j+1) = \frac{2}{3}\alpha_{i+1,j}(i+1) + \frac{1}{3}\alpha_{i,j+1}(j+1)$$
$$= \tfrac{2}{3}\cdot[3\frac{(-1)}{1!} - \frac{(-3)}{1!}] + \tfrac{1}{3}[3\frac{(-1)}{1!} - \frac{(-3)}{1!}] = -1\cdot 1 + 1\cdot 1$$
$$= T_{11}\alpha_{ij} + T_{12}\beta_{ij}$$
$$R_{21}\beta_{i+1,j}(i+1) + R_{22}\beta_{i,j+1}(j+1) = \frac{1}{3}\beta_{i+1,j}(i+1) + \frac{2}{3}\beta_{i,j+1}(j+1)$$
$$= \tfrac{1}{3}\cdot(-3) + 0 = 0 - 1\cdot 1 = T_{21}\alpha_{ij} + T_{22}\beta_{ij}$$

In sum, (24)-(25) are satisfied in this case.

In conclusion, the power series recursive equations (24)-(25) are also valid in our updated "more general" bivariate case. The result is very promising since it indicates that the power series recursive equations have potential applications in our PDE study. It will be an amazing result if the original distribution function can be derived given the power series coefficients.

3.4 Summary

In this chapter we have introduced power series method to take a deeper insight into the application of partial differential equation on studying phase type distributions. For ease of illustration, we are only concerned about two dimensional cases. The motivation of using power series is that we can obtain recursive equations of coefficients, thus they might help us to derive all coefficients if given only boundary coefficients. After obtaining these recursive equations, we test them in some concrete examples. The result is promising in that no matter how complicated the representations of power series coefficients might look, they all fulfil the recursive equations. Therefore on the other hand, if it is difficult to get a closed form of distribution or survival functions while relatively easy to get the marginal distributions, we have the opportunity to obtain the approximate values of distribution functions given the boundary power series coefficients. Thus it is very interesting to have some numerical test to see whether the idea works.

Chapter 4 Power series numerical test

So far we have seen that the power series recursive equations work in different examples. We would like to see if only given finite power series coefficients due to limited resource, whether we are able to obtain some approximate result of the true distribution/survival functions. Typically, we know all marginals of an MPH distribution are MPH distributions [6]. Therefore it is relatively easy to obtain all marginal distributions/survival functions. Accordingly, we could obtain all boundary values of power series coefficients ($\alpha_{0n}, \alpha_{m0}, \beta_{0n}, \beta_{m0} (m, n \geq 0)$) easily.

Before we take the analysis of the possibility of approximation, we will first see whether it is possible to obtain all coefficients given the boundary values.

4.1 Using recursive equations to obtain all power series coefficients

As before we know the power series representation of two conditional distributions in the bivariate case:

$$\overline{F_1}(y_1, y_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij} y_1{}^i y_2{}^j$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij} y_1{}^i y_2{}^j$$

And also the Kulkarni PDE

$$\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{m} q_{ij} \overline{F_j}$$

This leads to

$$\frac{\partial \overline{F_1}}{y_1} = T_{11}\overline{F_1} + T_{12}\overline{F_2} \qquad (1)$$

$$\frac{\partial \overline{F_2}}{y_2} = T_{21}\overline{F_1} + T_{22}\overline{F_2} \qquad (2)$$

Where we have assumed the reward matrix is the identity matrix.

As explained in previous chapters, we know (1) and (2) could give us some recursive equations in power series. In our following work, we assume the boundary power series coefficients are known. At this point, we have interest on whether we could obtain values of all coefficients given these boundary values. We will first illustrate this possibility by looking at the easiest Kibble model.

We assume the Kibble model has the following sub-intensity matrix

$$T = \begin{bmatrix} -1 & 1 \\ q & -1 \end{bmatrix}$$

From (1) and (2) we get the recursive equations:

$$\alpha_{i+1,j} \cdot (i+1) = (-\alpha_{ij} + \beta_{ij}) \qquad (26)$$

$$\beta_{i,j+1} \cdot (j+1) = (q\alpha_{ij} - \beta_{ij}) \qquad (27)$$

Specifically,

$$\alpha_{1n} \cdot 1 = -\alpha_{0n} + \beta_{0n} \qquad (28)$$

$$\beta_{m1} \cdot 1 = q\alpha_{m0} - \beta_{m0} \qquad (29)$$

So we can get all values of α_{1n}, β_{m1} (n, m>0) with the boundary values.

Moreover, $\alpha_{m1} \cdot m = -\alpha_{m-1,1} + \beta_{m-1,1} \qquad (m > 0) \qquad (30)$

On the right-hand side of (30), $\beta_{m-1,1}$ can be derived from (29). Therefore with the boundary value α_{01} we can get α_{11}. Plugging α_{11} into (30) we can also get α_{21}. This indicates that with $\alpha_{m-1,1}$ we can always get α_{m1}. In sum, using (29) and (30) we can obtain values of all α_{m1} (m>0).

Furthermore, $\beta_{1n} \cdot n = q\alpha_{1,n-1} - \beta_{1,n-1} \qquad (n > 0) \qquad (31)$

On the right-hand side of (31), $\alpha_{1,n-1}$ can be derived from (28). With the boundary value β_{10} we can get β_{11}. Plugging β_{11} into (31) we get β_{12}. Similarly, we can get all values of β_{1n} (n>0).

Up to now, we have obtained values of all $\alpha_{0n}, \alpha_{m0}, \beta_{0n}, \beta_{m0}, \alpha_{1n}, \alpha_{m1}, \beta_{1n}, \beta_{m1}\ (m, n \geq 0)$

By induction method, we know from (26) and (27) that, given all $\alpha_{m-1,n}, \beta_{m-1,n}$ we can get all α_{mn}; and given all $\alpha_{m,n-1}, \beta_{m,n-1}$ we can get all β_{mn}. In conclusion, with the given boundary values, we can get values of all coefficients $\alpha_{mn}, \beta_{mn} (m, n \geq 0)$

Now we want to investigate the general case. Suppose $Y = (Y_1, Y_2) \sim MPH^*(\alpha, T; R)$.

According to Kulkarni PDE, we get

$$R_{11} \frac{\partial \overline{F_1}}{y_1} + R_{21} \frac{\partial \overline{F_1}}{y_2} = T_{11} \overline{F_1} + T_{12} \overline{F_2} \qquad (22)$$

$$R_{12} \frac{\partial \overline{F_2}}{y_1} + R_{22} \frac{\partial \overline{F_2}}{y_2} = T_{21} \overline{F_1} + T_{22} \overline{F_2} \qquad (23)$$

Plugging (22) and (23) into power series, we get

$$R_{11} \alpha_{i+1,j} (i+1) + R_{21} \alpha_{i,j+1} (j+1) = T_{11} \alpha_{ij} + T_{12} \beta_{ij} \qquad (24)$$

$$R_{12} \beta_{i+1,j} (i+1) + R_{22} \beta_{i,j+1} (j+1) = T_{21} \alpha_{ij} + T_{22} \beta_{ij} \qquad (25)$$

We still want to see whether it is possible to obtain values of power series coefficients given all boundary values. Without loss of generality, we assume none of $R_{11}, R_{12}, R_{21}, R_{22}$ is 0.

By using (24) and (25), we get

$$\alpha_{1n} = \frac{T_{11} \alpha_{0n} + T_{12} \beta_{0n} - R_{21} \alpha_{0,n+1} (n+1)}{R_{11}}$$

$$\alpha_{m1} = \frac{T_{11} \alpha_{m0} + T_{12} \beta_{m0} - R_{11} \alpha_{m+1,0} (m+1)}{R_{21}}$$

$$\beta_{1n} = \frac{T_{21} \alpha_{0n} + T_{22} \beta_{0n} - R_{22} \beta_{0,n+1} (n+1)}{R_{12}}$$

$$\beta_{m1} = \frac{T_{21} \alpha_{m0} + T_{22} \beta_{m0} - R_{21} \beta_{m+1,0} (m+1)}{R_{22}}$$

Like the case in Kibble model, given all values of $\alpha_{0n}, \alpha_{m0}, \beta_{0n}, \beta_{m0} (m, n \geq 0)$, we are able to obtain all values of $\alpha_{1n}, \alpha_{m1}, \beta_{1n}, \beta_{m1} (m, n \geq 0)$.

By induction method, we can get that

$$\alpha_{mn} = \frac{T_{11}\alpha_{m-1,n} + T_{12}\beta_{m-1,n} - R_{21}\alpha_{m-1,n+1}(n+1)}{mR_{11}} \qquad (m>0) \qquad (32)$$

$$\beta_{mn} = \frac{T_{21}\alpha_{m-1,n} + T_{22}\beta_{m-1,n} - R_{22}\beta_{m-1,n+1}(n+1)}{mR_{12}} \qquad (m>0) \qquad (33)$$

since all coefficients on both right-hand side of (32) and (33) have been obtained in previous calculations. In conclusion, we could get all power series coefficients given all boundary values in the general case.

In principle the power series method could provide us with an analytical solution. The straightforward way is to insert all the power series coefficients into

$$\overline{F_1}(y_1, y_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij} y_1{}^i y_2{}^j$$

$$\overline{F_2}(y_1, y_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{ij} y_1{}^i y_2{}^j$$

However, by doing this we can not get the explicit expression. Due to the limited time distributed to the project, we are not able solve the differential equations directly. Thus we have to resort to numerical tests to get the approximated distribution/survival functions.

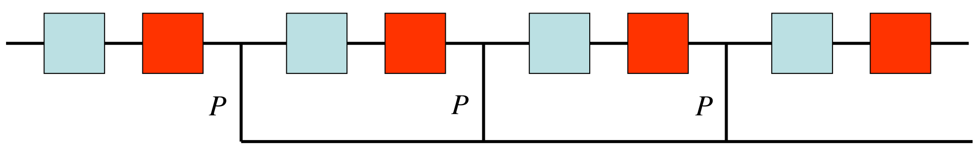
Besides only deriving the results in theory, it is very interesting to try examples and hopefully find some potential trend of the power series. We will give two concrete examples.

4.2 Numerical test in Kibble model

We will still refer to the case in chapter 2, in which the two survival functions are listed below:

$$\overline{F_1}(y_1,y_2)=\sum_{i=1}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!}) \quad (34)$$

$$\overline{F_2}(y_1,y_2)=\sum_{i=2}^{\infty}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!}) \quad (35)$$



Since we get this explicit expressions of both survival functions, we would like to see whether the survival function written by power series can be approximate to the original probabilistic representation if given the finite α_{mn} and β_{mn}.

The idea is illustrated below:

We know the probability that the Markov process repeats in state 1 and 2 for at least n cycles is $(1-p)^n$. We have interest in the case when $(1-p)^n < 10^{-6}$, that is $n > \frac{-6\log 10}{\log(1-p)}$.

We will choose the smallest value of n to fulfil the above conditions in our following numerical test.

The two survival functions can be approximately written by

$$\overline{F_1}(y_1,y_2)\approx\sum_{i=1}^{n}[p(1-p)^{i-1}(\sum_{k=0}^{i-1}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!})=\overline{F_1}(y_1,y_2\,|\,n)$$

$$\overline{F_2}(y_1,y_2)\approx\sum_{i=2}^{n}[p(1-p)^{i-1}(\sum_{k=0}^{i-2}\frac{(\lambda y_1)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_1)^l}{l!})(\sum_{k=0}^{i-1}\frac{(\lambda y_2)^k}{k!}\cdot\sum_{l=0}^{\infty}\frac{(-\lambda y_2)^l}{l!})=\overline{F_2}(y_1,y_2\,|\,n)$$

Given the boundary values of power series, we could calculate α_{mn} and β_{mn} for $1\le m,n\le u$, where u is an integer. These boundary values are listed below:

$\beta_{n0} = (-\lambda p)^n (1-p)/n!$
$\alpha_{n0} = \alpha_{0n} = (-\lambda p)^n / n!$

$\beta_{0n} = -(-\lambda)^n p(1-p^{n-1})/n!$ when $n \geq 2$
$\beta_{00} = 1-p$
$\beta_{01} = 0$

Now we could get the approximate values of two survival functions:

$$\overline{F_1}(y_1, y_2) \approx \sum_{i=0}^{u}\sum_{j=0}^{u} \alpha_{ij} y_1^{\,i} y_2^{\,j} = \overline{F_1}(y_1, y_2 \mid u, PS)$$

$$\overline{F_2}(y_1, y_2) \approx \sum_{i=0}^{u}\sum_{j=0}^{u} \beta_{ij} y_1^{\,i} y_2^{\,j} = \overline{F_2}(y_1, y_2 \mid u, PS)$$

We could further define $e_1 = \left| \frac{\overline{F_1}(y_1, y_2 \mid n) - \overline{F_1}(y_1, y_2 \mid u, PS)}{\overline{F_1}(y_1, y_2 \mid n)} \right|$

$$e_2 = \left| \frac{\overline{F_2}(y_1, y_2 \mid n) - \overline{F_2}(y_1, y_2 \mid u, PS)}{\overline{F_2}(y_1, y_2 \mid n)} \right|$$

as the according error compared to the probabilistic result.

We will start to work with the case of marginal distribution, which is to say $y_1 = 0$ or $y_2 = 0$.

The easiest case would be that p is approaching 1, in this case $\overline{F_1}(y_1, 0)$, $\overline{F_1}(0, y_2)$, $\overline{F_2}(0, y_2)$ can be approximated by $\exp(-\lambda y_1)$ or $\exp(-\lambda y_2)$, and $\overline{F_2}(y_1, 0)$ is almost 0.

Suppose $y_2 = 0$ or $y_1 = 0$, p=0.99 and $\lambda = 1$. We calculate the values of e_1, e_2 in two cases and list them in Table 1 and Table 2.

u/ y_1	0.5	0.75	1	1.5	2
1	0.1715 0.1715	0.4589 0.4589	0.9731 0.9731	3.1413 3.1413	8.0979 8.0979
2	0.0294 0.0294	0.1203 0.1203	0.3458 0.3458	1.7267 1.7267	6.0993 6.0994
3	0.0037 0.0037	0.0231 0.0231	0.0895 0.0895	0.6829 0.6829	3.2708 3.2708
4	3.7501e-04 3.7501e-04	0.0035 0.0035	0.0183 0.0183	0.2117 0.2117	1.3674 1.3674
5	3.1270e-05 2.9649e-05	4.4156e-04 4.3949e-04	0.0031 0.0031	0.0540 0.0540	0.4693 0.4693
10	1.6419e-08 1.6377e-06	2.2864e-08 2.0879e-06	8.2553e-08 2.6963e-06	7.6584e-06 1.1742e-05	2.8520e-04 2.9135e-04
20	1.6402e-08 1.6377e-06	2.0990e-08 2.0860e-06	2.6815e-08 2.6406e-06	4.3336e-08 4.1273e-06	6.8640e-08 6.2145e-06
50	1.6402e-08 1.6377e-06	2.0990e-08 2.0860e-06	2.6815e-08 2.6406e-06	4.3336e-08 4.1273e-06	6.8640e-08 6.2145e-06
100	1.6402e-08 1.6377e-06	2.0990e-08 2.0860e-06	2.6815e-08 2.6406e-06	4.3336e-08 4.1273e-06	6.8640e-08 6.2145e-06

Table 1. Values of e_1, e_2 when p=0.99 and $\lambda = 1$ and $y_2 = 0$

u/ y_2	0.5	0.75	1	1.5	2
1	0.1715 0.0982	0.4589 0.2078	0.9731 0.3557	3.1413 0.7846	8.0979 1.4466
2	0.0294 0.0377	0.1203 0.1285	0.3458 0.3154	1.7267 1.2030	6.0993 3.3977
3	0.0037 0.0074	0.0231 0.0388	0.0895 0.1298	0.6829 0.7747	3.2708 3.0291
4	0.0004 0.0010	0.0035 0.0080	0.0183 0.0363	0.2117 0.3322	1.3674 1.7669
5	3.1270e-05 1.0755e-04	0.0004 0.0013	0.0031 0.0078	0.0540 0.1083	0.4693 0.7782
10	1.6419e-08 1.0979e-06	2.2864e-08 1.1952e-06	8.2553e-08 1.0564e-06	7.6584e-06 3.0409e-05	0.0003 0.0010
20	1.6402e-08 1.0980e-06	2.0990e-08 1.2065e-06	2.6815e-08 1.3508e-06	4.3336e-08 1.7517e-06	6.8640e-08 2.3187e-06
50	1.6402e-08 1.0980e-06	2.0990e-08 1.2065e-06	2.6815e-08 1.3508e-06	4.3336e-08 1.7517e-06	6.8640e-08 2.3187e-06
100	1.6402e-08 1.0980e-06	2.0990e-08 1.2065e-06	2.6815e-08 1.3508e-06	4.3336e-08 1.7517e-06	6.8640e-08 2.3187e-06

Table 2. Values of e_1, e_2 when p=0.99 and $\lambda = 1$ and $y_1 = 0$

We can see that a very good approximation can be achieved when u is around 20. As u increases further, the errors are approximately the same (from the computer calculation). This indicates that the power series coefficients drop sharply when the indexes increase. We will test the system by changing the parameters a little bit.

Suppose $y_2 = 0$, p=0.5 and $\lambda = 0.5$. We calculate the values of e_1, e_2 and list them in the following table.

u/ y_1	0.5	0.75	1	1.5	2
1	0.0085 0.0085	0.0199 0.0199	0.0370 0.0370	0.0906 0.0906	0.1756 0.1756
2	3.5816e-04 3.5870e-04	0.0013 0.0013	0.0031 0.0031	0.0117 0.0117	0.0305 0.0305
3	1.0704e-05 1.0164e-05	5.9285e-05 5.8710e-05	1.9835e-04 1.9774e-04	0.0011 0.0011	0.0039 0.0039
4	8.2260e-07 1.3629e-06	2.8337e-06 3.4089e-06	1.0641e-05 1.1254e-05	8.5277e-05 8.5971e-05	3.9677e-04 3.9755e-04
5	5.3443e-07 1.0748e-06	5.0429e-07 1.0795e-06	1.9196e-07 8.0423e-07	4.6384e-06 3.9446e-06	3.2589e-05 3.1803e-05
10	5.4033e-07 1.0807e-06	5.7518e-07 1.1504e-06	6.1227e-07 1.2245e-06	6.9380e-07 1.3876e-06	7.8619e-07 1.5724e-06
20	5.4033e-07 1.0807e-06	5.7518e-07 1.1504e-06	6.1227e-07 1.2245e-06	6.9379e-07 1.3876e-06	7.8619e-07 1.5724e-06
50	5.4033e-07 1.0807e-06	5.7518e-07 1.1504e-06	6.1227e-07 1.2245e-06	6.9379e-07 1.3876e-06	7.8619e-07 1.5724e-06
100	5.4033e-07 1.0807e-06	5.7518e-07 1.1504e-06	6.1227e-07 1.2245e-06	6.9379e-07 1.3876e-06	7.8619e-07 1.5724e-06

Table 3. Values of e_1, e_2 when p=0.5 and $\lambda = 0.5$ and $y_2 = 0$.

We can see that the approximation is very good when the value of u is not less than 10. So we can trust the power series system to obtain the marginal survival function $\overline{F_1}(y_1, 0)$ and $\overline{F_2}(y_1, 0)$ in this case.

The marginal survival function $\overline{F_1}(0, y_2)$ and $\overline{F_2}(0, y_2)$ can also be obtained by power series method with good precision. See the values of e_1, e_2 below:

u/ y_2	0.5	0.75	1	1.5	2
1	0.0085 0.0140	0.0199 0.0301	0.0370 0.0514	0.0906 0.1084	0.1756 0.1832
2	3.5816e-04 0.0018	0.0013 0.0061	0.0031 0.0143	0.0117 0.0475	0.0305 0.1126
3	1.0704e-05 1.3796e-04	5.9285e-05 6.8764e-04	1.9835e-04 0.0022	0.0011 0.0110	0.0039 0.0353
4	8.2260e-07 6.4478e-06	2.8337e-06 5.5044e-05	1.0641e-05 2.3510e-04	8.5277e-05 0.0018	3.9677e-04 0.0079
5	5.3443e-07 1.2884e-06	5.0429e-07 4.6364e-06	1.9196e-07 2.1599e-05	4.6384e-06 2.4010e-04	3.2589e-05 0.0014
10	5.4033e-07 9.6703e-07	5.7518e-07 9.8239e-07	6.1227e-07 1.0027e-06	6.9380e-07 1.0559e-06	7.8619e-07 1.1010e-06
20	5.4033e-07 9.6703e-07	5.7518e-07 9.8239e-07	6.1227e-07 1.0027e-06	6.9379e-07 1.0570e-06	7.8617e-07 1.1284e-06
50	5.4033e-07 9.6703e-07	5.7518e-07 9.8239e-07	6.1227e-07 1.0027e-06	6.9379e-07 1.0570e-06	7.8617e-07 1.1284e-06
100	5.4033e-07 9.6703e-07	5.7518e-07 9.8239e-07	6.1227e-07 1.0027e-06	6.9379e-07 1.0570e-06	7.8617e-07 1.1284e-06

Table 4. Values of e_1, e_2 when p=0.5 and $\lambda = 0.5$ and $y_1 = 0$.

After investigating the case of marginal distribution, we would like to test the system further.

We would like to see the symmetrical case first, that is $y_1 = y_2$. Suppose p=0.5 and $\lambda = 1$, we calculate the corresponding errors, and list them below:

u/ $y_1 = y_2$	0.5	0.75	1	1.5	2
1	0.0189 0.0027	0.0187 0.0189	0.1597 0.0753	1.0818 0.3565	3.7120 1
2	5.5199e-04 0.0077	0.0024 0.0258	0.0148 0.0753	0.4963 0.5375	3.7120 2.7896
3	5.5510e-04 0.0010	0.0046 0.0031	0.0228 0.0017	0.3353 0.1755	3.1885 2.1931
4	6.2985e-05 9.1093e-05	5.4504e-04 4.6360e-04	0.0016 0.0019	0.0516 0.0583	1.3560 1.1491
5	6.2117e-06 7.0015e-06	7.0847e-05 2.4294e-05	4.2537e-04 6.9061e-05	0.0141 0.0121	0.4503 0.4035
10	7.4848e-07 1.2751e-06	9.1432e-07 1.4974e-06	1.1074e-06 1.7704e-06	2.0002e-06 2.6775e-06	4.0307e-05 2.6630e-05
20	7.4848e-07 1.2751e-06	9.1436e-07 1.4971e-06	1.1060e-06 1.7637e-06	1.5883e-06 2.4547e-06	2.2469e-06 3.4134e-06
50	7.4848e-07 1.2751e-06	9.1436e-07 1.4971e-06	1.1060e-06 1.7637e-06	1.5883e-06 2.4547e-06	2.2469e-06 3.4134e-06
100	7.4848e-07 1.2751e-06	9.1436e-07 1.4971e-06	1.1060e-06 1.7637e-06	1.5883e-06 2.4547e-06	2.2469e-06 3.4134e-06

Table 5. Values of e_1, e_2 when p=0.5 and $\lambda = 1$ and $y_1 = y_2$.

In this case, the power series system is much more reliable. The error is almost neglectable when u is bigger than 20.

So far we can see that we can trust the system when u is larger than 20. In the following test, we are only concerned about the case when u=20 or u=50.

We increase values of y_1, y_2 gradually to see the scope of errors. They are listed in Table 6-11.

u\ (y_1, y_2)	0.05 0.05	0.05 0.5	0.05 1	0.05 1.5	0.05 2
20	5.0098e-07 9.7840e-07	6.2435e-07 1.0270e-06	7.9827e-07 1.1532e-06	1.0216e-06 1.3475e-06	1.3084e-06 1.6155e-06
50	5.0098e-07 9.7840e-07	6.2435e-07 1.0270e-06	7.9827e-07 1.1532e-06	1.0216e-06 1.3475e-06	1.3084e-06 1.6155e-06

Table 6. Values of e_1, e_2 when p=0.5, $y_1 = 0.05$ and $\lambda = 1$

u\(y_1, y_2)	0.1 0.1	0.1 0.5	0.1 1	0.1 1.5	0.1 2
20	5.2573e-07 1.0048e-06	6.3675e-07 1.0519e-06	8.1074e-07 1.1787e-06	1.0342e-06 1.3740e-06	1.3211e-06 1.6434e-06
50	5.2573e-07 1.0048e-06	6.3675e-07 1.0519e-06	8.1074e-07 1.1787e-06	1.0342e-06 1.3740e-06	1.0342e-06 1.3740e-06

Table 7. Values of e_1, e_2 when p=0.5, $y_1 = 0.1$ and $\lambda = 1$

u\(y_1, y_2)	0.5 0.5	0.5 0.75	0.5 1	0.5 1.5	0.5 2
20	7.4848e-07 1.2751e-06	8.3091e-07 1.3319e-06	9.2447e-07 1.4070e-06	1.1509e-06 1.6114e-06	1.4415e-06 1.8936e-06
50	7.4848e-07 1.2751e-06	8.3091e-07 1.3319e-06	9.2447e-07 1.4070e-06	1.1509e-06 1.6114e-06	1.4415e-06 1.8936e-06

Table 8. Values of e_1, e_2 when p=0.5, $y_1 = 0.5$ and $\lambda = 1$

u\(y_1, y_2)	1 0.5	1 0.75	1 1	1 1.5	1 2
20	9.2447e-07 1.6244e-06	1.0093e-06 1.6841e-06	1.1060e-06 1.7637e-06	1.3408e-06 1.9819e-06	1.6422e-06 2.2840e-06
50	9.2447e-07 1.6244e-06	1.0093e-06 1.6841e-06	1.1060e-06 1.7637e-06	1.3408e-06 1.9819e-06	1.6422e-06 2.2840e-06

Table 9. Values of e_1, e_2 when p=0.5, $y_1 = 1$ and $\lambda = 1$

u\(y_1, y_2)	1.5 0.5	1.5 0.75	1.5 1	1.5 1.5	1.5 2
20	1.1509e-06 2.0727e-06	1.2393e-06 2.1355e-06	1.3408e-06 2.2203e-06	1.5883e-06 2.4547e-06	1.9065e-06 2.7812e-06
50	1.1509e-06 2.0727e-06	1.2393e-06 2.1355e-06	1.3408e-06 2.2203e-06	1.5883e-06 2.4547e-06	1.9065e-06 2.7812e-06

Table 10. Values of e_1, e_2 when p=0.5, $y_1 = 1.5$ and $\lambda = 1$

u\(y_1, y_2)	2 0.5	2 0.75	2 1	2 1.5	2 2
20	1.4415e-06 2.6479e-06	1.5346e-06 2.7142e-06	1.6422e-06 2.8047e-06	1.9065e-06 3.0580e-06	2.2469e-06 3.4134e-06
50	1.4415e-06 2.6479e-06	1.5346e-06 2.7142e-06	1.6422e-06 2.8047e-06	1.9065e-06 3.0580e-06	2.2469e-06 3.4134e-06

Table 11. Values of e_1, e_2 when p=0.5, $y_1 = 2$ and $\lambda = 1$

In conclusion, we can get very good approximation of $\overline{F_1}(y_1, y_2)$ and $\overline{F_2}(y_1, y_2)$ by $\overline{F_1}(y_1, y_2 \mid u, PS)$ and $\overline{F_2}(y_1, y_2 \mid u, PS)$ respectively, if we choose a large value of u.

With the success of numerical test in the previous Kibble model using identity reward matrix, we would like to test the power series method in another example.

Recall that $\sum_{j=1}^{k} r_j(i) \frac{\partial \overline{F_i}}{y_j} = \sum_{j=1}^{m} q_{ij} \overline{F_j}$

The PDE for the general bivariate case can be written as:

$$
\begin{aligned}
R_{11} \frac{\partial \overline{F_1}}{y_1} + R_{21} \frac{\partial \overline{F_1}}{y_2} &= T_{11} \overline{F_1} + T_{12} \overline{F_2} \\
R_{12} \frac{\partial \overline{F_2}}{y_1} + R_{22} \frac{\partial \overline{F_2}}{y_2} &= T_{21} \overline{F_1} + T_{22} \overline{F_2}
\end{aligned}
$$

The following power recursive equations are obtained:

$$
\begin{aligned}
R_{11} \alpha_{i+1,j}(i+1) + R_{21} \alpha_{i,j+1}(j+1) &= T_{11} \alpha_{ij} + T_{12} \beta_{ij} \\
R_{21} \beta_{i+1,j}(i+1) + R_{22} \beta_{i,j+1}(j+1) &= T_{21} \alpha_{ij} + T_{22} \beta_{ij}
\end{aligned}
$$

If we use reward matrix with non-zero element, we get two recursive equations as follows:

$$
\alpha_{mn} = \frac{T_{11} \alpha_{m-1,n} + T_{12} \beta_{m-1,n} - R_{21} \alpha_{m-1,n+1}(n+1)}{m R_{11}} \qquad (m > 0)
$$

$$
\beta_{mn} = \frac{T_{21} \alpha_{m-1,n} + T_{22} \beta_{m-1,n} - R_{22} \beta_{m-1,n+1}(n+1)}{m R_{12}} \qquad (m > 0)
$$

Therefore given all boundary power series coefficients, we can obtain all remaining power series coefficients.

The example is the same as from Chapter 2: Erlang (2) distribution

Consider $Y = (Y_1, Y_2) \sim MPH^*(\alpha, T; R)$, with

$$T=\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \text{ and } R=\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$$

Another representation of Y could be conjured:

$$Y=\begin{bmatrix}Y_1\\ Y_2\end{bmatrix}=\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}\begin{bmatrix}Z_1\\ Z_2\end{bmatrix}$$

Where (Z_1, Z_2) are independent exp(1) distributed random variables.

We are interested in the case that $\frac{1}{2}y_2 < y_1 < 2y_2$, since all other cases are too simple to consider. In this case we have

$$\overline{F_1}(y_1,y_2)=3\exp(-y_1-y_2)-\exp(-3y_1)-\exp(-3y_2)$$
$$=3\sum_{k=0}^{\infty}\frac{(-y_1)^k}{k!}\sum_{k=0}^{\infty}\frac{(-y_2)^k}{k!}-\sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!}-\sum_{k=0}^{\infty}\frac{(-3y_2)^k}{k!}$$
$$\overline{F_2}(y_1,y_2)=\exp(-3y_1)=\sum_{k=0}^{\infty}\frac{(-3y_1)^k}{k!}$$

The boundary power series coefficients are listed below:

$$\alpha_{m0}=3\frac{(-1)^m}{m!}-\frac{(-3)^m}{m!}, \beta_{m0}=\frac{(-3)^m}{m!}, \text{ m>0}$$
$$\alpha_{0n}=3\frac{(-1)^n}{n!}-\frac{(-3)^n}{n!}, \beta(0,n)=0, \text{ n>0}$$
$$\alpha_{00}=1, \beta_{00}=1$$

Using the recursive equations and these boundary values, we can calculate α_{mn} and β_{mn} for $1\le m,n\le u$, where u is an integer.

Similarly,

$$\overline{F_1}(y_1,y_2)\approx\sum_{i=0}^{u}\sum_{j=0}^{u}\alpha_{ij}y_1{}^{i}y_2{}^{j}=\overline{F_1}(y_1,y_2\,|\,u,PS)$$
$$\overline{F_2}(y_1,y_2)\approx\sum_{i=0}^{u}\sum_{j=0}^{u}\beta_{ij}y_1{}^{i}y_2{}^{j}=\overline{F_2}(y_1,y_2\,|\,u,PS)$$

Furthermore we define $e_1=\left|\frac{\overline{F_1}(y_1,y_2)-\overline{F_1}(y_1,y_2\,|\,u,PS)}{\overline{F_1}(y_1,y_2)}\right|$

$$e_2=\left|\frac{\overline{F_2}(y_1,y_2)-\overline{F_2}(y_1,y_2\,|\,u,PS)}{\overline{F_2}(y_1,y_2)}\right|$$

We will start to try y_1, y_2 with values nearly 0. Values of errors are listed in Table 1.

u\\(y_1, y_2)	0.05 0.0251	0.05 0.03	0.05 0.05	0.05 0.075	0.05 0.099
50	3.3481e-16 0	0 0	0 0	3.3702e-16 0	1.1318e-16 0
100	3.3481e-16 0	0 0	0 0	3.3702e-16 0	1.1318e-16 0

Table 12. Values of e_1, e_2 with small values of $y_1, y_2 (\approx 0)$

The results are very promising especially for $\overline{F_2}(y_1, y_2 | u, PS)$, which can replace $\overline{F_2}(y_1, y_2)$ completely. We will further increase values of y_1, y_2 gradually. See results in the table below.

u\\(y_1, y_2)	0.5 0.3	0.5 0.5	0.5 0.75	0.5 0.9	0.5 0.99
50	6.1826e-16 3.7318e-16	3.3777e-16 3.7318e-16	4.1817e-16 3.7318e-16	9.8806e-16 3.7318e-16	1.6584e-15 3.7318e-16
100	6.1826e-16 3.7318e-16	3.3777e-16 3.7318e-16	4.1817e-16 3.7318e-16	9.8806e-16 3.7318e-16	1.6584e-15 3.7318e-16
u\\(y_1, y_2)	1 0.51	1 0.75	1 1	1 1.5	1 1.99
50	8.4021e-16 1.3937e-16	1.5161e-15 1.3937e-16	1.4492e-15 1.3937e-16	1.4525e-14 1.3937e-16	2.9299e-14 1.3937e-16
100	8.4021e-16 1.3937e-16	1.5161e-15 1.3937e-16	1.4492e-15 1.3937e-16	1.4525e-14 1.3937e-16	2.9299e-14 1.3937e-16
u\\(y_1, y_2)	2 1.01	2 1.5	2 2	2 3	2 3.99
50	3.3165e-14 7.0929e-13	1.4526e-13 7.0929e-13	2.2154e-13 7.0929e-13	5.5876e-12 7.0929e-13	8.2959e-10 7.0929e-13
100	3.3165e-14 7.0929e-13	1.4526e-13 7.0929e-13	2.2154e-13 7.0929e-13	5.5876e-12 7.0929e-13	1.7277e-10 7.0929e-13

Table 13. Values of e_1, e_2 (we let y_1 change from 0.5 to 2)

In conclusion, we can get very good approximation of $\overline{F_1}(y_1, y_2)$ and $\overline{F_2}(y_1, y_2)$ by $\overline{F_1}(y_1, y_2 | u, PS)$ and $\overline{F_2}(y_1, y_2 | u, PS)$ respectively, if we choose a large value of u.

4.4 Summary

The aim of this chapter is to see the possibilities of getting back to the original survival function of phase type distributed random variables if the power series coefficients are known. Theoretically the answer is yes. However, in most cases, we are not able to get a general explicit representation of power series in terms indexes i and j. What we always have is the marginal distribution, which indicates that we can obtain all boundary power series coefficients. Using the general recursive equations, we have seen that we can get all the remaining coefficients. At this point, we have interest in the possibilities of approximating the survival functions given finite power series coefficients. We have used two examples to see the performance, which shows that we can get a very good approximation.

Chapter 5 Conclusion

In this project we have introduced the core definition and properties of phase type distributions. One of the essential applications of phase type distribution can be found by the combination of continuous-time Markov chain and reward system. Since phase type distributions play an important role in applied probability models, it is important to study its distribution. Usually it is difficult to compute this distribution directly. There are several computation techniques for the distributions in MPH*, of which we have a particular interest in the PDE method. Several concrete examples have been shown. The probabilistic representations of the survival functions satisfy the partial differential equations derived from the general case.

The aim of the project is to find out the possibilities of obtaining distribution or survival function by the computation technique of the PDEs. To do this, we have introduced power series method to get a deeper insight into the application of partial differential equations when studying phase type distributions. We have obtained the recursive equations of power series coefficients, and verified that we are able to derive all coefficients if given only the boundary values. Several concrete examples have been shown to work in this framework of recursive equations.

In most cases, it is not possible to get all power series coefficients in an explicit form. Also it is difficult to get a closed form of distribution or survival functions while relatively easy to get the marginal distributions. So alternatively, we are able to obtain the approximate values of distribution functions given the boundary power series coefficients. We have used two examples to see the performance, of which the key indicators are the errors defined as the difference between the probabilistic result and the approximate values. The performances show that we can get a very good approximation.

However, the limitation of the power series method is that it will be very difficult to get a closed form of the survival functions. More specifically, what we get is an infinite sum of power series, which doesn't have an explicit representation normally. Therefore we are restricted to the "approximate" value of the distribution functions.

The ideal goal is that we could solve the differential equations directly, but that's not an easy task. The potential improving area might lie in other methods in complex analysis and partial differential equations properties.

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